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Switching design for the robust stability of nonlinear uncertain stochastic switched discrete-time systems with interval time-varying delay

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Abstract

This paper is concerned with robust stability of nonlinear uncertain stochastic switched discrete time-delay systems with interval time-varying delay. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust stability for the nonlinear uncertain stochastic switched discrete time-delay system with interval time-varying delay is designed via linear matrix inequalities.

Keywords. Switching design, nonlinear uncertain stochastic switched discrete system, time-varying delay, robust stability, Lyapunov function, linear matrix inequality.

1 Introduction

Time delay is often a source of instability and poor performance, and is encountered in various engineering systems, such as chemical processes and long transmission lines in pneumatic systems. Time-delay systems have received much attention in recent years, and various topics concerning time-delay systems have been addressed; see, e.g., [1-10], and the references cited therein. As an important class of hybrid systems, switched systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models, such as manufacturing, communication networks, automotive engineering control and chemical processes (see, e.g., [1–3] and the references therein). On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4–8]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [9–11]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays (see, e.g., [1–3] and the references therein). It was shown in [5, 7, 12] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [13], but the result was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme. To the best of the author's knowledge, the stability for linear discrete-time systems with both time-varying delays and nonlinear uncertain stochastic discrete switch system has not been fully investigated (see, e.g., [14–20] and the references therein), which are important in both theories and applications. This motivates our research.

This paper studies robust stability problem for nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to robustly stable of the nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delay. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the robust stability of the nonlinear uncertain stochastic switched discrete-time delay system with interval time-varying delay. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for robust stability in terms of LMIs.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the robust stability is presented in Section 3.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or x^Ty; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)-$ dimension. N^+ denotes the set of all non-negative integers; A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.

Matrix A is semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n; A$ is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a nonlinear uncertain stochastic switched discrete-time delay systems with interval time-varying delay of the form

$$\begin{aligned} x(k+1) = (A_\gamma + \Delta A_\gamma(k))x(k) + (B_\gamma + \Delta B_\gamma(k))x(k-d(k)) + f(k, x(k-d(k))) \\ + \sigma(x(k), x(k-d(k)), k)\omega(k), \quad k \in N^+, \quad x(k) = v_k, \quad k = -d_2, -d_2+1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i, B_i, i = 1, 2, ..., N$ are given constant matrices.

The nonlinear perturbations $f(k, x(k-d(k)))$ satisfies the following condition

$$f^T(k, x(k-d(k)))f(k, x(k-d(k))) \leq \beta^2 x^T(k-d(k))x(k-d(k)), \tag{2.2}$$

where β is positive constant. For simplicity, we denote $f(k, x(k-d(k))$ by f, respectively.

The time-varying uncertain matrices $\Delta A_i(k)$ and $\Delta B_i(k)$ are defined by:

$$\Delta A_i(k) = E_{ia}F_{ia}(k)H_{ia}, \quad \Delta B_i(k) = E_{ib}F_{ib}(k)H_{ib},$$

where $E_{ia}, E_{ib}, H_{ia}, H_{ib}$ are known constant real matrices with appropriate dimensions.
$F_{ia}(k), F_{ib}(k)$ are unknown uncertain matrices satisfying

$$F^T_{ia}(k)F_{ia}(k) \leq I, \quad F^T_{ib}(k)F_{ib}(k) \leq I, \quad k = 0, 1, 2, ..., \tag{2.3}$$

where I is the identity matrix of appropriate dimention, $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E[\omega(k)] = 0, \quad E[\omega^2(k)] = 1, \quad E[\omega(i)\omega(j)] = 0 (i \neq j), \tag{2.4}$$

and σ: $R^n \times R^n \times R \to R^n$ is the continuous function, and is assumed to satisfy that

$$\begin{aligned}&\sigma^T(x(k), x(k-d(k)), k)\sigma(x(k), x(k-d(k)), k) \leq \rho_1 x^T(k)x(k) + \rho_2 x^T(k-d(k))x(k-d(k),\\ &x(k), x(k-d(k) \in R^n,\end{aligned} \tag{2.5}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known constant scalars. The time-varying function $d(k) : N^+ \to N^+$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k \in N^+$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The nonlinear uncertain stochastic switched system with interval time-varying delay (2.1) is robustly stable in the mean square if there exists a positive definite scalar function $V(k, x(k) : R^n \times R^n \to R$ and a switching rule $\gamma(.)$ such that

$$E[\Delta V(k, x(k))] = E[V(k+1, x(k+1)) - V(k, x(k))] < 0,$$

along any trajectory of solution of the system (2.1) for all uncertainties which satisfy (2.3).

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

Proposition 2.1. [22] *The system $\{J_i\}, i = 1, 2, \ldots, N$, is strictly complete if there exist $\delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strict completeness.

Proposition 2.2. (Cauchy inequality) *For any symmetric positive definite marix $N \in M^{n\times n}$ and $a, b \in R^n$ we have*

$$\pm a^T b \leq a^T N a + b^T N^{-1} b.$$

Proposition 2.3. [23] *Let E, H and F be any constant matrices of appropriate dimensions and $F^T F \leq I$. For any $\epsilon > 0$, we have*

$$EFH + H^T F^T E^T \leq \epsilon EE^T + \epsilon^{-1} H^T H.$$

3 Main results

Let us set

$$\begin{aligned}
&W_i(S_1,S_2,P,Q)=\begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14}\\ * & W_{i22} & W_{i23} & W_{i24}\\ * & * & W_{i33} & W_{i34}\\ * & * & * & W_{i44}\end{bmatrix},\\
&W_{i11}=Q-P,\\
&W_{i12}=S_1-S_1A_i,\\
&W_{i13}=-S_1B_i,\\
&W_{i14}=-S_1-S_2A_i,\\
&W_{i22}=P+S_1+S_1^T+S_1E_{ib}E_{ib}^TS_1^T+H_{ia}^TH_{ia},\\
&W_{i23}=-S_1B_i,\\
&W_{i24}=S_2-S_1,\\
&W_{i33}=-Q+S_2E_{ib}E_{ib}^TS_2^T+H_{ia}^TH_{ia}+H_{ib}^TH_{ib}+\rho_2 I,\\
&W_{i34}=-S_2B_i,\\
&W_{i44}=-S_2-S_2^T+H_{ia}^TH_{ia}+H_{ib}^TH_{ib},\\
&J_i(S_1,S_2,Q)=(d_2-d_1)Q-S_1A_i-A_i^TS_1^T+S_1E_{ia}E_{ia}^TS_1^T\\
&\qquad\qquad +S_1E_{ib}E_{ib}^TS_1^T+S_2E_{ia}E_{ia}^TS_2^T+H_{ia}^TH_{ia}+\rho_1 I,\\
&\alpha_i=\{x\in R^n:\quad x^TJ_i(S_1,S_2,Q)x<0\},\ i=1,2,...,N,\\
&\bar\alpha_1=\alpha_1,\quad \bar\alpha_i=\alpha_i\setminus\bigcup_{j=1}^{i-1}\bar\alpha_j,\quad i=2,3,\dots,N.
\end{aligned}\tag{3.1}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The nonlinear uncertain stochastic switched system with interval time-varying delay (2.1) is robustly stable if there exist symmetric positive definite matrices $P>0,Q>0$ and matrices S_1,S_2 satisfying the following conditions:*

(i) $\exists\delta_i\ge 0, i=1,2,\dots,N, \sum_{i=1}^N\delta_i>0: \sum_{i=1}^N\delta_iJ_i(S_1,S_2,Q)<0.$
(ii) $W_i(S_1,S_2,P,Q)<0,\quad i=1,2,...,N.$

The switching rule is chosen as $\gamma(x(k))=i$, whenever $x(k)\in\bar\alpha_i$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k)=V_1(k)+V_2(k)+V_3(k),$$

where

$$V_1(k)=x^T(k)Px(k),\quad V_2(k)=\sum_{i=k-d(k)}^{k-1}x^T(i)Qx(i),$$

$$V_3(k)=\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1}x^T(l)Qx(l),$$

We can verify that

$$\lambda_1\|x(k)\|^2\le V(k).\tag{3.2}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f_i(k, x(k-d(k)))\, \omega(k)]^T$ and

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P & 0 & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system (2.1) and taking the mathematical expectation, we obtained

$$\begin{aligned} E[\Delta V_1(k)] &= E[x^T(k+1)Px(k+1) - x^T(k)Px(k)] \\ &= E[\xi^T(k)H\xi(k) - 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix}]. \end{aligned} \tag{3.3}$$

because of

$$\begin{aligned} \xi^T(k)H\xi(k) &= x(k+1)Px(k+1), \\ 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \end{pmatrix} &= x^T(k)Px(k). \end{aligned}$$

Using the expression of system (2.1)

$$\begin{aligned} 0 &= -S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_1(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_1f \\ &\quad + S_1\sigma\omega(k), \\ 0 &= -S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_2(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_2f \\ &\quad + S_2\sigma\omega(k), \\ 0 &= -\sigma^Tx(k+1) + \sigma^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + \sigma^T(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + \sigma^Tf \\ &\quad + \sigma^T\sigma\omega(k), \end{aligned}$$

we have
$E[-2\xi^T(k)G^T$

$$\times \begin{pmatrix} 0.5x(k) \\ \begin{matrix}[-S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_1(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_1f \\ +S_1\sigma\omega(k)]\end{matrix} \\ \begin{matrix}[-S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + S_2(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + S_2f \\ +S_2\sigma\omega(k)]\end{matrix} \\ \begin{matrix}[-\sigma^Tx(k+1) + \sigma^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) + \sigma^T(B_i + E_{ib}F_{ib}(k)H_{ib})x(k-d(k)) + \sigma^Tf \\ +\sigma^T\sigma\omega(k)]\end{matrix} \end{pmatrix}]$$

$$= E[-\xi^T(k)G^T \begin{pmatrix} 0.5I & 0 & 0 & 0 & 0 \\ S_1A_i + S_1E_{ia}F_{ia}(k)H_{ia} & -S_1 & S_1B_i + S_1E_{ib}F_{ib}(k)H_{ib} & S_1 & S_1\sigma \\ S_2A_i + S_2E_{ia}F_{ia}(k)H_{ia} & -S_2 & S_2B_i + S_2E_{ib}F_{ib}(k)H_{ib} & S_2 & S_2\sigma \\ \sigma^TA_i + \sigma^TE_{ia}F_{ia}(k)H_{ia} & -\sigma^T & \sigma^TB_i + \sigma^TE_{ib}F_{ib}(k)H_{ib} & \sigma^T & \sigma^T\sigma \end{pmatrix} \xi(k)$$

$$-\xi^T(k)\begin{pmatrix} 0.5I & 0 & 0 & 0 & 0 \\ S_1A_i+S_1E_{ia}F_{ia}(k)H_{ia} & -S_1 & S_1B_i+S_1E_{ib}F_{ib}(k)H_{ib} & S_1 & S_1\sigma \\ S_2A_i+S_2E_{ia}F_{ia}(k)H_{ia} & -S_2 & S_2B_i+S_2E_{ib}F_{ib}(k)H_{ib} & S_2 & S_2\sigma \\ \sigma^TA_i+\sigma^TE_{ia}F_{ia}(k)H_{ia} & -\sigma^T & \sigma^TB_i+\sigma^TE_{ib}F_{ib}(k)H_{ib} & \sigma^T & \sigma^T\sigma \end{pmatrix}^T G\xi(k)].$$

Therefore, from (3.3) it follows that

$$\begin{aligned} E[\Delta V_1(k)] = E[&x^T(k)[-P-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}-A_i^TS_1^T-H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \\ &+2x^T(k)[S_1-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1) \\ &+2x^T(k)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x^T(k)[-S_1-S_2A_i-S_2E_{ia}F_{ia}(k)H_{ia}]f(k,x(k-d(k))) \\ &+2x^T(k)[-S_1\sigma-\sigma^TA_i-\sigma^TE_{ia}F_{ia}(k)H_{ia}]\omega(k) \\ &+x(k+1)[P+S_1+S_1^T]x(k+1) \\ &+2x(k+1)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x(k+1)[S_2-S_1]f(k,x(k-d(k))) \\ &+2x(k+1)[\sigma^T-S_1\sigma]\omega(k) \\ &+2x^T(k-d(k))[-S_3B_i-S_2E_{ib}F_{ib}(k)H_{ib}]f(k,x(k-d(k))) \\ &+2x^T(k-d(k))[-\sigma^TB_i-\sigma^TE_{ib}F_{ib}(k)H_{ib}]\omega(k) \\ &+f(k,x(k-d(k)))^T[-S_2-S_2^T]f(k,x(k-d(k))) \\ &+2f(k,x(k-d(k)))^T(k)[-S_2\sigma-\sigma^T]\omega(k) \\ &+\omega^T(k)[-\sigma^T\sigma]\omega(k)]. \end{aligned}$$

By asumption (2.4), we have

$$\begin{aligned} E[\Delta V_1(k)] = E[&x^T(k)[-P-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}-A_i^TS_1^T-H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \\ &+2x^T(k)[S_1-S_1A_i-S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1) \\ &+2x^T(k)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x^T(k)[-S_1-S_2A_i-S_2E_{ia}F_{ia}(k)H_{ia}]f(k,x(k-d(k))) \\ &+x(k+1)[P+S_1+S_1^T]x(k+1) \\ &+2x(k+1)[-S_1B_i-S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k)) \\ &+2x(k+1)[S_2-S_1]f(k,x(k-d(k))) \\ &+2x^T(k-d(k))[-S_2B_i-S_2E_{ib}F_{ib}(k)H_{ib}]f(k,x(k-d(k))) \\ &+f(k,x(k-d(k)))^T[-S_2-S_2^T]f(k,x(k-d(k))) \\ &+\omega^T(k)[-\sigma^T\sigma]\omega(k)]. \end{aligned}$$

Applying Propositon 2.2, Propositon 2.3, condition (2.3) and asumption (2.5), the following estimations hold

$$-S_1E_{ia}F_{ia}(k)H_{ia}-H_{ia}^TF_{ia}^T(k)E_{ia}^TS_1^T \le S_1E_{ia}E_{ia}^TS_1^T+H_{ia}^TH_{ia},$$
$$-2x^T(k)S_1E_{ia}F_{ia}(k)H_{ia}x(k+1) \le x^T(k)S_1E_{ia}E_{ia}^TS_1^Tx(k)+x(k+1)^TH_{ia}^TH_{ia}x(k+1),$$
$$-2x^T(k)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k)) \le x^T(k)S_1E_{ib}E_{ib}^TS_1^Tx(k)+x(k-d(k))^TH_{ib}^TH_{ib}x(k-d(k)),$$
$$-2x^T(k)S_2E_{ia}F_{ia}(k)H_{ia}f \le x^T(k)S_2E_{ia}E_{ia}^TS_2^Tx(k)+f^TH_{ia}^TH_{ia}f,$$

$$-2x(k-d(k))^T(k)S_2E_{ib}F_{ib}(k)H_{ib}f \le x(k-d(k))^T(k)S_2E_{ib}E_{ib}^TS_2^Tx(k-d(k)) + f^TH_{ib}^TH_{ib}f,$$
$$-2x^T(k+1)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k)) \le x^T(k+1)S_1E_{ib}E_{ib}^TS_1^Tx(k+1) + x(k-d(k))^TH_{ib}^TH_{ib}x(k-d(k)),$$
$$-\sigma^T(x(k),x(k-d(k)),k)\sigma(x(k),x(k-d(k)),k) \le \rho_1x^T(k)x(k) + \rho_2x^T(k-d(k))x(k-d(k).$$

Therefore, we have

$$\begin{aligned} E[\Delta V_1(k)] \le E[&x^T(k)[-P - S_1A_i - A_i^TS_1^T + S_1E_{ia}E_{ia}^TS_1^T + S_1E_{ib}E_{ib}^TS_1^T \\ &+ S_2E_{ia}E_{ia}^TS_2^T + H_{ia}^TH_{ia} + \rho_1 I]x(k) \\ &+ 2x^T(k)[S_1 - S_1A_i]x(k+1) \\ &+ 2x^T(k)[-S_1B_i]x(k-d(k)) \\ &+ 2x^T(k)[-S_1 - S_2A_i]f(k,x(k-d(k))) \\ &+ x(k+1)[P + S_1 + S_1^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia}]x(k+1) \\ &+ 2x(k+1)[-S_1B_i]x(k-d(k)) \\ &+ 2x(k+1)[S_2 - S_1]f(k,x(k-d(k))) \\ &+ x^T(k-d(k))[S_2E_{ib}E_{ib}^TS_3^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib} + \rho_2 I]x(k-d(k)) \\ &+ 2x^T(k-d(k))[-S_3B_i]f(k,x(k-d(k))) \\ &+ f(k,x(k-d(k)))^T[-S_2 - S_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib}]f(k,x(k-d(k)))]. \end{aligned} \tag{3.4}$$

The difference of $V_2(k)$ is given by

$$\begin{aligned} E[\Delta V_2(k)] &= E[\sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)] \\ &= E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \\ &\quad + \sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i)]. \end{aligned} \tag{3.5}$$

Since $d(k) \ge d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i) \le 0,$$

and hence from (3.5) we have

$$E[\Delta V_2(k)] \le E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))]. \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}
E[\Delta V_3(k)] &= E[\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j}^{k} x^T(l)Qx(l) - \sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1} x^T(l)Qx(l)] \\
&= E[\sum_{j=-d_2+2}^{-d_1+1}[\sum_{l=k+j}^{k-1} x^T(l)Qx(l) + x^T(k)Q(\xi)x(k) \\
&\quad - \sum_{l=k+j}^{k-1} x^T(l)Qx(l) - x^T(k+j-1)Qx(k+j-1)]] \qquad (3.7)\\
&= E[\sum_{j=-d_2+2}^{-d_1+1}[x^T(k)Qx(k) - x^T(k+j-1)Qx(k+j-1)]] \\
&= E[(d_2-d_1)x^T(k)Qx(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Qx(j)].
\end{aligned}$$

Since $d(k) \le d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i) \le 0,$$

we obtain from (3.6) and (3.7) that

$$E[\Delta V_2(k) + \Delta V_3(k)] \le E[(d_2-d_1+1)x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))]. \qquad (3.8)$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$E[\Delta V(k)] \le E[x^T(k)J_i(S_1,S_2,Q)x(k) + \psi^T(k)W_i(S_1,S_2,P,Q)\psi(k)], \qquad (3.9)$$

where

$$\psi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f(k,x(k-d(k)))]^T,$$

$$\begin{aligned}
W_i(S_1,S_2,P,Q) &= \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14} \\ * & W_{i22} & W_{i23} & W_{i24} \\ * & * & W_{i33} & W_{i34} \\ * & * & * & W_{i44} \end{bmatrix}, \\
W_{i11} &= Q - P, \\
W_{i12} &= S_1 - S_1A_i, \\
W_{i13} &= -S_1B_i, \\
W_{i14} &= -S_1 - S_2A_i, \\
W_{i14} &= -S_1 - S_2A_i, \\
W_{i22} &= P + S_1 + S_1^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia}, \\
W_{i23} &= -S_1B_i, \\
W_{i24} &= S_2 - S_1, \\
W_{i33} &= -Q + S_2E_{ib}E_{ib}^TS_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib} + \rho_2 I, \\
W_{i34} &= -S_2B_i, \\
W_{i44} &= -S_2 - S_2^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib}, \\
J_i(S_1,S_2,Q) &= (d_2-d_1)Q - S_1A_i - A_i^TS_1^T + 2S_1E_{ia}E_{ia}^TS_1^T + S_1E_{ib}E_{ib}^TS_1^T \\
&\quad + S_2E_{ia}E_{ia}^TS_2^T + H_{ia}^TH_{ia} + \rho_1 I.
\end{aligned}$$

Therefore, we finally obtain from (3.9) and the condition (ii) that

$$E[\Delta V(k)] < E[x^T(k)J_i(S_1, S_2, Q)x(k)], \quad \forall i = 1, 2, ..., N, k = 0, 1, 2, ...$$

We now apply the condition (i) and Proposition 2.1., the system $J_i(S_1, S_2, Q)$ is strictly complete, and the sets α_i and $\bar{\alpha}_i$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_i = R^n \backslash \{0\}, \quad \bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset, i \neq j.$$

Therefore, for any $x(k) \in R^n, k = 1, 2, ...,$ there exists $i \in \{1, 2, \dots, N\}$ such that $x(k) \in \bar{\alpha}_i$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_i$, from the condition (3.9) we have

$$E[\Delta V(k)] \leq E[x^T(k)J_i(S_1, S_2, Q)x(k)] < 0, \quad k = 1, 2, ...,$$

which, combining the condition (3.2), and Definition 2.1., concludes the proof of the theorem in the mean square.

Remark 3.1. Note that the results proposed in [5, 7, 12] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [13] was limited to constant delays. In [21], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4 Conclusion

This paper has proposed a switching design for the robust stability of nonlinear uncertain stochastic switched discrete time-delay systems with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stability for the nonlinear uncertain stochastic switched discrete time-delay system with interval time-varying delay is designed via linear matrix inequalities.

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References

[1] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag., **19**(1999), 57-70.

[2] A.V. Savkin and R.J. Evans, Hybrid Dynamical Systems: Controller and Sensor Switching Problems, Springer, New York, 2001.

[3] Z. Sun and S.S. Ge, Switched Linear Systems: Control and Design, Springer, London, 2005.

[4] F. Gao, S. Zhong and X. Gao, Delay-dependent stability of a type of linear switching systems with discrete and distributed time delays, Appl. Math. Computation, **196**(2008), 24-39.

[5] G Rajchakit, T Rojsiraphisal, M Rajchakit, Robust stability and stabilization of uncertain switched discrete-time systems, Advances in Difference Equations 2012, 2012:134. doi:10.1186/1687-1847-2012-134

[6] M Rajchakit, G Rajchakit, Mean square robust stability of stochastic switched discrete-time systems with convex polytopic uncertainties, Journal of Inequalities and Applications 2012, 2012:135. doi:10.1186/1029-242X-2012-135

[7] S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

[8] D.H. Ji, J.H. Park, W.J. Yoo and S.C. Won, Robust memory state feedback model predictive control for discrete-time uncertain state delayed systems, Appl. Math. Computation, **215**(2009), 2035-2044.

[9] K. Ratchagit and V.N. Phat, Stability criterion for discrete-time systems, J. Ineq. Appl., **2010**(2010), 1-6.

[10] G.S. Zhai, B. Hu, K. Yasuda, and A. Michel, Qualitative analysis of discrete- time switched systems. In: Proc. of the American Control Conference, 2002, 1880-1885.

[11] Manlika Rajchakit, Piyapong Niamsup, Grienggrai Rajchakit, A switching rule for exponential stability of switched recurrent neural networks with interval time-varying delay, Advances in Difference Equations 2013, 2013:44. doi:10.1186/1687-1847-2013-44

[12] Dong, H, Wang, Z, Ho, DWC, Gao, H: , Robust H_∞ filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: the finite-horizon case. IEEE Trans. Sig. Process. **59**, 3048–3057 (2011)

[13] K. Ratchagit, A switching rule for the asymptotic stability of discrete-time systems with convex polytopic uncertainties, Asian-European J. Math., **5**(2012), 1250025 (12 pages).

[14] Jun Li, Weigen Wu, Jimin Yuan, Qianrong Tan, and Xing Yin, Delay-Dependent Stability Criterion of Arbitrary Switched Linear Systems with Time-Varying Delay, Discrete Dynamics in Nature and Society, **2010**(2010).

[15] M. Rajchakit and G. Rajchakit, Mean Square Exponential Stability of Stochastic Switched System with Interval Time-Varying Delays, Abstract and Applied Analysis, vol. 2012, Article ID 623014, 12 pages, 2012. doi:10.1155/2012/623014

[16] Wang, Z, Wei, G, Feng, G: Reliable H_∞ control for discrete-time piecewise linear systems with infinite distributed delays. Automatica **45**, 2991–2994 (2009)

[17] M. Rajchakit and G. Rajchakit, LMI approach to robust stability and stabilization of nonlinear uncertain discrete-time systems with convex polytopic uncertainties, Advances in Difference Equations, **2012**(2012).

[18] Wang, Y, Wang, Z, Liang, J: A delay fractioning approach to global synchronization of delayed complex networks with stochastic disturbances. Phys. Lett. A **372**, 6066–6073 (2008)

[19] Wang, Z, Wang, Y, Liu, Y: Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays. IEEE Trans. Neural Netw. **21**, 11–25 (2010)

[20] V. N. Phat and K. Ratchagit, Stability and stabilization of switched linear discrete-time systems with interval time-varying delay, Nonlinear Analysis: Hybrid Systems, **5**(2011), 605-612.

[21] W.A. Zhang, Li Yu, Stability analysis for discrete-time switched time-delay systems, Automatica, **45**(2009), 2265-2271.

[22] F. Uhlig, A recurring theorem about pairs of quadratic forms and extensions, Linear Algebra Appl., **25**(1979), 219-237.

[23] R.P. Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

Stabilization of switched discrete-time systems with convex polytopic uncertainties

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Abstract

This paper is concerned with robust stabilization of switched discrete time-delay systems with convex polytopic uncertainties. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust stabilization for the system with convex polytopic uncertainties is designed via linear matrix inequalities.

Keywords. Switching design, convex polytopic uncertainties, discrete system, robust stabilization, Lyapunov function, linear matrix inequality.

AMS (MOS) Subject Classification. 34D20, 93D20, 37C75.

1 Introduction

In many physical phenomena and practical applications, such as autonomous transmission systems, computer disc drivers, room temperature control, power electronics, chaos generators (see, e.g., [1–3] and the references therein), they are governed by more than one dynamical systems (differential or difference equations) governed by switching laws to determine which subsystem will be activated on a certain time interval. Such systems are called switched systems. On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4–7]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [8–10]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. It was shown in [5, 7, 11] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [12], but the result was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme. To the best of our knowledge, the stabilization of discrete-time systems with both convex polytopic uncertainties and switch system, non-differentiable time-varying delays has not been fully studied yet (see, e.g., [1, 4–27] and the references therein), which are important in both theories and applications. This motivates our research.

This paper studies robust stabilization problem for switched linear discrete systems with convex polytopic uncertainties with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to robust stabilization the system. By using improved Lyapunov-Krasovskii functionals

combined with LMIs technique, we propose new criteria for the robust stabilization of the system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for robust stabilization in terms of of LMIs.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the robust stabilization is presented in Section 3.

2 Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)-$ dimension. A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.
Matrix A is semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a linear switched control discrete-time systems with convex polytopic uncertainties with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= A_{\gamma(x(k))}(\zeta)x(k) + B_{\gamma(x(k))}(\zeta)u(k), \quad k = 0, 1, 2, ... \\ x(k) &= v_k, \quad k = -d_2, -d_2 + 1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m, m \leq n$, is the control input, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \dots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system.
We consider a delayed feedback control law

$$u(k) = C_{\gamma(x(k))}(\zeta)x(k - d(k)), k = -h_2, ..., 0, \tag{2.2}$$

and $C_{\gamma(x(k))}(\zeta)$ is the controller gain to be determined. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i(\zeta), B_i(\zeta), C_i(\zeta), i = 1, 2, ..., N$ are given constant matrices. The system matrices are subjected to uncertainties and belong to the polytope Ω given by

$$\Omega = \{[A_i, B_i, C_i](\zeta) := \sum_{j=1}^{N} \zeta_j [A_{ij}, B_{ij}, C_{ij}], \quad \sum_{j=1}^{N} \zeta_j = 1, \zeta_j \geq 0\},$$

where A_{ij}, B_{ij}, C_{ij}, $i, j = 1, 2, ..., N$, are given constant matrices with appropriate dimensions. The time-varying function $d(k)$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k = 0, 1, 2, ...$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Applying the feedback controller (2.2) to the system (2.1), the closed-loop discrete time-delay system is

$$x(k+1) = A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k - d(k)), \quad k = 0, 1, 2, ... \tag{2.3}$$

Definition 2.1. The system (2.1) is robustly stablilizable if there exist a switching function $\gamma(.)$ and a delayed feedback control (2.2) such that the zero solution of the system (2.3) is asymptotically stable for all uncertainties in Ω.

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

Proposition 2.1. [28] *The system* $\{J_i\}, i = 1, 2, \ldots, N$, *is strictly complete if there exist* $\delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ *such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ *then the above condition is also necessary for the strict completeness.*

Proposition 2.2. *For real numbers* $\zeta_j \geq 0, j = 1, 2, ..., N$, $\sum_{j=1}^{N} \zeta_j = 1$, *the following inequality hold*

$$(N-1) \sum_{j=1}^{N} \zeta_j^2 - 2 \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l \geq 0.$$

Proof. The proof is followed from the completing the square:

$$(N-1) \sum_{j=1}^{N} \zeta_j^2 - 2 \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l = \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} (\zeta_j - \zeta_l)^2 \geq 0.$$

3 Main results

Let us set

$$\|x_k\| = \sup_{s \in [-d_2, 0]} \|x(k+s)\|,$$

$$W_{ijj}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{ij}^T R_j & -R_j^T B_{ij} C_{ij} \\ R_j - R_j^T A_{ij} & P_j + R_j + R_j^T & -R_j^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_j & -C_{ij}^T B_{ij}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ijl}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{il}^T R_j & -R_j^T B_{il} C_{il} \\ R_j - R_j^T A_{il} & P_j + R_j + R_j^T & -R_j^T B_{il} C_{il} \\ -C_{il}^T B_{il}^T R_j & -C_{il}^T B_{il}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ilj}(P,Q,R) = \begin{pmatrix} Q_l - P_l & R_l^T - A_{ij}^T R_l & -R_l^T B_{ij} C_{ij} \\ R_l - R_l^T A_{ij} & P_l + R_l + R_l^T & -R_l^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_l & -C_{ij}^T B_{ij}^T R_l & -Q_l \end{pmatrix},$$

$$\mathcal{R} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P(\zeta) = \sum_{j=1}^{N} \zeta_j P_j, \quad Q(\zeta) = \sum_{j=1}^{N} \zeta_j Q_j, \quad R(\zeta) = \sum_{j=1}^{N} \zeta_j R_j, \quad \lambda_1 = \lambda_{\min}(P),$$

$$J_{ijj}(R,Q) := (d_2-d_1)Q_j - A_{ij}^T R_j - R_j^T A_{ij},$$

$$J_{ijl}(R,Q) := (d_2-d_1)Q_j - A_{il}^T R_j - R_j^T A_{il},$$

$$J_{ilj}(R,Q) := (d_2-d_1)Q_l - A_{ij}^T R_l - R_l^T A_{ij},$$

$$\begin{aligned}
\alpha_{ijj} =& \{x \in R^n: \quad x^T J_{ijj}(R,Q)x < 0,\}, \quad i=1,2,...,N, j=1,2,...,N,\\
\alpha_{ijl} =& \{x \in R^n: \quad x^T J_{ijl}(R,Q)x < 0,\}, \quad i=1,2,...,N, j=1,2,...,N-1; l=j+1,...,N,\\
\alpha_{ijl} =& \{x \in R^n: \quad x^T J_{ilj}(R,Q)x < 0,\}, \quad i=1,2,...,N, j=1,2,...,N-1; l=j+1,...,N,\\
\bar{\alpha}_{1jj} =& \alpha_{1jj}, \quad \bar{\alpha}_{ijj} = \alpha_{ijj} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ijj}, \quad i=2,3,\ldots,N, j=1,2,\ldots,N,\\
\bar{\alpha}_{1jl} =& \alpha_{1jl}, \quad \bar{\alpha}_{ijl} = \alpha_{ijl} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ijl}, \quad i=2,3,\ldots,N, j=1,2,...,N-1; l=j+1,...,N,\\
\bar{\alpha}_{1lj} =& \alpha_{1lj}, \quad \bar{\alpha}_{ilj} = \alpha_{ilj} \setminus \bigcup_{i=1}^{i-1} \bar{\alpha}_{ilj}, \quad i=2,3,\ldots,N, j=1,2,...,N-1; l=j+1,...,N.
\end{aligned} \tag{3.1}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The switched control system with convex polytopic uncertainties (2.1) is stabilizable by the delayed feedback control (2.2) if there exist symmetric matrices $P_i > 0, Q_i > 0, \mathcal{R} \geq 0, i = 1,2...,N$ and matrix $R_i, i = 1,2...,N$ satisfying the following conditions*

(i) $\exists \delta_i \geq 0, \quad \sum_{i=1}^N \delta_i > 0 : \sum_{i=1}^N \delta_i J_{ijj} < 0,$ *and* $J_{ijj} + \mathcal{R} < 0, \quad i = 1,2,\ldots,N,$
$j = 1,2,\ldots,N.$

(ii) $\exists \delta_i \geq 0, \quad \sum_{i=1}^N \delta_i > 0 : \sum_{i=1}^N [\delta_i J_{ijl} + \delta_i J_{ilj}] < 0,$ *and* $J_{ijl} + J_{ilj} - \frac{2}{N-1}\mathcal{R} < 0,$
$i = 1,2,\ldots,N, \quad j = 1,2,\ldots,N-1, \quad l = j+1,\ldots,N.$

(iii) $W_{ijj} + \mathcal{R} < 0, \quad i = 1,2,...,N, \quad j = 1,2,...,N.$

(iv) $W_{ijl} + W_{ilj} - \frac{2}{N-1}\mathcal{R} < 0, \quad i = 1,2,...,N, \quad j = 1,2,...,N-1; \quad l = j+1,...,N.$

The switching rule is chosen as $\gamma(x(k)) = i$, whenever $x(k) \in \bar{\alpha}_{ijl}$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = x^T(k)P(\zeta)x(k), \quad V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i),$$

$$V_3(k) = \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Q(\zeta)x(l),$$

We can verify that

$$\lambda_1 \|x(k)\|^2 \leq V(k). \tag{3.2}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))]^T$, and

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P(\zeta) & 0 & 0 \\ R(\zeta) & R(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system is given by

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)P(\zeta)x(k+1) - x^T(k)P(\zeta)x(k) \\ &= \xi^T(k)H(\zeta)\xi(k) - 2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

because of

$$\xi^T(k)H(\zeta)\xi(k) = x(k+1)P(\zeta)x(k+1).$$

Using the expression of system (2.3)

$$0 = -x(k+1) + A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k-d(k)),$$

we have

$$\begin{aligned} &-2\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5x(k) \\ -x(k+1) + A_i(\zeta)x(k) + B_i(\zeta)C_i(\zeta)x(k-d(k)) \\ 0 \end{pmatrix}\xi(k) \\ &= -\xi^T(k)G^T(\zeta)\begin{pmatrix} 0.5I & 0 & 0 \\ A_i(\zeta) & -I & B_i(\zeta)C_i(\zeta) \\ 0 & 0 & 0 \end{pmatrix}\xi(k) - \xi^T(k)\begin{pmatrix} 0.5I & A_i(\zeta)^T & 0 \\ 0 & -I & 0 \\ 0 & (B_i(\zeta)C_i(\zeta))^T & 0 \end{pmatrix}G(\zeta)\xi(k). \end{aligned}$$

Therefore, from (3.3) it follows that

$$\Delta V_1(k) = \xi^T(k)W_i(P(\zeta), Q(\zeta), R(\zeta))\xi(k), \tag{3.4}$$

where

$$\begin{aligned} W_i(P(\zeta), Q(\zeta), R(\zeta)) = &\begin{pmatrix} 0 & 0 & 0 \\ 0 & P(\zeta) & 0 \\ 0 & 0 & 0 \end{pmatrix} - G^T(\zeta)\begin{pmatrix} 0.5I & 0 & 0 \\ A_i(\zeta) & -I & B_i(\zeta)C_i(\zeta) \\ 0 & 0 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0.5I & A_i^T(\zeta) & 0 \\ 0 & -I & 0 \\ 0 & (B_i(\zeta)C_i(\zeta))^T & 0 \end{pmatrix}G(\zeta). \end{aligned}$$

The difference of $V_2(k)$ is given by

$$\begin{aligned} \Delta V_2(k) &= \sum_{i=k+1-d(k+1)}^{k} x^T(i)Q(\zeta)x(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Q(\zeta)x(i) \\ &= \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)) \\ &\quad + \sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i). \end{aligned} \tag{3.5}$$

Since $d(k) \geq d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q(\zeta)x(i) \leq 0,$$

and hence from (3.5) we have

$$\Delta V_2(k) \leq \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) + x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)). \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}
\Delta V_3(k) &= \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j}^{k} x^T(l)Q(\zeta)x(l) - \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Q(\zeta)x(l) \\
&= \sum_{j=-d_2+2}^{-d_1+1} [\sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) + x^T(k)Q(\zeta)(\xi)x(k) \\
&\quad - \sum_{l=k+j}^{k-1} x^T(l)Q(\zeta)x(l) - x^T(k+j-1)Q(\zeta)x(k+j-1)] \\
&= \sum_{j=-d_2+2}^{-d_1+1} [x^T(k)Q(\zeta)x(k) - x^T(k+j-1)Q(\zeta)x(k+j-1)] \\
&= (d_2-d_1)x^T(k)Q(\zeta)x(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Q(\zeta)x(j).
\end{aligned} \tag{3.7}$$

Since $d(k) \leq d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Q(\zeta)x(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Q(\zeta)x(i) \leq 0,$$

we obtain from (3.6) and (3.7) that

$$\Delta V_2(k) + \Delta V_3(k) \leq (d_2-d_1+1)x^T(k)Q(\zeta)x(k) - x^T(k-d(k))Q(\zeta)x(k-d(k)). \tag{3.8}$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$\Delta V(k) \leq x^T(k)J_i(R(\zeta),Q(\zeta))x(k) + \xi^T(k)W_i(P(\zeta),Q(\zeta),R(\zeta))\xi(k), \tag{3.9}$$

where

$$W_i(P(\zeta),Q(\zeta),R(\zeta)) = \begin{pmatrix} Q(\zeta)-P(\zeta) & R^T(\zeta)-A_i^T(\zeta)R(\zeta) & -R^T(\zeta)B_i(\zeta)C_i(\zeta) \\ R(\zeta)-R^T(\zeta)A_i(\zeta) & P(\zeta)+R(\zeta)+R^T(\zeta) & -R^T(\zeta)B_i(\zeta)C_i(\zeta) \\ -C_i^T(\zeta)B_i^T(\zeta)R(\zeta) & -C_i^T(\zeta)B_i^T(\zeta)R(\zeta) & -Q(\zeta) \end{pmatrix},$$

and

$$J_i(R(\zeta),Q(\zeta)) = (d_2-d_1)Q(\zeta) - A_i^T(\zeta)R(\zeta) - R^T(\zeta)A_i(\zeta).$$

Let us denote

$$W_{ijj}(P,Q,R) = \begin{pmatrix} Q_j-P_j & R_j^T-A_{ij}^TR_j & -R_j^TB_{ij}C_{ij} \\ R_j-R_j^TA_{ij} & P_j+R_j+R_j^T & -R_j^TB_{ij}C_{ij} \\ -C_{ij}^TB_{ij}^TR_j & -C_{ij}^TB_{ij}^TR_j & -Q_j \end{pmatrix},$$

$$W_{ijl}(P,Q,R) = \begin{pmatrix} Q_j - P_j & R_j^T - A_{il}^T R_j & -R_j^T B_{il} C_{il} \\ R_j - R_j^T A_{il} & P_j + R_j + R_j^T & -R_j^T B_{il} C_{il} \\ -C_{il}^T B_{il}^T R_j & -C_{il}^T B_{il}^T R_j & -Q_j \end{pmatrix},$$

$$W_{ilj}(P,Q,R) = \begin{pmatrix} Q_l - P_l & R_l^T - A_{ij}^T R_l & -R_l^T B_{ij} C_{ij} \\ R_l - R_l^T A_{ij} & P_l + R_l + R_l^T & -R_l^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_l & -C_{ij}^T B_{ij}^T R_l & -Q_l \end{pmatrix},$$

$$J_{ijj}(R,Q) := (d_2 - d_1)Q_j - A_{ij}^T R_j - R_j^T A_{ij},$$

$$J_{ijl}(R,Q) := (d_2 - d_1)Q_j - A_{il}^T R_j - R_j^T A_{il},$$

$$J_{ilj}(R,Q) := (d_2 - d_1)Q_l - A_{ij}^T R_l - R_l^T A_{ij},$$

$$(A_i^T R)_{jl} := A_{il}^T R_j + A_{ij}^T R_l, \quad (R^T A_i)_{jl} = R_j^T A_{il} + R_l^T A_{ij},$$

$$(R^T B_i C_i)_{jl} = R_j^T B_{il} C_{il} + R_l^T B_{ij} C_{ij}, \quad (C_i^T B_i^T R)_{jl} = C_{il}^T B_{il}^T R_j + C_{ij}^T B_{ij}^T R_l,$$

$$P_{jl} = P_j + P_l, \quad Q_{jl} = Q_j + Q_l, \quad R_{jl} = R_j + R_l.$$

From the convex combination of the expression of $P(\zeta), Q(\zeta), R(\zeta), A(\zeta), B(\zeta), C(\zeta)$, we have

$$\begin{aligned} W_i(P(\zeta),Q(\zeta),R(\zeta)) &= \sum_{j=1}^{N} \zeta_j^2 \begin{pmatrix} Q_j - P_j & R_j^T - A_{ij}^T R_j & -R_j^T B_{ij} C_{ij} \\ R_j - R_j^T A_{ij} & P_j + R_j + R_j^T & -R_j^T B_{ij} C_{ij} \\ -C_{ij}^T B_{ij}^T R_j & -C_{ij}^T B_{ij}^T R_j & -Q_j \end{pmatrix} \\ &+ \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l \begin{pmatrix} Q_j - P_j + Q_l - P_l & R_{jl}^T - (A_i^T R)_{jl} & -(R^T B_i C_i)_{jl} \\ R_{jl} - (R^T A_i)_{jl} & P_{jl} + R_{jl} + R_{jl}^T & -(R^T B_i C_i)_{jl} \\ -(C_i^T B_i^T R)_{jl} & -(C_i^T B_i^T R)_{jl} & -Q_{jl} \end{pmatrix} \\ &= \sum_{j=1}^{N} \zeta_j^2 W_{ijj}(P,Q,R) + \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l [W_{ijl}(P,Q,R) + W_{ilj}(P,Q,R)]. \end{aligned}$$

$$\begin{aligned} J_i(R(\zeta),Q(\zeta)) &= \sum_{j=1}^{N} \zeta_j^2 (d_2 - d_1)Q_j - A_{ij}^T R_j - R_j^T A_{ij} \\ &+ \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l (d_2 - d_1) Q_{jl} - (A_i^T R)_{jl} - (R^T A_i)_{jl} \\ &= \sum_{j=1}^{N} \zeta_j^2 J_{ijj}(Q,R) + \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l [J_{ijl}(Q,R) + J_{ilj}(Q,R)]. \end{aligned}$$

Then the conditions (i)-(iv) give

$$W_i(P(\zeta),Q(\zeta),R(\zeta)) < -\sum_{j=1}^{N} \zeta_j^2 \mathcal{R} + \frac{2}{N-1} \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l \mathcal{R} \le 0,$$

$$J_i(R(\zeta),Q(\zeta)) < -\sum_{j=1}^{N} \zeta_j^2 \mathcal{R} + \frac{2}{N-1} \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l \mathcal{R} \le 0,$$

because of Proposition 2.2:

$$(N-1) \sum_{j=1}^{N} \zeta_j^2 - 2 \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} \zeta_j \zeta_l = \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} (\zeta_j - \zeta_l)^2 \ge 0.$$

Therefore, we finally obtain from (3.9) and the condition (iii), (iv) that

$$\Delta V(k) < x^T(k)J_i(R(\zeta), Q(\zeta))x(k), \quad \forall i = 1, 2, ..., N, k = 0, 1, 2, ...$$

We now apply the condition (i), (ii), and Proposition 2.1., the system $J_i(R(\zeta), Q(\zeta))$ is strictly complete, and the sets α_{ijl} and $\bar{\alpha}_{ijl}$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_{ijl} = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_{ijl} = R^n \backslash \{0\}, \quad \bar{\alpha}_{ijl} \cap \bar{\alpha}_{tjl} = \emptyset, i \neq t.$$

Therefore, for any $x(k) \in R^n, k = 0, 1, 2, ...$, there exists $i \in \{1, 2, \dots, N\}$ such that $x(k) \in \bar{\alpha}_{ijl}$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_{ijl}$, from the condition (3.9) we have

$$\Delta V(k) \leq x^T(k)J_i(R(\zeta), Q(\zeta))x(k) < 0, \quad k = 1, 2, ...,$$

which, combining the condition (3.2) and the Lyapunov stability theorem [29], concludes the proof of the theorem.

Remark 3.1. Note that theresult sproposed in [4,5,6] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [9] was limited to constant delays. In [10], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4 Conclusion

This paper has proposed a switching design for the robust stabilization of switched linear discrete-time systems with convex polytopic uncertainties with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stabilization for the system with convex polytopic uncertainties is designed via linear matrix inequalities.

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References

[1] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag., **19**(1999), 57-70.

[2] A.V. Savkin and R.J. Evans, Hybrid Dynamical Systems: Controller and Sensor Switching Problems, Springer, New York, 2001.

[3] Z. Sun and S.S. Ge, Switched Linear Systems: Control and Design, Springer, London, 2005.

[4] F. Gao, S. Zhong and X. Gao, Delay-dependent stability of a type of linear switching systems with discrete and distributed time delays, Appl. Math. Computation, **196**(2008), 24-39.

[5] C.H. Lien, K.W. Yu, Y.J. Chung, Y.F. Lin, L.Y. Chung and J.D. Chen, Exponential stability analysis for uncertain switched neutral systems with interval-time-varying state delay, Nonlinear Analysis: Hybrid systems, **3**(2009),334–342.

[6] V.N. Phat, T. Bormat and P. Niamsup, Switching design for exponential stability of a class of nonlinear hybrid time-delay systems, Nonlinear Analysis: Hybrid Systems, **3**(2009), 1-10.

[7] G Rajchakit, T Rojsiraphisal, M Rajchakit, Robust stability and stabilization of uncertain switched discrete-time systems, Advances in Difference Equations 2012, 2012:134. doi:10.1186/1687-1847-2012-134

[8] S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

[9] D.H. Ji, J.H. Park, W.J. Yoo and S.C. Won, Robust memory state feedback model predictive control for discrete-time uncertain state delayed systems, Appl. Math. Computation, **215**(2009), 2035-2044.

[10] K. Ratchagit and V.N. Phat, Stability criterion for discrete-time systems, J. Ineq. Appl., **2010**(2010), 1-6.

[11] G.S. Zhai, B. Hu, K. Yasuda, and A. Michel, Qualitative analysis of discrete- time switched systems. In: Proc. of the American Control Conference, 2002, 1880-1885.

[12] V.N. Phat, Robust stability and stabilizability of uncertain linear hybrid systems with state delays, IEEE Trans. CAS II, **52**(2005), 94-98.

[13] K. Ratchagit, A switching rule for the asymptotic stability of discrete-time systems with convex polytopic uncertainties, Asian-European J. Math., **5**(2012), 1250025 (12 pages).

[14] W.A. Zhang, Li Yu, Stability analysis for discrete-time switched time-delay systems, Automatica, **45**(2009), 2265-2271.

[15] V. N. Phat and K. Ratchagit, Stability and stabilization of switched linear discrete-time systems with interval time-varying delay, Nonlinear Analysis: Hybrid Systems, **5**(2011), 605-612.

[16] M Rajchakit, G Rajchakit, Mean square robust stability of stochastic switched discrete-time systems with convex polytopic uncertainties, Journal of Inequalities and Applications 2012, 2012:135. doi:10.1186/1029-242X-2012-135

[17] Huaiqin Wu, Ning Li, Kewang Wang, Guohua Xu, and Qiangqiang Guo, Global Robust Stability of Switched Interval Neural Networks with Discrete and Distributed Time-Varying Delays of Neural Type, Mathematical Problems in Engineering, **2012**(2012).

[18] Manlika Rajchakit, Piyapong Niamsup, Grienggrai Rajchakit, A switching rule for exponential stability of switched recurrent neural networks with interval time-varying delay, Advances in Difference Equations 2013, 2013:44. doi:10.1186/1687-1847-2013-44

[19] Binbin Du and Xiaojie Zhang, Delay-Dependent Stability Analysis and Synthesis for Uncertain Impulsive Switched System with Mixed Delays, Discrete Dynamics in Nature and Society, **2011**(2011).

[20] M. De la Sen and A. Ibeas, Stability Results of a Class of Hybrid Systems under Switched Continuous-Time and Discrete-Time Control, Discrete Dynamics in Nature and Society, **2009**(2009).

[21] M. Rajchakit and G. Rajchakit, Mean Square Exponential Stability of Stochastic Switched System with Interval Time-Varying Delays, Abstract and Applied Analysis, vol. 2012, Article ID 623014, 12 pages, 2012. doi:10.1155/2012/623014

[22] Jinxing Lin and Chunxia Fan, Exponential Admissibility and Dynamic Output Feedback Control of Switched Singular Systems with Interval Time-Varying Delay, Mathematical Problems in Engineering, **2010**(2010).

[23] M. Rajchakit and G. Rajchakit, LMI approach to robust stability and stabilization of nonlinear uncertain discrete-time systems with convex polytopic uncertainties, Advances in Difference Equations, **2012**(2012).

[24] Jun Li, Weigen Wu, Jimin Yuan, Qianrong Tan, and Xing Yin, Delay-Dependent Stability Criterion of Arbitrary Switched Linear Systems with Time-Varying Delay, Discrete Dynamics in Nature and Society, **2010**(2010).

[25] Liguo Zhang, Hongfeng Li, and Yangzhou Chen, Robust Stability Analysis and Synthesis for Switched Discrete-Time Systems with Time Delay, Discrete Dynamics in Nature and Society, **2010**(2010).

[26] M. De la Sen, On the Characterization of Hankel and Toeplitz Operators Describing Switched Linear Dynamic Systems with Point Delays, Abstract and Applied Analysis, **2009**(2009).

[27] K. Ratchagit and V. N. Phat, Robust Stability and Stabilization of Linear Polytopic Delay-Difference Equations with Interval Time-Varying Delays, Neural, Parallel, and Scientific Computations, **19**(2011), 361-372.

[28] F. Uhlig, A recurring theorem about pairs of quadratic forms and extensions, Linear Algebra Appl., **25**(1979), 219-237.

[29] R.P. Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

A preconditioner for block two-by-two symmetric indefinite matrices *

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Abstract

A new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices is presented in this paper. The proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner. Here, we call it the product preconditioner. Results concerning the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix are analyzed. Numerical experiments are used to illustrate the efficiency of the proposed product preconditioner.

Key words: Product preconditioner; Symmetric indefinite matrices; Krylov subspace method

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1 Introduction

Recently, a large amount of work has been devoted to the problem of solving linear systems in saddle point form. Here, our concern is to construct a new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices whose (1,1) and (2,2) block are nonsingular. Often this kind of linear systems in saddle point form is likely to generate from a wide range of applications, such as the Helmholtz equation

$$\begin{cases} \Delta u + (2\pi)^2 u = f(x, y), & (x, y) \in \Omega \cup \Re_2^+, \\ u = 0, & (x, y) \in \partial(\Omega \cup \Re_2^+), \end{cases} \tag{1}$$

with radiation boundary condition

$$\lim_{r\to\infty} r(\frac{\partial u}{\partial \eta} - \imath 2\pi u) = 0, \tag{2}$$

where $\Omega = [0, 1] \times [-1, 0]$ is a unit square domain, $\Re_2^+$ denotes the upper half-space and $\imath$ is the imaginary unit in (2), see [1, 2] for details.

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By using a finite difference discretization to the Helmholtz equation (1) on the uniform grid of Ω, we obtain the linear system in saddle point form

$$\mathcal{A}u = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$, $u = [x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$ and $b = [f^{\mathrm{T}}, g^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$, with $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$, are the unknown and given right-hand side vectors, respectively. Then the coefficient matrix $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ is a nonsingular, symmetric and possibly indefinite matrix, and our main aim is to solve the linear system (3) of $n + m$ linear equations with $n + m$ unknowns.

Iterative procedure is a convenient numerical solution method for computing the linear system (3). Often we have Uzawa's algorithms [3, 4] and multigrid methods [5, 6]. In particular, Krylov subspace methods have become more and more popular for solving the linear system (3), such as the conjugate gradient (CG) and biconjugate gradient stabilized (Bi-CGSTAB) methods, minimal residual method (MINRES), generalized minimal residual (GMRES) and quasi-minimal residual (QMR) methods which have been considered in [7–14].

However, these iterative methods are all likely to suffer from slow convergence for some large linear systems which come from many practical applications like the computational fluid dynamics and structural mechanics. Thus it is necessary to use the idea of preconditioning such that the preconditioned matrix has a tightly clustered eigenvalues, see [1, 15–22] and the references therein.

More precisely, we see that a kind of triangular preconditioner has been proposed by Elman and Silvester [14] and Elman [23] when the (2,2) block matrix $C = 0$. These triangular preconditioners were extended by Kay, Loghin and Wathen [24], Cao [25] and Simoncini [26] to the case where C is symmetric positive or negative semidefinite. In addition, Keller, Gould and Wathen [18] presented a constraint preconditioner for the case $C = 0$, in which they discussed the eigenvalue distribution and form of the eigenvectors of the constraint preconditioned matrix and its minimal polynomial. Thereafter, Dollar and Wathen [19] and Dollar [22] studied an approximation factorization constraint preconditioner by combining with the conjugate gradient method, and extended the idea of [18] by allowing the matrix C to be symmetric and positive semidefinite. Furthermore, we found block diagonal, triangular and constraint preconditioners had been discussed by Siefert and De Sturler [17], Murphy, Golub and Wathen [15], De Sturler and Liesen [16], and Cao [20, 21] for the numerical solution of nonsymmetric or generalized saddle point problems. More preconditioning techniques for solving the linear system in saddle point form can be found in an excellent survey written by Benzi, Golub and Liesen [1].

In this paper, we are concerned with investigating a new preconditioner for the symmetric indefinite linear system (3). The proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner [22]. We call it the product preconditioner. The idea used to develop the product preconditioner can trace back to [27]. Benzi has used the idea in [27] to solve Markov chain problems, see [28, 29]. Results concerning the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix are given in this paper. Numerical experiments with preconditioned GMRES method [30] on certain problem serve to illustrate the efficiency and stability of the proposed product preconditioner.

The remainder of this paper is organized as follows. In Section 2, we first briefly introduce the background material on stationary iterations and matrix splittings, and then construct the

product preconditioner. In Section 3, we analyze the eigensolution distribution of the product preconditioned matrix. Numerical experiments with various preconditioned GMRES methods are presented in Section 4. Finally, conclusions are made in Section 5.

2 Background and product preconditioner

In this section, we first briefly introduce the background material on stationary iterations and matrix splittings from [27, 28, 31], and then construct the product preconditioner.

2.1 Stationary iterations and matrix splittings

Consider the solution of a large sparse linear system of the form $\mathcal{A}u = b$, where $\mathcal{A}$ is a square and nonsingular, symmetric indefinite matrix, and b is the given right-hand vector. Stationary iterative method is likely to be an attractive method by using a splitting of the coefficient matrix $\mathcal{A}$, denoted as

$$\mathcal{A} = M - N,$$

where M is a nonsingular matrix. Then the splitting gives rise to the stationary iterative method

$$u_{k+1} = Tu_k + c, \quad k = 0, 1, \cdots, \tag{4}$$

where $T = M^{-1}N$ is called the iterative matrix, $c = M^{-1}b$, and u_0 is a given initial guess. It is well known that the iterative method (4) converges for any initial guess u_0 if and only if its spectral radius $\rho(T) < 1$ [31].

Recently, Benzi and Szyld have defined a related approach by the alternating iterations

$$\begin{cases} u_{k+1/2} = M_1^{-1}N_1u_k + M_1^{-1}b, \\ u_{k+1} = M_2^{-1}N_2u_{k+1/2} + M_2^{-1}b, \end{cases} \quad k = 0, 1, \cdots, \tag{5}$$

in an excellent paper [27], where $\mathcal{A} = M_1 - N_1 = M_2 - N_2$ are splittings of $\mathcal{A}$, both M_1 and M_2 matrices are nonsingular, and u_0 is defined as above. Not only the existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods were proved, but also the convergence theory of some alternating iterations were analyzed in [27]. In addition, Benzi and Szyld have constructed a splitting $\mathcal{A} = M - N$ based on the nonsingular matrix M_1 and M_2. The splitting is given by (see Eq. (10) in [27])

$$M^{-1} = M_2^{-1}(M_1 + M_2 - \mathcal{A})M_1^{-1}. \tag{6}$$

Evidently, the matrix $M_1 + M_2 - \mathcal{A}$ must be nonsingular for (6) to be well defined.

2.2 Product preconditioner

Now, we construct the product preconditioner as the multiplication of two fairly simple preconditioners from the derivation of the alternating iterations in [27]. The first preconditioner is the famous block Jacobi preconditioner

$$M_{bj} = \begin{pmatrix} A & O \\ O & -C \end{pmatrix}. \tag{7}$$

Note that M_{bj} is nonsingular since both A and C are invertible.

The second preconditioner is the popular nonsingular constraint preconditioner

$$M_{sc} = \begin{pmatrix} G & B \\ B^{\mathrm{T}} & -C \end{pmatrix} \tag{8}$$

discussed in [22], where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, but is not equal to A. In practice, G is often taken to be the diagonal matrix formed with the diagonal entries of A, i.e., $G = \mathrm{diag}(\mathrm{diag}(A))$. Note that the Schur complement matrices $-(C+B^{\mathrm{T}}A^{-1}B)$ and $-(C+B^{\mathrm{T}}G^{-1}B)$ are nonsingular since matrix $\mathcal{A}$ in (3) and M_{sc} in (8) are nonsingular (proof can be found in [20]).

According to the alternating iterations (5) and equation (6), the product preconditioner M_{ps} is given by

$$M_{ps}^{-1} = M_{sc}^{-1}(M_{bj} + M_{sc} - \mathcal{A})M_{bj}^{-1}, \tag{9}$$

where the matrix

$$M_{bj} + M_{sc} - \mathcal{A} = \begin{pmatrix} G & O \\ O & -C \end{pmatrix}$$

is invertible. Hence, M_{ps}^{-1} is well defined. From equation (9), we have the product preconditioner

$$M_{ps} = M_{bj}(M_{bj} + M_{sc} - \mathcal{A})^{-1}M_{sc} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix}. \tag{10}$$

Also, we can rewrite

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix} = \begin{pmatrix} I & O \\ B^{\mathrm{T}}A^{-1} & I \end{pmatrix}\begin{pmatrix} A & AG^{-1}B \\ O & -(C + B^{\mathrm{T}}G^{-1}B) \end{pmatrix},$$

then, we have

$$M_{ps}^{-1} = \begin{pmatrix} A^{-1} - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}B^{\mathrm{T}}A^{-1} & G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1} \\ (C + B^{\mathrm{T}}G^{-1}B)^{-1}B^{\mathrm{T}}A^{-1} & -(C + B^{\mathrm{T}}G^{-1}B)^{-1} \end{pmatrix}.$$

Finally, the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ can be expressed as

$$M_{ps}^{-1}\mathcal{A} = \begin{pmatrix} I & A^{-1}B - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \\ O & (C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \end{pmatrix}. \tag{11}$$

3 Properties of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$

In this section, we focus on analyzing the eigenvalue distribution and form of the eigenvectors of the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$.

3.1 Eigenvalue distribution

In this section, we consider the eigenvalue distribution of the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$. It is well known that the convergence of an iterative method has close relation to the distribution of the eigenvalues of the coefficient matrix for symmetric matrix systems. Hence,

a desired eigenvalue distribution is wished to obtain by the applications of preconditioning techniques. We prove a result of this type as follows.

Theorem 1. *Let $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ defined in (3) be a nonsingular and symmetric indefinite matrix. Preconditioning $\mathcal{A}$ by the product preconditioner*

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix},$$

where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, $G \neq A$, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. Then the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ has

- *an eigenvalue at 1 with multiplicity n;*
- *m eigenvalues which are defined by the generalized eigenvalue problem $(C + B^{\mathrm{T}}A^{-1}B)y = \lambda(C + B^{\mathrm{T}}G^{-1}B)y$.*

Proof. Suppose λ is the eigenvalue of $M_{ps}^{-1}\mathcal{A}$, and $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$ is the corresponding eigenvector. Besides, from (11), we have the preconditioned matrix

$$M_{ps}^{-1}\mathcal{A} = \begin{pmatrix} I & A^{-1}B - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \\ O & (C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B) \end{pmatrix},$$

where $A^{-1}B - G^{-1}B(C + B^{\mathrm{T}}G^{-1}B)^{-1}(C + B^{\mathrm{T}}A^{-1}B)$ is irrelevant to the results in Theorem 1. Hence, by making use of the related knowledge in linear algebra, we obtain the results in Theorem 1 immediately. □

3.2 Eigenvector distribution

To our knowledge, the termination of a Krylov subspace method is not only related to the distribution of eigenvalues of the preconditioned matrix, but also to the number of corresponding linearly independent eigenvectors. Hence, for completeness of this paper, we establish the relationship between eigenvalues and eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ and discuss its eigenvector distribution. The following analysis is similar to the discussions in [4, 18, 22].

We start this part from the generalized eigenvalue problem

$$\begin{pmatrix} A & B \\ B^{\mathrm{T}} & -C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{12}$$

where λ is the eigenvalue of $M_{ps}^{-1}\mathcal{A}$, and $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$ is the corresponding eigenvector. By calculations, we obtain

$$Ax + By = \lambda Ax + \lambda AG^{-1}By \tag{13}$$

and

$$B^{\mathrm{T}}x - Cy = \lambda(B^{\mathrm{T}}x - Cy). \tag{14}$$

From (14), we obtain $(1 - \lambda)(B^{\mathrm{T}}x - Cy) = 0$. Hence, either $\lambda = 1$ or $B^{\mathrm{T}}x - Cy = 0$ holds true. In the former case, we have

$$By = AG^{-1}By. \tag{15}$$

Evidently, equation (15) is satisfied by $y = 0$, and thus there are n linearly independent eigenvectors of the form $(x^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}$ associated with the unit eigenvalue. On the other hand, there

may exist $y \neq 0$ which satisfies (15). Then, without loss of generality, we suppose that there are i $(0 \leq i \leq m)$ linearly independent eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where the components y result from the eigenvalue problem $By = AG^{-1}By$.

Now, suppose $\lambda \neq 1$, then we have $B^{\mathrm{T}}x - Cy = 0$, which implies $y = C^{-1}B^{\mathrm{T}}x$ since C is nonsingular. Substituting this into equation (13), we get the generalized eigenvalue problem

$$(A + BC^{-1}B^{\mathrm{T}})x = \lambda(A + AG^{-1}BC^{-1}B^{\mathrm{T}})x, \tag{16}$$

where x is impossible to be equal to a zero vector. Since if $x = 0$, then we have $y = 0$, which is conflict with the known condition $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} \neq 0$. Therefore, we suppose there exist j $(0 \leq j \leq n)$ linearly independent eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where components x arise from the eigenvalue problem (16) with $y = C^{-1}B^{\mathrm{T}}x$.

We conclude this subsection with the following theorem.

Theorem 2. *Let $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ defined in (3) be a nonsingular and symmetric indefinite matrix. Preconditioning $\mathcal{A}$ by the product preconditioner*

$$M_{ps} = \begin{pmatrix} A & AG^{-1}B \\ B^{\mathrm{T}} & -C \end{pmatrix},$$

where $G \in \mathbb{R}^{n\times n}$ is an approximation of A, $G \neq A$, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. Then the product preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ has $n+m$ eigenvalues as given in Theorem 1 and $n+i+j$ linearly independent eigenvectors. There are

- *n eigenvectors of the form $[x^{\mathrm{T}}, 0^{\mathrm{T}}]^{\mathrm{T}}$ that correspond to case $\lambda = 1$;*
- *$\exists\, i$ $(0 \leq i \leq m)$ eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where the components y construct a basis of the generalized eigenvalue problem $By = AG^{-1}By$ and $\lambda = 1$;*
- *$\exists\, j$ $(0 \leq j \leq n)$ eigenvectors of the form $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$ that correspond to case $\lambda \neq 1$.*

Proof. According to the analysis above, we have obtained the specific form of the eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$. Now, our aim is to prove that the $n+i+j$ eigenvectors are linearly independent, that is, we need to show that

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} + \begin{pmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{pmatrix}$$

$$+ \begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} a_1^{(3)} \\ \vdots \\ a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{17}$$

implies that the vectors $a^{(k)}$ $(k = 1, 2, 3)$ are zero vectors. Multiplying (17) by the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$, and recalling that the first matrix in (17) arises from the case $\lambda_k = 1$ $(k = 1, \cdots, n)$, the second matrix from the case $\lambda_k = 1$ $(k = 1, \cdots, i)$, where the components y are

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basis vectors of the generalized eigenvalue problem $By = \lambda AG^{-1}By$, and the last matrix from the case $\lambda_k \neq 1$ $(k = 1, \cdots, j)$. We have

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} + \begin{pmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{pmatrix}$$

$$+ \begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} \lambda_1^{(3)} a_1^{(3)} \\ \vdots \\ \lambda_j^{(3)} a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{18}$$

Subtracting (17) from (18), we obtain

$$\begin{pmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{pmatrix} \begin{pmatrix} (\lambda_1^{(3)} - 1) a_1^{(3)} \\ \vdots \\ (\lambda_j^{(3)} - 1) a_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the components $x_k^{(3)}$ $(k = 1, \cdots, j)$ are linearly independent eigenvectors which arise from the generalized eigenvalue problem (16) and $y_k^{(3)} = C^{-1}B^{\mathrm{T}}x_k^{(3)}$ $(k = 1, \cdots, j)$. Thus we have

$$(\lambda_k - 1)a_k^{(3)} = 0, \quad k = 1, \cdots, j.$$

As a result of the eigenvalues λ_k $(k = 1, \cdots, j)$ are nonunit. We obtain $a_k^{(3)} = 0$ $(k = 1, \cdots, j)$. In addition, we know the components $y_k^{(2)}$ $(k = 1, \cdots, i)$ are basis vectors of the equation $By = AG^{-1}By$, which implies that $y_k^{(2)}$ $(k = 1, \cdots, i)$ are linearly independent. Thus we have $a_k^{(2)} = 0$ $(k = 1, \cdots, i)$.

Therefore, substituting $a_k^{(2)} = 0$ $(k = 1, \cdots, i)$ and $a_k^{(3)} = 0$ $(k = 1, \cdots, j)$ into (17), then equation (17) simplifies to

$$\begin{pmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_n^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Clearly, $a_k^{(1)} = 0$ $(k = 1, \cdots, n)$ follows from the linear independence of $x_k^{(1)}$ $(k = 1, \cdots, n)$. Summarizing the discussions above, we obtain $a^{(k)} = 0$ $(k = 1, 2, 3)$. $\square$

4 Numerical experiments

In this section, we report on numerical results obtained with a Matlab 7. 0.1 implementation on a Window-XP with 2.93GHz 64-bit processor and 2GB memory. The main goal is to test the product preconditioner (PS) defined in (10) and to compare it with the block diagonal preconditioner (BD)

$$M_{bd} = \begin{pmatrix} G & O \\ O & -(C + B^{\mathrm{T}}G^{-1}B) \end{pmatrix}, \tag{19}$$

presented in [16, 17, 24, 25], the block triangular preconditioner (BT)

$$M_{bt} = \begin{pmatrix} G & B \\ O & -(C + B^{\mathrm{T}}G^{-1}B) \end{pmatrix}, \tag{20}$$

considered in [11, 16, 23–26] and the constraint (SC) preconditioners given in (8) by the computing time (CPU), iteration step (IT) and relative residual error (RES).

There are various strategies to choose G in PS, SC, BD and BT preconditioners. In our computations, we not only take G to be the diagonal matrix formed with the diagonal entries of A, i.e., $G = \mathrm{diag}(\mathrm{diag}(A))$, but also to be the tridiagonal matrix of the (1,1) block matrix of A, that is, $G = \mathrm{tridiag}(A)$. As a representative iterative solver we used GMRES [30] with the right preconditioning in our experiments. All iterations are started from the zero vector, and terminated when $\mathrm{RES} = \|b - Au\|_2/\|b\|_2 \leq 10^{-9}$.

The test problem is the Helmholtz equation (1), together with radiation boundary condition (2), see [2, 9] for details. By using a finite difference discretization to equation (1) on the uniform grid of Ω, we obtain the nonsingular and symmetric indefinite linear system (3), where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ are nonsingular, $B \in \mathbb{R}^{n\times m}$. To be more precise, we have matrix

$$A = K \otimes I + I \otimes K + I \otimes D, \quad B = -(I \otimes e_n), \quad C = I - hT,$$

with $K = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{p\times p}$, $D = -4\pi^2 h^2 I$, $I \in \mathbb{R}^{p\times p}$ an identity matrix, $e_n = [0, 0, \cdots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^p$, $h = 1/(p + 1)$, and $T \in \mathbb{R}^{p\times p}$ a Toeplitz matrix which results from the generating function $f(\theta) = 2|\theta|(\theta^2 - 1)$. Hence, we have $n = p^2$, $m = p$, and the order of the coefficient matrix $\mathcal{A}$ is $n + m$. Moreover, we choose the right-hand vector $b = [f^{\mathrm{T}}, g^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m}$ such that the exact solution of system (3) is $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}} = [1, 1, \cdots, 1]^{\mathrm{T}}$, and GMRES(50) with at most 50 restarts is used in our experiments thus the number 2500 in Table 1 and Table 2 means that the corresponding preconditioned GMRES method does not converge in 2500 iterations.

h		1/32	1/48	1/64	1/80	1/90
$n+m$		992	2256	4032	6320	8010
BD	IT	123	325	794	530	1108
	CPU	0.4530	2.5470	10.7350	12.6410	33.2660
	RES	8.4763e-10	9.7414e-10	9.7214e-10	9.9784e-10	9.9993e-10
BT	IT	98	165	554	785	731
	CPU	0.3440	1.3750	7.7040	18.0790	22.9220
	RES	6.9060e-10	9.6006e-10	9.6952e-10	9.8453e-10	9.9223e-10
SC	IT	98	217	341	850	976
	CPU	0.3280	1.5930	4.4210	17.6250	27.5470
	RES	7.4747e-10	9.4590e-10	9.7868e-10	9.9417e-10	9.9499e-10
PS	IT	8	9	10	10	10
	CPU	0.1250	0.4530	1.2650	2.7180	4.0620
	RES	9.2136e-10	4.2030e-10	9.2256e-11	1.5617e-10	1.9037e-10

Table 1: IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \mathrm{tridiag}(A)$.

h		1/32	1/48	1/64	1/80	1/90
$n+m$		992	2256	4032	6320	8010
BD	IT	261	615	1085	1316	2500
	CPU	0.4220	2.0320	5.8590	11.6560	28.7190
	RES	8.7031e-10	9.9129e-10	9.9364e-10	9.9653e-10	8.1554e-09
BT	IT	211	603	969	2003	1403
	CPU	0.3280	2.0160	5.2500	18.4530	16.2660
	RES	9.5016e-10	9.7251e-10	9.9961e-10	9.9769e-10	9.9885e-10
SC	IT	200	410	1115	1025	1470
	CPU	0.3430	1.3900	6.3750	9.5320	18.0310
	RES	9.8438e-10	9.6321e-10	9.9669e-10	9.9912e-10	9.9463e-10
PS	IT	9	9	10	10	10
	CPU	0.0780	0.1720	0.3430	0.5320	0.7190
	RES	6.2956e-11	6.5957e-10	1.2888e-10	2.1624e-10	2.6212e-10

Table 2: IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \mathrm{diag}(\mathrm{diag}(A))$.

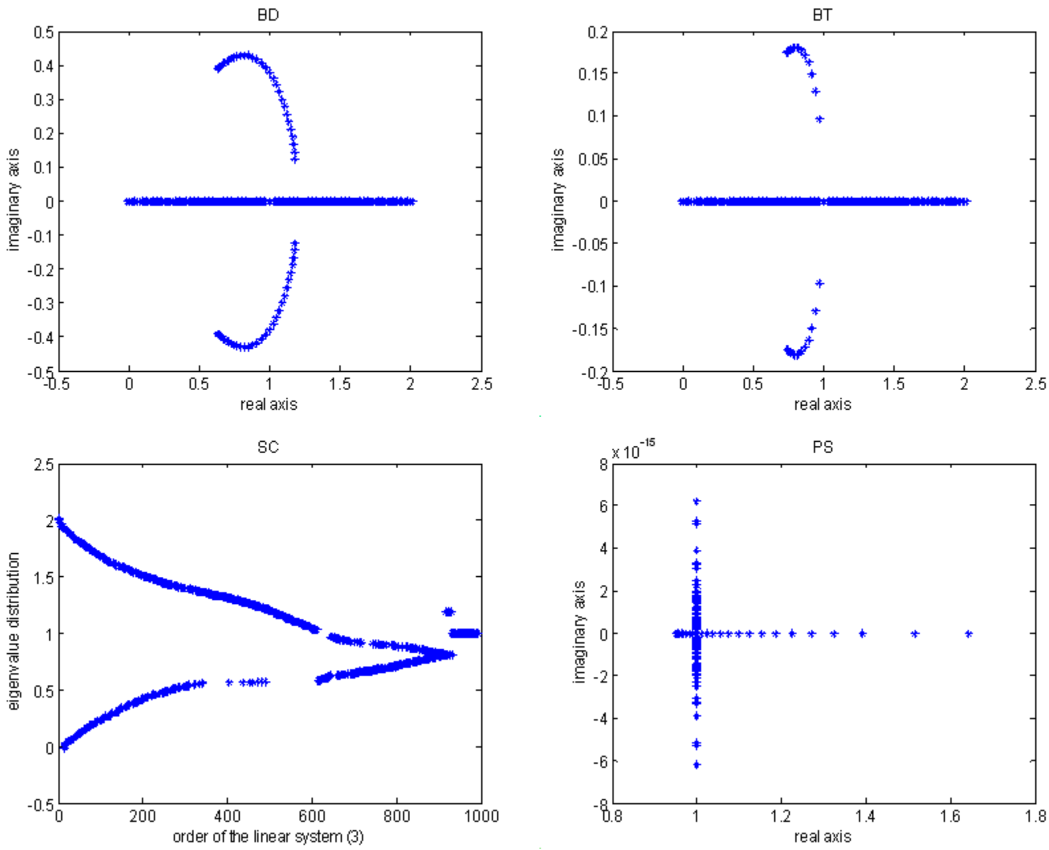

Figure 1: Comparisons of the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for this Helmholtz equation when $G = \mathrm{tridiag}(A)$ and $n+m = 992$.

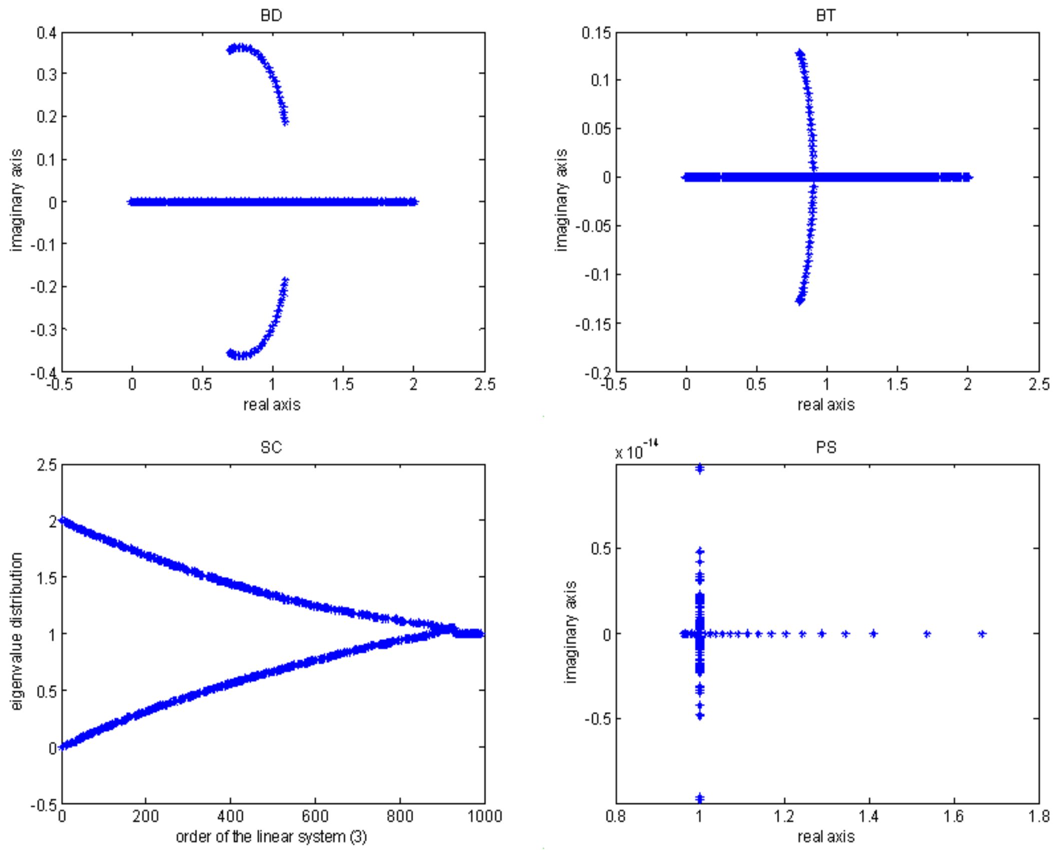

Figure 2: Comparisons of the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for this Helmholtz equation when $G = \mathrm{diag}(\mathrm{diag}(A))$ and $n + m = 992$.

Table 1 supplies the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \mathrm{tridiag}(A)$. As we have seen from Table 1, the PS preconditioned GMRES method has given the best iteration counts. For the BD, BT and SC preconditioned GMRES methods, their iteration counts have been reduced by around 96%. In terms of the computing time, the PS preconditioned GMRES method costs much less than these of the BD, BT and SC preconditioned GMRES methods. In addition, the precision of the relative residual error for the PS preconditioned GMRES method is higher than these of the BD, BT and SC preconditioned GMRES methods, except for the case that $n + m = 992$.

Table 2 provides the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods for this Helmholtz equation when $G = \mathrm{diag}(\mathrm{diag}(A))$. From Table 2, it is not difficult to find that, for this approximate (1,1) block matrix G, all the iteration counts, computing time and the relative residual error of the PS preconditioned GMRES method are better than these of the BD, BT and SC preconditioned GMRES methods.

Both the numerical results in Table 1 and Table 2 have shown that the PS preconditioned GMRES method is superior to the BD, BT and SC preconditioned GMRES methods in obtaining a considerable reduction of iteration counts. These results have confirmed our theoretical analysis in previous sections. That is, the convergence of a Krylov subspace method under

preconditioning has relation to the spectral properties of the preconditioned matrix.

For obtaining an intuitive comparison, Figure 1 and Figure 2 have plotted the eigenvalue distribution of the BD, BT, SC and PS preconditioned matrices for $G = \text{tridiag}(A)$ and $G = \text{diag}(\text{diag}(A))$ with the chosen order of the nonsingular and symmetric indefinite linear system (3) is 992, respectively.

5 Conclusions

We have proposed and investigated a new preconditioner for the numerical solution of block two-by-two symmetric indefinite matrices whose (1,1) and (2,2) blocks are nonsingular. As we have seen in this paper, the proposed preconditioner is constructed as the product of two fairly simple preconditioners: one is the famous block Jacobi preconditioner, and the other is the popular constraint preconditioner. Here, we call it the product preconditioner, and denote it as PS preconditioner. Results concerning the eigenvalue distribution and form of the eigenvectors of the preconditioned matrix $M_{ps}^{-1}\mathcal{A}$ are discussed in Section 3, respectively. Numerical experiments with preconditioned GMRES method on the problem (1) are used to illustrate the efficiency and stability of the proposed product preconditioner. Moreover, we have confirmed our theoretical analysis by comparing the IT, CPU and RES of the BD, BT, SC and PS preconditioned GMRES methods in Table 1 and Table 2.

References

[1] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta Numer.*, 14 (2005), pp. 1-137.

[2] G. Bao, W.-W. Sun, A fast algorithm for the electromagnetic scattering from a large cavity, *SIAM J. Sci. Comput.*, 27 (2005), pp. 553-574.

[3] H.C. Elman, G.H. Golub, Inexact and preconditioned Uzawa algorithms for saddle point problems, *SIAM J. Numer. Anal.*, 31 (1994), pp. 1645-1661.

[4] J. Bramble, J. Pasciak, A. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems, *Tech. Rep.*, Department of Mathematics, Cornel University, Ithaca, NY, 1994.

[5] G. Wittum, Multigrid methods for Stokes and Navier-Stokes equations. Transforming smoothers: Algorithms and numerical results, *Numer. Math.*, 54 (1989), pp. 543-563.

[6] H.C. Elman, Multigrid and Krylov subspace methods for the discrete Stokes equations, *Internat. J. Numer. Methods Fluids*, 227 (1996), pp. 1645-1661.

[7] J.Y. Yuan, A.N. Iusem, Preconditioned conjugate gradient methods for generalized least squares problems, *J. Comput. Appl. Math.*, 71 (1996), pp. 287-297.

[8] T. Rusten, R. Winther, A preconditioned iterative method for saddle point problems, *SIAM J. Matrix Anal. Appl.*, 13 (1992), pp. 887-904.

[9] A. Wathen, D. Silvester, Fast iterative solution of stabilized Stokes systems. Part I. Using simple diagonal preconditioners, *SIAM J. Num. Anal.*, 30 (1993), pp. 630-649.

[10] D. Silvester, A. Wathen, Fast iterative solution of stabilized Stokes systems. Part II. Using general block preconditioners, *SIAM J. Num. Anal.*, 31 (1994), pp. 1352-1367.

[11] L.F. Pavarino, Preconditioned mixed spectral element methods for elasticity and Stokes problems, *SIAM J. Sci. Comput.*, 19 (1998), pp. 1941-1957.

[12] G.H. Golub, A.J. Wathen, An iteration for indefinite systems and its application to the Navier-Stokes equations, *SIAM J. Sci. Comput.*, 19 (1998), pp. 530-539.

[13] J. Peters, V. Reichelt, A. Reusken, Fast iterative solvers for discrete stokes equations, *SIAM J. Sci. Comput.*, 27 (2005), pp. 646-666.

[14] H.C. Elman, D.J. Silvester, Fast nonsymmetric iterations and preconditioning for Navier-Stokes equations, *SIAM J. Sci. Comput.*, 17 (1996), pp. 33-46.

[15] M.F. Murphy, G.H. Golub, A.J. Wathen, A note on preconditioning for indefinite linear systems, *SIAM J. Sci. Comput.*, 21 (2000), pp. 1969-1972.

[16] E. De Sturler, J. Liesen, Block-diagonal and constraint preconditioners for nonsysmmetric indefinite linear systems, Part I: theory, *SIAM J. Sci. Comput.*, 26 (2005), pp. 1598-1619.

[17] G. Siefert, E. De Sturler, Preconditioners for generalized saddle-point problems, *SIAM J. Numer. Anal.*, 44 (2006), pp. 1275-1296.

[18] C. Keller, N.I.M. Gould, A.J. Wathen, Constraint preconditioning for indefinite linear systems, *SIAM J. Matrix Anal. Appl.*, 21 (2000), pp. 1300-1317.

[19] H.S. Dollar, A.J. Wathen, Approximation factorization constraint preconditioners for saddle-point matrices, *SIAM J. Sci. Comput.*, 27 (2006), pp. 1555-1572.

[20] Z.-H. Cao, Constraint Schur complement preconditioners for nonsymmetric saddle point problems, *Appl. Numer. Math.*, 59 (2009), pp. 151-169.

[21] Z.-H. Cao, A note on constraint preconditioning for nonsymmetric indefinite matrices, *SIAM J. Matrix Anal. Appl.*, 24 (2002), pp. 121-125.

[22] H.S. Dollar, Constraint-style preconditioners for regularized saddle point problems, *SIAM J. Matrix Anal. Appl.*, 29 (2007), pp. 672-684.

[23] H.C. Elman, Preconditioning for the steady-state Navier-Stokes equations with low viscosity, *SIAM J. Sci. Comput.*, 20 (1999), pp. 1299-1316.

[24] D. Kay, D. Loghin, A. Wathen, A preconditioner for the steady-state Navier-Stokes equations, *SIAM J. Sci. Comput.*, 24 (2002), pp. 237-256.

[25] Z.-H. Cao, Positive stable block triangular preconditioners for symmetric saddle point problems, *Appl. Numer. Math.*, 57 (2007), pp. 899-910.

[26] V. Simoncini, Block triangular preconditioners for symmetric saddle-point problems, *Appl. Numer. Math.*, 49 (2004), pp. 63-80.

[27] M. Benzi, D.B. Szyld, Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods, *Numer. Math.*, 76 (1997), pp. 309-321.

[28] M. Benzi, B. Uçar, Product preconditioning for Markov chain problems, in *Proceedings of the 2006 Markov Anniversary Meeting*, A. N. Langville and W. J. Stewart, eds., Raleigh, NC, 2006, (Charleston, SC), Boson Books, pp. 239-256.

[29] M. Benzi, B. Uçar, Block triangular preconditioners for M-matrices and Markov chains, *ETNA*, 26 (2007), pp. 209-227.

[30] Y. Saad, M.H. Schultz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, *SIAM J. Sci. Statist. Comput.*, 7 (1986), pp. 856-869.

[31] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed., SIAM, Philadelphia, PA, 2003.

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HYERS-ULAM STABILITY OF A GENERAL DIAGONAL SYMMETRIC FUNCTIONAL EQUATION

CHOONKIL PARK AND HAMID REZAEI*

ABSTRACT. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability for the symmetric functional equation $f(\varphi_1(x,y,z)) = \varphi_2(f(x), f(y), f(z))$ in Banach spaces. As a consequence, we obtain some stability results in the sense of Hyers-Ulam-Rassias.

1. INTRODUCTION

The stability theory of functional equations originated from the well-known Ulam's problem [15], concerning the stability of homomorphisms in metric groups. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X_1 and X_2 be Banach spaces. Assume that $f : X_1 \to X_2$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X_1$ and some $\varepsilon > 0$. Then there exists a unique additive mapping $T : X_1 \to X_2$ such that $\|f(x) - T(x)\| \le \varepsilon$ for all $x \in X_1$. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [14] for linear mappings, considering the Cauchy difference to be unbounded.

Theorem 1.1. ([14]) *Let X_1 be a normed space and X_2 a Banach space. Let $f : X_1 \to X_2$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X_1$, where $\theta > 0$ and $p \in [0, 1)$. Then there exists a unique additive mapping $A : X_1 \to X_2$ such that $\|f(x) - A(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$ for all $x \in X_1$.

A generalization of the Th.M. Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [13] followed the innovative approach of the Th.M. Rassias Theorem [14] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q = 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 5, 8, 9]).

In this paper, we introduce the following functional equation

$$f(\varphi_1(x,y,z)) = \varphi_2(f(x), f(y), f(z)). \tag{1.2}$$

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Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.2) in Banach spaces.

2. Hyers-Ulam stability of (1.2): direct method

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2), where $\varphi_i : X_i \times X_i \times X_i \to X_i, i = 1, 2$, are mappings such that

$$\varphi_i(\varphi_i(x, x, x), \varphi_i(y, y, y)) = \varphi_i(\varphi_i(x, y, z), \varphi_i(x, y, z), \varphi_i(x, y, z)). \tag{2.1}$$

Let us call such mappings *diagonal symmetric* on X_i. For example

(1) Let X be a vector space, and $\varphi : X \times X \times X \to X$ be a function that

$$\varphi(\lambda x, \lambda y, \lambda z) = \lambda\varphi(x, y, z) \quad (x, y, z \in X)$$

for every scalar λ and $\varphi(x, x, x) = \alpha x$ for some scalar α, then φ is diagonal symmetric on X.

(2) Let X be a vector space, and $\varphi : X\times X\times X \to X$ defined by $\varphi(x, y, z) = ax+by+cz+d$, where a, b, c, d are scalars and $x, y, z \in X$. Then it is easy to check that φ is diagonal symmetric.

Theorem 2.1. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x, x, x)$ for $i = 1, 2$ and suppose that T_2 is an invertible bounded linear operator on X_2. Let $\beta : X_1 \times X_1 \times X_1 \to [0, +\infty)$ be a function with this property that there exists some $0 < \lambda < 1$ such that*

$$\|T_2^{-1}\|\beta(T_1x, T_1y, T_1z) \leq \lambda\beta(x, y, z)$$

for all $x, y, z \in X_1$. If $f : X_1 \to X_2$ is a mapping satisfying

$$\|f(\varphi_1(x, y, z)) - \varphi_2(f(x), f(y), f(z))\| < \beta(x, y, z) \tag{2.2}$$

for all $x, y, z \in X_1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leq \frac{\|T_2^{-1}\|\beta(x, x, x)}{1 - \lambda}, \tag{2.3}$$

$$A(\varphi_1(x, y, z)) = \varphi_2(A(x), A(y), A(z)) \tag{2.4}$$

for all $x, y, z \in X_1$.

Proof. Letting $z = y = x$ (2.2), we get

$$\|fT_1(x) - T_2f(x)\| \leq \beta(x, x)$$

for all $x \in X_1$. It follows from (2.1) that

$$\varphi_i(T_ix, T_iy, T_iz) = T_i(\varphi_i(x, y, z)) \tag{2.5}$$

for all $x, y, z \in X_i$ and $i = 1, 2$. Let $q_n(x) := T_2^{-n}f(T_1^nx)$ for all $n \geq 1$ and all $x \in X_1$. Then

$$\begin{aligned}\|q_{n+1}(x) - q_n(x)\| &= \|T_2^{-n-1}f(T_1^{n+1}x) - T_2^{-n}f(T_1^nx)\| \\ &\leq \|T_2^{-n-1}\|\|fT_1(T_1^nx) - T_2f(T_1^nx)\| \\ &\leq \|T_2^{-1}\|^{n+1}\beta(T_1^nx, T_1^nx) \leq \|T_2^{-1}\|\lambda^n\beta(x, x, x).\end{aligned}$$

Here, in the last inequality, the contractive property of β is used. Hence

$$\|q_{n+1}(x) - q_n(x)\| \leq \|T_2^{-1}\|\lambda^n \beta(x, x, x),$$

and so the sequence $\{q_n(x)\}$ is a Cauchy sequence for each x. Since X_2 is complete, there exists a limit mapping $A(x) := \lim_{n\to\infty} q_n(x)$. Now by induction on n, we prove that

$$\|q_n(x) - f(x)\| \leq \sum_{i=0}^{n-1} \|T_2^{-1}\|\lambda^i \beta(x, x, x) \tag{2.6}$$

for all $n \in \mathbb{N}$ and all $x \in X_1$. Fix $x \in X_1$. Note that

$$\begin{aligned} \|q_1(x) - f(x)\| &= \|T_2^{-1} f(T_1(x)) - f(x)\| \\ &\leq \|T_2^{-1}\| \|f(T_1(x)) - T_2(f(x))\| \leq \|T_2^{-1}\|\beta(x, x, x). \end{aligned}$$

Now suppose (2.6) holds for a fixed n. Then

$$\begin{aligned} \|q_{n+1}(x) - f(x)\| &\leq \|q_{n+1}(x) - q_n(x)\| + \|q_n(x) - f(x)\| \\ &\leq \|T_2^{-1}\|\lambda^n \beta(x, x) + \sum_{i=0}^{n-1} \|T_2^{-1}\|\lambda^i \beta(x, x, x) \\ &= \sum_{i=0}^{n} \|T_2^{-1}\|\lambda^i \beta(x, x, x). \end{aligned}$$

Letting $n \to +\infty$ in (2.6), we get

$$\|A(x) - f(x)\| \leq \frac{\|T_2^{-1}\|\beta(x, x, x)}{1 - \lambda}$$

for all $x \in X_1$.

Now we prove that A satisfies (2.4). Replacing x, y, z in (2.2) with $T_1^n x, T_1^n y, T_1^n z$, respectively, we get

$$\begin{aligned} \|f(\varphi_1(T_1^n x, T_1^n y, T_1^n y)) - \varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z))\| \\ \leq \beta(T_1^n x, T_1^n y, T_1^n z). \end{aligned} \tag{2.7}$$

It follows from (2.5) that

$$\varphi_1(T_1^n x, T_1^n y, T_1^n z) = T_1^n(\varphi_1(x, y, z)) \tag{2.8}$$

for all $x, y, z \in X_1$, and

$$\varphi_2(T_2^n x, T_2^n y, T_1^n z) = T_2^n(\varphi_2(x, y, z))$$

for all $x, y, z \in X_2$. Replacing x, y, z by $T_2^{-n}x, T_2^{-n}y, T_1^n z$, respectively, in the last above relation, we get

$$\varphi_2(x, y, z) = T_2^n(\varphi_2(T_2^{-n}x, T_2^{-n}y, T_2^{-n}z))$$

and then replacing x, y, z by $f(T_1^n x), f(T_1^n y), f(T_1^n z)$, respectively, we get

$$\varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z)) = T_2^n(\varphi_2(T_2^{-n}(f(T_1^n x)), T_2^{-n}(f(T_1^n y)), T_2^{-n}(f(T_1^n z)))).$$

By the definition of q_n, we obtain

$$\varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z)) = T_2^n(\varphi_2(q_n(x), q_n(y), q_n(z))). \tag{2.9}$$

It follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned}
&\|q_n(\varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&= \|T_2^{-n} f(T_1^n \varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&\le \|T_2^{-1}\|^n \|f(T_1^n \varphi_1(x,y,z)) - T_2^n \varphi_2(q_n(x), q_n(y), q_n(z))\| \\
&= \|T_2^{-1}\|^n \|f(\varphi_1(T_1^n x, T_1^n y, T_1^n z)) - \varphi_2(f(T_1^n x), f(T_1^n y), f(T_1^n z))\| \\
&\le \|T_2^{-1}\|^n \beta(T_1^n x, T_1^n y, T_1^n z) \\
&\le \lambda^n \beta(x,x,x).
\end{aligned}$$

Therefore,

$$\|q_n(\varphi_1(x,y,z)) - \varphi_2(q_n(x), q_n(y), q_n(z))\| \le \lambda^n \beta(x,x,x)$$

for all $x, y \in X_1$ and all $n \in \mathbb{N}$. Applying the continuity of φ, considering $0 < \lambda < 1$, and letting $n \to +\infty$ in the last inequality, we obtain (2.4).

Now we prove that A is a unique mapping satisfying (2.3) and (2.4). Assume that there exists another mapping $A' : X \to X$ satisfying (2.3) and (2.4). Letting $y = x$ in (2.4), we get $AT_1(x) = T_2 A(x)$ and $A'T_1(x) = T_2 A'(x)$ and more generally

$$AT_1^n(x) = T_2^n A(x) \text{ and } A'T_1^n(x) = T_2^n A'(x).$$

Hence

$$A(x) = T_2^{-n} A(T_1^n(x)) \text{ and } A'(x) = T_2^{-n} A'(T_1^n(x))$$

for all $x \in X$ and $n \in \mathbb{N}$. By the triangle inequality, (2.3) and (2.10), we obtain

$$\begin{aligned}
\|A(x) - A'(x)\| &= \|T_2^{-n} A(T_1^n x) - T_2^{-n} A'(T_1^n x)\| \\
&\le \|T_2^{-1}\|^n \|A(T_1^n x) - A'(T_1^n x)\| \\
&\le \|T_2^{-1}\|^n (\|A(T_1^n x) - f(T_1^n x)\| + \|f(T_1^n x) - A'(T_1^n x)\|) \\
&\le \|T_2^{-1}\|^n (2 \frac{\|T_2^{-1}\| \beta(T_1^n x, T_1^n x)}{1-\lambda}) \\
&\le 2\|T_2^{-1}\| (\frac{\|T_2^{-1}\|^n \beta(T_1^n x, T_1^n x)}{1-\lambda}) \\
&\le 2\|T_2^{-1}\| \frac{\lambda^n \beta(x,x)}{1-\lambda}
\end{aligned}$$

for all $x \in X_1$ and all $n \in \mathbb{N}$. Letting $n \to +\infty$, we get $A(x) = A'(x)$ for all $x \in X_1$. $\square$

The proof of the following theorem is similar and we omit it:

Theorem 2.2. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x,x,x)$ for $i = 1, 2$ and suppose that T_2 is a bounded linear operator on X_2 and T_1 is invertible on X_1. Let $\beta : X_1 \times X_1 \times X_1 \to [0, +\infty)$ be a function with this property that there exists some $0 < \lambda < 1$ such that*

$$\|T_2\| \beta(T_1^{-1}x, T_1^{-1}y, T_1^{-1}z) \le \lambda \beta(x,y,z)$$

for all $x, y, z \in X_1$. If $f : X_1 \to X_2$ is a mapping satisfying

$$\|f(\varphi_1(x,y,z)) - \varphi_2(f(x), f(y), f(z))\| < \beta(x,y,z)$$

for all $x, y, z \in X_1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leq \frac{\|T_1^{-1}\|\beta(x,x,x)}{1-\lambda}$$
$$A(\varphi_1(x,y,z)) = \varphi_2(A(x), A(y), A(z))$$

for all $x, y, z \in X_1$.

Corollary 2.3. *Assume that X_1 is a normed space and X_2 is a Banach space and that φ_1, φ_2 are continuous diagonal symmetric mappings on X_1, X_2, respectively. Put $T_i(x) := \varphi_i(x,x,x)$ for $i = 1, 2$ and suppose that T_2 is an invertible bounded linear operator on X_2. Let $f : X_1 \to X_2$ be a mapping for which there exist some $\theta_1, \theta_2 > 0$, and $p \geq 0$ such that*

$$\|f(\varphi_1(x,y,z)) - \varphi_2(f(x), f(y), f(z))\| < \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|x\|^{p/3}\|y\|^{p/3}\|z\|^{p/3})$$

for all $x, y, z \in X_1$. If $\|T_2^{-1}\|\|T_1\|^p < 1$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leq \theta\|T_2^{-1}\|\frac{(2\theta_1 + \theta_2)\|x\|^p}{1 - \|T_2^{-1}\|\|T_1\|^p},$$

$$A(\varphi_1(x,y,z)) = \varphi_2(A(x), A(y), A(z))$$

for all $x, y, z \in X_1$.

Proof. Let
$$\beta(x,y,z) := \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|x\|^{p/3}\|y\|^{p/3}\|z\|^{p/3})$$
for $x, y \in X_1$, and $\lambda := \|T_2^{-1}\|\|T_1\|^p$. Then
$$\|T_2^{-1}\|\beta(T_1x, T_1y) \leq \lambda\beta(x,y,z)$$
for all $x, y, z \in X_1$. This completes the proof. □

Consider the following choices of $\varphi_1, \varphi_2, T_1$ and T_2:
(1) $\varphi_1(x,y,z) = \varphi_2(x,y,z) = \frac{x+y+z}{2}$ and $T_1(x) = T_2(x) = \frac{3x}{2}$;

(2) $\varphi_1(x,y,z) = xyz$, $\varphi_2(x,y,z) = x+y+z$,$T_1(x) = x^3$ and $T_2(x) = 3x$;
to deduce the following corollary:

Corollary 2.4. *The following functional equations has Hyers-Ulam stability in the sense of Theorem 2.1 and 2.2:*

(i) $2f(\frac{x+y+z}{2}) = f(x) + f(y) + f(z)$, $f : X_1 \to X_2$ where X_1 is a vector space and X_2 is a Banach space.

(ii) $f(xyz) = f(x) + f(y) + f(z)$, $f : X_1 \to X_2$ where X_1 is any abelian semigroup and X_2 is a Banach space.

Let A be a C^*-algebra and $a \in A$ a self-adjoint element, i.e., $a = a^*$. Then a is said to be positive if it is of the form $a = bb^*$ for some $a \in A$. The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [4]).

It is well-known that for a positive element a and a positive integer n there exists a unique positive element $x \in A^+$ such that $a = x^n$. We denote x by $\sqrt[n]{a}$. Then the functional

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equation (1.1) is Hyers-Ulam stable in the sense of Theorem 2.1 and 2.2 when the mapping $\varphi : A^+ \times A^+ \times A^+ \to A^+$ is one of the following choices:

(1) $\varphi(x,y,z) = \sqrt[n]{ax^2+by^2+cz^2}$ where $a+b+c>1$,
(2) $\varphi(x,y,z) = \sqrt[n]{x^n+y^n+z^n}$,

3. Hyers-Ulam stability of (1.2): fixed point method

We now introduce one of the fundamental results of the fixed point theory.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies

- $d(x,y) = 0$ if and only if $x = y$,
- $d(x,y) = d(y,x)$ for all $x,y \in X$,
- $d(x,z) \leq d(x,y) + d(y,z)$,

for all $x,y,z \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.2) in Banach spaces.

Theorem 3.1. ([10]) *Let $(\mathcal{X}, d)$ be a generalized complete metric space. Assume that $\Lambda : \mathcal{X} \to \mathcal{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$, i.e.,*

$$d(\Lambda g, \Lambda h) \leq L d(g,h)$$

for all $g,h \in \mathcal{X}$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}f, \Lambda^{n_0}f) < +\infty$ for some $f \in \mathcal{X}$, then the following statements are true:

(1) The sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ;

(2) A is the unique fixed point of Λ in $\mathcal{X}^* = \{g \in \mathcal{X} : d(\Lambda^{n_0}f, g) < +\infty\}$;

(3) If $g \in X^*$, then

$$d(g, A) \leq \frac{1}{1-L} d(\Lambda g, g).$$

Radu [12] proved the Hyers-Ulam stability of the additive Cauchy equation (1.1) by using fixed point method (see [2]).

In the following, Theorem 2.1 is proved by the fixed point method.

Theorem 3.2. *Let $X_1, X_2, \varphi_1, \varphi_2, T_1, T_2, \beta$ be given as in Theorem 2.1. If $f : X_1 \to X_2$ is a mapping satisfying (2.2), then there exists a unique mapping $A : X_1 \to X_2$ satisfying (2.3) and $AT_1(x) = T_2A(x)$ for all $x \in X_1$.*

Proof. Letting $y = x$ in (2.2), we get

$$\|fT_1(x) - T_2f(x)\| \leq \beta(x,x,x)$$

for all $x \in X_1$. Consider the set $\mathcal{X} := \{f : f : X_1 \to X_2 \text{ is a function }\}$ and define the generalized metric on $\mathcal{X}$ by

$$d(g,h) = \inf\left\{\mu \in (0,+\infty) : \|g(x) - h(x)\| \leq \mu\beta(x,x) \text{ for all } x \in X_1\right\}.$$

where, as usual, inf $= +\infty$. It is easy to show that $(\mathcal{X}, d)$ is complete (see [11]). Now we consider the linear mapping $\Lambda : \mathcal{X} \to \mathcal{X}$ such that

$$\Lambda g(x) = T_2^{-1} g(T_1 x)$$

for all $x \in X_1$. For given $g, h \in \mathcal{X}$,

$$\|\Lambda g(x) - \Lambda f(x)\| = \|T_2^{-1} g(T_1 x) - T_2^{-1} h(T_1 x)\| \leq \|T_2^{-1}\| \beta(T_1 x, T_1 x, T_1 x) \leq \lambda \beta(x, x, x)$$

for all $x \in X_1$. By the definition of d,

$$d(\Lambda f, \Lambda g) \leq \lambda d(f, g).$$

Note that

$$\begin{aligned} \|f(x) - \Lambda f(x)\| &= \|f(x) - T_2^{-1} f(T_1 x)\| \\ &\leq \|T_2^{-1}\| \|T_1 f(x) - f(T_1 x)\| \leq \|T_2^{-1}\| \beta(x, x, x) \end{aligned}$$

for all $x \in X_1$, and so $d(\Lambda f, f) \leq \|T_2^{-1}\| < +\infty$. By the preceding theorem, there exists a mapping $A : X_1 \to X_2$ satisfying the following conditions:

(1) A is a fixed point of Λ, i.e., $T_2^{-1} A T_1 = \Lambda A = A$ whence $A(T_1(x)) = T_2(A(x))$ for all $x \in X_1$. Moreover, A is a unique fixed point of Λ in the set $\mathcal{X}^* := \{g \in \mathcal{X} : d(f, g) < +\infty\}$ which implies that

$$\|f(x) - A(x)\| \leq \mu \beta(x, x, x).$$

(2) $d(\Lambda^n f, A) \to 0$ as $n \to +\infty$, i.e., $A(x) = \lim_n T_2^{-n} f(T_1^n x)$.

(3) By (3) of the preceding theorem, we conclude that

$$d(f, A) \leq \frac{1}{1 - \lambda} d(f, \Lambda f) < \frac{1}{1 - \lambda} \|T_2^{-1}\|,$$

and so

$$\|f(x) - A(x)\| \leq \frac{\|T_2^{-1}\| \beta(x, x, x)}{1 - \lambda},$$

as desired. In order to prove that A satisfies (2.4), we can proceed exactly as in the proof of Theorem 2.2 to show that $A : X_1 \to X_2$ is indeed a mapping satisfying (2.4). □

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64–66.

[2] L. Cadariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, Grazer Math. Ber. **346** (2004), 43–52.

[3] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, Hackensacks, New Jersey, 2002.

[4] J. Dixmier, C^*-Algebras, North-Holland Publ. Co., Amsterdam, New York and Oxford, 1977.

[5] G.L. Forti, Elementary remarks on Ulam-Hyers stability of linear functional equations, J. Math. Anal. Appl. **328** (2007), 109–118.

[6] P. Găvruta, An answer to a question of Th.M. Rassias and J. Tabor on mixed stability of mappings, Bul. Stiint. Univ. Politeh. Timis. Ser. Mat. Fiz. **42** (1997), 1–6.

[7] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA **27** (1941) 222–224.
[8] D.H. Hyers, G. Isac and Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston, Basel, Berlin, 1998.
[9] S. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York, 2011.
[10] B. Margolis and J.B. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[11] D. Mihet and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. **343** (2008), 567–572.
[12] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory **4** (2003), 91–96.
[13] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. **46** (1982), 126–130.
[14] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[15] S.M. Ulam, A Collection of Mathematical Problems, Interscience Tracts in Pure and Applied Mathematics, No. 8, Interscience, New York, NY, USA, 1960.

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GENERAL DECAY OF SOLUTIONS FOR A SINGULAR NONLOCAL VISCOELASTIC PROBLEM WITH NONLINEAR DAMPING AND SOURCE

YUN SUN, GANG LI, AND WENJUN LIU

Abstract. This paper deals with a singular nonlocal viscoelastic problem with nonlinear damping and source terms. We establish a general decay rate result without imposing any restrictive growth assumption on the damping term.

1. Introduction

In this paper, we investigate the following one-dimensional viscoelastic equation

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x \mathrm{d}s + h(u_t) = b|u|^{p-2}u, \ x \in (0,\ell), t \in (0,\infty), \\ u(\ell,t) = 0, \displaystyle\int_0^\ell xu(x,t)\mathrm{d}x = 0 & t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in [0,\ell], \end{cases} \tag{1.1}$$

where $\ell < \infty$, $b > 0$, $p > 2$, g and h are specific functions which will be given later.

This type of evolution problems are generally encountered when the data on the boundary can not be measured directly, but their average values are known. For the case of singular type, we can refer to [8, 9, 10, 11, 14] for the existence, uniqueness and blow-up results. Here, it is worth mentioning that many results concerning decay have been extensively studied for the case of classical conditions. Under the condition $-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t)$, the exponential or polynomial decay results were obtained in [3, 4, 5, 6]. Later, some authors relaxed these conditions by considering only $g'(t) \le -\xi g(t)$ or $g'(t) \le -\xi g^r(t)$, for all $t \ge 0$ and some $\xi > 0$ (see [1, 2, 15]). In [12, 13], the condition has been replaced by $g'(t) \le -\xi(t)g(t)$, where $\xi(t)$ is a positive function. This allows the authors to obtain general rates of decay than just exponential or polynomial type.

Motivated by [11, 13], we study problem (1.1) in this paper and intend to establish a general decay result under certain conditions, without imposing any restrictive growth assumption on the damping term. The paper is organized as follows. In Section 2 we present some assumptions and known results needed for our work. Section 3 is devoted to the proof of some lemmas and the decay result: Theorem 2.4 .

2. Preliminaries and main result

In this section we first introduce some functional spaces and present some assumptions and known results which will be used throughout this paper, and then state our main result.

Let $L_x^p = L_x^p(0,\ell)$ be the weighted Banach space equipped with the norm $\|u\|_p = \left(\int_0^\ell x|u|^p \mathrm{d}x\right)^{\frac{1}{p}}$. In particular, when $p = 2$, we denote $H = L_x^2(0,\ell)$ to be the weighted Hilbert space of square

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integrable functions having the finite norm $\|u\|_H = \left(\int_0^\ell xu^2\mathrm{d}x\right)^{\frac{1}{2}}$. We take $V = V_x^{1,1}(0,\ell)$ to be the weighted Hilbert space equipped with the norm $\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{\frac{1}{2}}$, and $V_0 = \{v \in V \text{ such that } v(\ell) = 0\}$.

For the functionals g and h we give the following assumptions as in [13]:

(H1) $g(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0,$$

and there exists a nondecreasing differentiable function $\xi(t)$ such that

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0 \qquad \textit{and} \quad \int_0^{+\infty} \xi(t)\mathrm{d}t = \infty$$

(H2) $h : \mathbb{R} \mapsto \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1([0,+\infty))$, with $h_0(0) = 0$, and positive constants c_1, c_2, and ϵ such that

$$h_0(|s|) \le |h(s)| \le h_0^{-1}(s), \quad \forall\ |s| \le \epsilon, \tag{2.1}$$

$$c_1|s| \le |h(s)| \le c_2|s|, \quad \forall\ |s| \ge \epsilon. \tag{2.2}$$

Remark 1. *Hypothesis* (H2) *implies that $sh(s) > 0$, for all $s \neq 0$.*

Lemma 2.1. ([11]) *For any v in V_0, we have*

$$\int_0^\ell x(v(x))^2\mathrm{d}x \le C_* \int_0^\ell x\,(v_x(x))^2\,\mathrm{d}x.$$

Lemma 2.2. ([11]) *For any v in V_0, $2 < p < 4$, we have*

$$\int_0^\ell x(v(x))^p\mathrm{d}x \le C_p\|v_x\|_H^p,$$

where C_p is a constant depending on p only.

Lemma 2.3. ([11, Theorem 2.3]) *Suppose that $2 < p < 3$ and* (H1) *and* (H2) *hold. Then for any u_0 in V_0 and u_1 in H, problem* (1.1) *has a unique local solution*

$$u \in C(0,t_*;V_0) \cap C^1(0,t_*;H)$$

for $t_* > 0$ small enough.

Now we introduce the functionals for $I(t)$ and $E(t)$:

$$I(t) := I(u(t)) = \Big(1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_0^\ell xu_x^2\mathrm{d}x + (g \circ u_x)(t) - b\int_0^\ell x|u(t)|^p\mathrm{d}x, \tag{2.3}$$

$$\begin{aligned} E(t) := E(u(t)) = {} & \frac{1}{2}\Big(1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_0^\ell xu_x^2\mathrm{d}x + \frac{1}{2}(g \circ u_x)(t) \\ & - \frac{b}{p}\int_0^\ell x|u(t)|^p\mathrm{d}x + \frac{1}{2}\int_0^\ell xu_t^2\mathrm{d}x, \end{aligned} \tag{2.4}$$

where

$$(g \circ u_x)(t) = \int_0^\ell \int_0^t xg(t-s)|u_x(x,t) - u_x(x,s)|^2\mathrm{d}s\mathrm{d}x.$$

Remark 2. *Multiplying Eq.* (1.1) *by xu_t and integrating over $(0,\ell)$, we can easily get*

$$E'(t) = \frac{1}{2}(g' \circ u_x)(t) - \frac{1}{2}g(t)\int_0^\ell xu_x^2(x,t)\mathrm{d}x - \int_0^\ell xu_t h(u_t)\mathrm{d}x \le 0, \quad \forall\ t \ge 0. \tag{2.5}$$

Our main result of this paper reads as follows.

Theorem 2.4. *Suppose that* (H1) *and* (H2) *hold,* $2 < p < 3$*, if* $(u_0, u_1) \in V_0 \times H$ *such that*

$$\beta = \frac{bC_p}{l}\left(\frac{2p}{(p-2)l}E(u_0,u_1)\right)^{\frac{p-2}{2}} < 1, \quad I(u_0) > 0. \tag{2.6}$$

Then, there exists a constant $C > 0$ *such that, for* t *large, the solution of* (1.1) *satisfies*

$$E(t) \le C\left(H_0^{-1}\left(\frac{1}{\int_0^t \xi(s)\mathrm{d}s}\right)\right)^2 \quad where \quad H_0(s) = sh_0(s). \tag{2.7}$$

Moreover, if we define $J(s) = \frac{h_0(s)}{s}$*, which is strictly increasing with* $J(0) = 0$*, then we can improve* (2.7) *to the following estimate:*

$$E(t) \le C\left(h_0^{-1}\left(\frac{1}{\int_0^t \xi(s)\mathrm{d}s}\right)\right)^2. \tag{2.8}$$

For the proof of the above theorem, we use the following lemma.

Lemma 2.5. ([7]) *Let* $E : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function and* $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ *be a strictly increasing* C^1 *function, with* $\sigma(t) \to +\infty$ *as* $t \to +\infty$*. Assume that there exist* $p, q \ge 0$ *and* $c > 0$ *such that*

$$\int_S^\infty \sigma'(t)E(t)^{1+p}\mathrm{d}t \le cE(s)^{1+p} + \frac{cE(s)}{\sigma^q}, \quad 1 \le S < +\infty.$$

Then there exist positive constants κ *and* ω *such that*

$$E(t) \le \kappa e^{-\omega\sigma(t)} \qquad \forall\, t \ge 1, if p = q = 0$$
$$E(t) \le \frac{\kappa}{\sigma(t)^{\frac{1+q}{p}}} \qquad \forall\, t \ge 1, if p > 0.$$

3. General decay of solutions

In this section we prove our main result. For this purpose we establish several lemmas.

Lemma 3.1. ([11, Lemma 4.1 and Lemma 4.2]) *Under the assumptions of Theorem 2.4, we conclude that* $I(u(t)) > 0, \forall\, t > 0$ *and the solution is global and bounded. Furthermore, the following inequality holds*

$$l\int_0^\mu xu_x^2\mathrm{d}x \le \left(\frac{2p}{p-2}\right)E(u_0,u_1), \quad \forall t > 0. \tag{3.1}$$

Lemma 3.2. *For all* $u \in V_0$*, there exists* $C_* > 0$ *such that*

$$\int_0^\ell x\left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^2\mathrm{d}x \le (1-l)C_*(g\circ u_x)(t).$$

Proof. Using Cauchy-Schwarz's inequality, (H1) and Lemma 2.1, we can easily obtain the result.

We define the following functionals

$$\mathcal{L}(t) := N_1E(t) + N_2K(t) + \chi(t), \tag{3.2}$$

where

$$K(t) := -\int_0^\ell xu_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,$$
$$\chi(t) := \int_0^\ell xuu_t\mathrm{d}x,$$

N_1 and N_2 are positive constants to be chosen later.

Lemma 3.3. *Suppose that* (H1) *holds and* $p > 2$. *Let* u *be the solution of problem* (1.1). *Then there exist positive constants* $\alpha_1, \alpha_2 > 0$ *such that*

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t). \tag{3.3}$$

Proof. Straightforward computations, Young's inequality and Lemma 3.2 lead to

$$\begin{aligned}\mathcal{L}(t) \le & \left[\frac{N_1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)+\frac{C_*}{2}\right]\int_0^\ell xu_x^2\mathrm{d}x + \left(\frac{N_1}{2}+\frac{N_2}{2}+\frac{1}{2}\right)\int_0^\ell xu_t^2\mathrm{d}x \\ & + \left[\frac{N_1}{2}+\frac{N_2}{2}(1-l)C_*\right](g\circ u_x)(t) - \frac{bN_1}{p}\int_0^\ell x|u|^p\mathrm{d}x \\ \le & \alpha_1\left(\int_0^\ell xu_x^2\mathrm{d}x + \int_0^\ell xu_t^2\mathrm{d}x + (g\circ u_x)(t) - \frac{b}{p}\int_0^\ell x|u|^p\mathrm{d}x\right),\end{aligned} \tag{3.4}$$

for some $\alpha_1 > 0$. On the other hand,

$$\begin{aligned}\mathcal{L}(t) \ge & \frac{1}{2}(N_1 l - C_*)\int_0^\ell xu_x^2\mathrm{d}x + \frac{1}{2}(N_1 - N_2 - 1)\int_0^\ell xu_t^2\mathrm{d}x \\ & + \frac{1}{2}\left[N_1 - N_2(1-l)C_*\right](g\circ u_x)(t) - \frac{bN_1}{p}\int_0^\ell x|u|^p\mathrm{d}x.\end{aligned} \tag{3.5}$$

Choose $N_2 > 1$ and then take N_1 satisfying

$$N_1 > \max\left\{\frac{C_*}{l}, N_2+1, N_2(1-l)C_*\right\}. \tag{3.6}$$

Then we completes the proof.

Lemma 3.4. *Suppose that* (H1) *and* (H2) *hold and* $p > 2$, *let* $(u_0, u_1) \in V_0 \times H$ *be given. If* u *is the solution of* (1.1), *then we have*

$$\chi'(t) \le -\frac{l}{2}\int_0^\ell xu_x^2\mathrm{d}x + \int_0^\ell xu_t^2\mathrm{d}x + C(g\circ u_x)(t) + C\int_0^\ell xh^2(u_t)\mathrm{d}x + b\int_0^\ell x|u|^p\mathrm{d}x. \tag{3.7}$$

Proof. By exploiting problem (1.1) and integrating by parts, we get

$$\begin{aligned}\chi'(t) = & \int_0^\ell xu_t^2\mathrm{d}x - \int_0^\ell xu_x^2\mathrm{d}x + \left(\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell xu_x^2\mathrm{d}x \\ & + \int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\mathrm{d}x - \int_0^\ell xuh(u_t)\mathrm{d}x + b\int_0^\ell x|u|^p\mathrm{d}x.\end{aligned} \tag{3.8}$$

Using Young's and Poincaré's inequalities and Lemma 3.2, we obtain

$$\begin{aligned}& \int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\mathrm{d}x \\ \le & \delta\int_0^\ell xu_x^2\mathrm{d}x + \frac{1}{4\delta}\int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(s)-u_x(t)\right)\mathrm{d}s\right)^2\mathrm{d}x \\ \le & \delta\int_0^\ell xu_x^2\mathrm{d}x + \frac{C}{\delta}(g\circ u_x)(t),\end{aligned} \tag{3.9}$$

$$-\int_0^\ell xuh(u_t)\mathrm{d}x \le \delta\int_0^\ell xu^2\mathrm{d}x + \frac{1}{4\delta}\int_0^\ell xh^2(u_t)\mathrm{d}x \le \delta C_*\int_0^\ell xu_x^2\mathrm{d}x + \frac{1}{4\delta}\int_0^\ell xh^2(u_t)\mathrm{d}x. \tag{3.10}$$

Combining (3.8)-(3.10), and choosing δ small enough such that $\delta \le \frac{l}{2(1+C_*)}$, then (3.7) is obtained.

Lemma 3.5. *Under the assumptions* (H1) *and* (H2), *suppose* $2 < p < 3$, *then the functional* K *satisfies, along the solution, the estimate*

$$\begin{aligned}K'(t) \le &-\left(\int_0^t g(s)\mathrm{d}s-\delta\right)\int_0^\ell xu_t^2\mathrm{d}x+\left(\delta+\frac{\delta l^2}{bC_p}\right)\int_0^\ell xu_x^2\mathrm{d}x+\left(C+\frac{C}{\delta}\right)(g\circ u_x)(t)\\ &-\frac{C}{\delta}(g'\circ u_x)(t)+C\int_0^\ell xh^2(u_t)\mathrm{d}x,\quad \forall\ 0<\delta<1. \end{aligned}\tag{3.11}$$

Proof. By direct computations and (1.1), we get

$$\begin{aligned}K'(t)=&\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\mathrm{d}x\\ &+\int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\right)^2\mathrm{d}x-\int_0^\ell xu_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\ &-\left(\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell xu_t^2\mathrm{d}x+\int_0^\ell xh(u_t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\ &-b\int_0^\ell x|u|^{p-2}u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \end{aligned}\tag{3.12}$$

By Young's inequality and Lemma 3.2, we have

$$\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_0^\ell xu_x\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\mathrm{d}x\le\delta\int_0^\ell xu_x^2\mathrm{d}x+\frac{C}{\delta}(g\circ u_x)(t),\tag{3.13}$$

$$\int_0^\ell x\left(\int_0^t g(t-s)\left(u_x(t)-u_x(s)\right)\mathrm{d}s\right)^2\mathrm{d}x\le C(g\circ u_x)(t),\tag{3.14}$$

$$-\int_0^\ell xu_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\le\delta\int_0^\ell xu_t^2\mathrm{d}x-\frac{C}{\delta}(g'\circ u_x)(t),\tag{3.15}$$

$$\int_0^\ell xh(u_t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\le C\int_0^\ell xh^2(u_t)\mathrm{d}x+C(g\circ u_x)(t),\tag{3.16}$$

As for the sixth term, using Lemma 2.2, (2.6) and (3.1), we get

$$\begin{aligned}&-b\int_0^\ell x|u|^{p-2}u\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x\\ \le& b\delta\int_0^\ell x|u|^{2p-2}\mathrm{d}x+\frac{C}{2\delta}(g\circ u_x)(t)\le b\delta C_p\left(\int_0^\ell xu_x^2\mathrm{d}x\right)^{p-2}\left(\int_0^\ell xu_x^2\mathrm{d}x\right)+\frac{C}{2\delta}(g\circ u_x)(t)\\ \le& b\delta C_p\left[\frac{2pE(u_0,u_1)}{(p-2)l}\right]^{p-2}\left(\int_0^\ell xu_x^2\mathrm{d}x\right)\le\frac{\delta l^2}{bC_p}\int_0^\ell xu_x^2\mathrm{d}x+\frac{C}{2\delta}(g\circ u_x)(t). \end{aligned}\tag{3.17}$$

Combining (3.12)-(3.17), the assertion of the lemma is established.

Now select N_1, N_2 large so that (3.3) remains valid and $\frac{l}{4N_2\left(1+\frac{l^2}{bC_p}\right)}\le\frac{l}{2(1+C_*)}$. Set $g_0=\int_0^{t_0}g(s)\mathrm{d}s$ for some fixed $t_0>0$. By combining (2.5), (3.2), (3.7) and (3.11), we take $\delta=\frac{l}{4N_2\left(1+\frac{l^2}{bC_p}\right)}$ and obtain, for all $t\ge t_0$,

$$\begin{aligned}\mathcal{L}'(t)\le&-\frac{l}{4}\int_0^\ell xu_x^2\mathrm{d}x-\left(N_2g_0-\frac{l}{4}-1\right)\int_0^\ell xu_t^2\mathrm{d}x+\left(\frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l}+C\right)(g\circ u_x)(t)\\ &+\left(\frac{1}{2}N_1-\frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l}\right)(g'\circ u_x)(t)+(CN_2+C)\int_0^\ell xh^2(u_t)\mathrm{d}x+b\int_0^\ell x|u|^p\mathrm{d}x.\end{aligned}$$

At this point, since N_2 large enough, so we can have $k := N_2 g_0 - \frac{l}{4} - 1 > 0$, then N_1 large enough so that (3.6) remains valid and $\frac{1}{2}N_1 - \frac{4CN_2^2\left(1+\frac{l^2}{bC_p}\right)}{l} > 0$. Thus, using (H1), it turns out that

$$\mathcal{L}'(t) \leq -\frac{l}{4}\int_0^\ell x u_x^2 \mathrm{d}x - k\int_0^\ell x u_t^2 \mathrm{d}x + C(g \circ u_x)(t) + C\int_0^\ell x h^2(u_t)\mathrm{d}x + b\int_0^\ell x|u|^p \mathrm{d}x,$$

which implies

$$E(t) \leq -m\mathcal{L}'(t) + C(g \circ u_x)(t) + C\int_0^\ell x h^2(u_t)\mathrm{d}x, \quad \forall\ t \geq t_0. \tag{3.18}$$

Proof of Theorem 2.4. (Sketch) Define $\phi(t) = 1 + \int_1^t \frac{1}{h_0\left(\frac{1}{s}\right)}\mathrm{d}s$, $\forall\ t \geq 1$ and $\sigma(t) = \phi^{-1}\left(\int_0^t \xi(s)\mathrm{d}s\right)$, for $\forall\ t \geq t_1 \geq t_0$. Then continue as that of [13, Theorem 3.5] we can complete the proof.

References

[1] S. Berrimi and S. A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, Nonlinear Anal. **64** (2006), no. 10, 2314–2331.

[2] S. Berrimi and S. A. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, Electron. J. Differential Equations **2004** (2004), no. 88, 10 pp.

[3] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, Electron. J. Differential Equations **2002** (2002), no. 44, 14 pp.

[4] M. M. Cavalcanti et al., Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, Differential Integral Equations **14** (2001), no. 1, 85–116.

[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay for a non-linear viscoelastic equation with strong damping, Math. Methods Appl. Sci. **24** (2001), no. 14, 1043–1053.

[6] M. M. Cavalcanti and H. P. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, SIAM J. Control Optim. **42** (2003), no. 4, 1310–1324.

[7] P. Martinez, A new method to obtain decay rate estimates for dissipative systems with localized damping, Rev. Mat. Complut. **12** (1999), no. 1, 251–283.

[8] S. Mesloub and A. Bouziani, On a class of singular hyperbolic equation with a weighted integral condition, Int. J. Math. Math. Sci. **22** (1999), no. 3, 511–519.

[9] S. Mesloub and F. Mesloub, Solvability of a mixed nonlocal problem for a nonlinear singular viscoelastic equation, Acta Appl. Math. **110** (2010), no. 1, 109–129.

[10] S. Mesloub and S. A. Messaoudi, A three point boundary value problem hyperbolic equation, Electron. J. Differ. Equ. **2002** (2002), no. 62, 13 pp.

[11] S. Mesloub and S. A. Messaoudi, Global existence, decay, and blow up of solutions of a singular nonlocal viscoelastic problem, Acta Appl. Math. **110** (2010), no. 2, 705–724.

[12] S. A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal. **69** (2008), no. 8, 2589–2598.

[13] S. A. Messaoudi and M. I. Mustafa, On the control of solutions of viscoelastic equations with boundary feedback, Nonlinear Anal. Real World Appl. **10** (2009), no. 5, 3132–3140.

[14] S. Wu, Blow-up of solutions for a singular nonlocal viscoelastic equation, J. Partial Differ. Equ. **24** (2011), no. 2, 140–149.

[15] Z. Zhang et al., A note on decay properties for the solutions of a class of partial differential equation with memory, J. Appl. Math. Comput. **37** (2011), no. 1-2, 85–102.

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Measuring fuzziness of generalized fuzzy rough sets induced by pseudo-operations

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Abstract: Rough sets is a new mathematical tool to handle imprecision, vagueness and uncertainty in data analysis. But, in Pawlak's rough set model, equivalence relation is a key and primitive notion and this equivalence relation seems to be a very stringent condition that limited the application domain of the rough sets. Various fuzzy generalizations of rough approximations have been made over the years. In this paper, we consider pseudo-operation of the following form: $x \oplus y = g^{-1}(g(x) + g(y))$, where g is a positive strictly monotone generating function and g^{-1} is its pseudo-inverse. Using this type of pseudo-operation, the pseudo-generalized fuzzy rough sets are presented and some properties of the pseudo fuzzy rough approximation operators are investigated. Moreover, we define a measure of fuzziness based on pseudo-generalized fuzzy rough sets with the new pseudo-lower and pseudo-upper approximations.

Keywords: Fuzzy sets; Rough sets; Pseudo-operations; Approximation operators

1. Introduction

The theory of rough set was originally proposed by Pawlak [1] as a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis. By using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. However, in Pawlak's rough set model, an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. Generalizations of rough set theory were considered by scholars in order to deal with complex practical problems [2-7].

There are at least two approaches for the development of definitions of lower and upper approximation operators, namely, the constructive and axiomatic approaches. In the constructive approach, some authors have extended equivalence relation to tolerance relations [8], similarity relations [9], ordinary binary relations [7,10], and others [11-13]. Meanwhile, some authors have relaxed the partition of universe to the covering and obtain the covering-based rough sets [4,14-20]. In addition, generalizations of rough sets to the fuzzy environment have also been made [2,5,21-26]. By introducing the lower and upper approximations in fuzzy set theory, Dubois and

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Prade [27] formulated rough fuzzy sets and fuzzy rough sets, they constructed a pair of lower and upper approximation operators for fuzzy sets with respect to fuzzy similarity relation by using the t-norm Min and its dual conorm Max. By using a residual implication (for short, R-implication) to define the lower approximation operator, Morsi and Yakout [28] generalized the fuzzy rough sets in the sense of Dubois and Prade. Later, Radzikowska and Kerre [29] proposed a more general approach to the fuzzification of a rough set. This approach is based on a border implication $\mathcal{I}$ (not necessarily a R-implication) and a triangular norm $\mathcal{T}$. Recently, Mi et al. [30] presented the generalized fuzzy rough sets determined by a triangular norm, Ouyang et al. [31] discussed fuzzy rough sets based on tolerance relations.

In the axiomatic approaches, a set of axioms is used to characterize the approximations. Lin and Liu [32] proposed six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the lower and upper approximations are the same as the abstract operators. The most important axiomatic studies for crisp rough sets were made by Yao [7,10,33]. Recently, the research of the axiomatic approach has also been extended to approximation operators in the fuzzy environment [28,30,34-37].

In some problems with uncertainty in the theory of probabilistic metric spaces, fuzzy logics and fuzzy measures, the pseudo-operations such as pseudo-additions and pseudo-multiplications are used [38-40]. Pseudo-analysis [38-47] has been applied in different fields, e.g., measure theory, integration, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. Interestingly, by using the Aczel's theorem [48], the pseudo-additions and pseudo-multiplications could be transferred into the corresponding results of reals such as the addition operator and multiplication operator. This can bring us the convenience of calculation.

We note that there are some literatures about pseudo integrals [7,8,10,25,35], but little literatures about rough set model based on pseudo-operations. The main purpose of this paper is to present a general framework for the study of fuzzy rough approximation operators based on pseudo-operations. By using the pseudo-operations, the pseudo-lower and pseudo-upper approximation operators are defined. Meanwhile, some properties of the proposed pseudo fuzzy rough approximation operators are investigated. Connections between the new and the existing fuzzy rough approximation operators are also discussed. Compared with the previous rough set models based on triangular norms [28-30,37], the pseudo-generalized fuzzy rough set proposed in this paper has its advantage to calculate its lower and upper approximations conveniently.

The remainder of this paper is organized as follows. In section 2, we recall some basic concepts of fuzzy sets, fuzzy relation, rough sets and pseudo-operations. In section 3, the pseudo-generalized fuzzy rough sets are presented. Some properties of the proposed pseudo fuzzy rough approximation operators are also investigated in this section. In section 4, the fuzziness of pseudo-generalized fuzzy rough sets is given. Section 5 presents conclusions.

2. Preliminaries

2.1 Fuzzy sets

Let U be a universe. Fuzzy set A is a mapping from U into the unit interval $[0, 1]$:

$$A : U \to [0, 1],$$

where for each $x \in U$, we call $A(x)$ the membership degree of x in A. If $U = \{x_1, x_2, \cdots, x_n\}$, then the fuzzy set A on U can be expressed by $\sum_{i=1}^{n} A(x_i)/x_i$. Additionally, the fuzzy power set, i.e., the set of all fuzzy sets in the universe U is denoted by $\mathcal{F}(U)$ [49].

For fuzzy sets $A, B \in \mathcal{F}(U)$,

$A \subseteq B \Leftrightarrow A(x) \leq B(x)$;

$(A \cap B)(x) = A(x) \wedge B(x) = \min\{A(x), B(x)\}$;

$(A \cup B)(x) = A(x) \vee B(x) = \max\{A(x), B(x)\}$;

$(\sim A)(x) = 1 - A(x)$, where $\sim A$ is the complement of A.

2.2 Fuzzy relation

Let U and W be two nonempty sets. The Cartesian product of U and W is denoted by $U \times W$. A fuzzy relation R from U to W is a fuzzy subset of $U \times W$, i.e., $R \in \mathcal{F}(U \times W)$, and $R(x, y)$ is called the degree of relation between x and y. In particular, if $U = W$, we call R a fuzzy relation on U. Usually, a fuzzy relation can be expressed by a fuzzy matrix.

2.3 Rough sets

In traditional Pawlak rough set theory, the pair (U, R) is called an approximation space (it is also called Pawlak approximation space), where U is a finite and non-empty set called the universe and R is an equivalence relation on U, i.e., R is reflexive, symmetrical and transitive. The relation R decomposes the set U into a disjoint class in such a way that two elements x and y are in the same class iff $(x, y) \in R$. Suppose R is an equivalence relation on U. With respect to R, we can define an equivalence class of an element x in U as follows:

$$[x]_R = \{y | (x, y) \in R\}.$$

The quotient set of U by the relation R is denoted by U/R, and

$$U/R = \{X_1, X_2, \cdots, X_m\}.$$

where X_i $(i = 1, 2, \cdots, m)$ is an equivalence class of R.

Given an arbitrary set $X \subseteq U$, it may not be possible to describe X precisely in the approximation space (U, R). One may characterize X by a pair of lower and upper approximations defined as follows:

$\underline{R}X = \{x \in U | [x]_R \subseteq X\} = \cup\{Y \in U/R | Y \subseteq X\}$;

$\overline{R}X = \{x \in U | [x]_R \cap X \neq \phi\} = \cup\{Y \in U/R | Y \cap X \neq \phi\}$.

The pair $(\underline{R}X, \overline{R}X)$ is referred to as a rough set of X.

2.4 Pseudo-operations

Throughout this paper, we only consider the case of pseudo-addition and present the fuzzy generalized rough sets using pseudo-addition. For the case of pseudo-multiplication, the discussion can be given similarly.

Definition 2.1 An operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a pseudo-addition if it satisfies the following axioms:

(1) Associativity: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in [0,\infty]$.

(2) Monotonicity: $a \oplus b \leq c \oplus d$ whenever $0 \leq a \leq c \leq \infty, 0 \leq b \leq d \leq \infty$.

(3) 0 is neutral element: $a \oplus 0 = 0 \otimes a = a$ for all $a \in [0,\infty]$.

(4) Continuity: for any sequences $(a_n)_{n\in N}$, $(b_n)_{n\in N}$ in $[0,\infty]^N$ such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ it holds $\lim\limits_{n\to\infty} a_n \oplus b_n = a \oplus b$.

(5) Commutativity: $a \oplus b = b \oplus a$ for all $a, b \in [0,\infty]$.

Lemma 2.1 (Aczel's theorem) Let g be a positive strictly monotone function defined on $[a,b] \subseteq (-\infty,+\infty)$ such that $0 \in Ran(g)$. The generalized generated pseudo-addition $\oplus$ and the generalized generated pseudo-multiplication $\odot$ are given by

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

$$x \odot y = g^{-1}(g(x)g(y)),$$

where g^{-1} is pseudo-inverse function for function g: $g^{-1}(y) = \sup\{x \in [a,b] | g(x) < y\}$ if g is a non-decreasing function and $g^{-1}(y) = \sup\{x \in [a,b] | g(x) > y\}$ if g is a non-increasing function.

Example 2.2 Suppose that $g(x) = 1 - x$ $(x \in [0,1])$, then its pseudo-inverse is

$$g^{-1}(x) = \begin{cases} 1 - x, & x \in [0,1], \\ 0, & x \in [1,+\infty). \end{cases}$$

And $x \oplus y = g^{-1}(g(x) + g(y)) = \max\{0, x + y - 1\}$, this is Lukasiewicz t-norm.

3. Construction of pseudo fuzzy rough approximation operators

Definition 3.1 Let U and W be two nonempty sets, R a fuzzy relation from U to W, then (U,W,R) is called a fuzzy approximation space. $g : [0,1] \to [0,+\infty)$ is a strictly decreasing function such that $g(1) = 0$ and $g(x) + g(y) \in Ran(g) \cup [g(0^+),+\infty)$ for all $(x,y) \in [0,1]^2$. Then for any $A \in \mathcal{F}(W)$, the pseudo-lower approximation $\underline{R}_\oplus(A)$ and the pseudo-upper approximation $\overline{R}_\oplus(A)$ of A are defined as follows, respectively:

$$\underline{R}_\oplus(A)(x) = \bigwedge_{y\in W} \{1 - R(x,y) \oplus (1 - A(y))\} = \bigwedge_{y\in W} \{1 - g^{-1}(g(R(x,y)) + g(1 - A(y)))\}, x \in U;$$

$$\overline{R}_\oplus(A)(x) = \bigvee_{y\in W} \{R(x,y) \oplus A(y)\} = \bigvee_{y\in W} \{g^{-1}(g(R(x,y)) + g(A(y)))\}, x \in U.$$

The pair $(\underline{R}_\oplus(A), \overline{R}_\oplus(A))$ is called a pseudo-generalized fuzzy rough set. $\underline{R}_\oplus$ and $\overline{R}_\oplus$ are referred to as the pseudo-lower and pseudo-upper fuzzy rough approximation operators, respectively.

Example 3.1 Suppose that (U,W,R) is a fuzzy approximation space, where U and W are two sets called object set and attribute set. Let $U = \{x_1, x_2, x_3\}$, $W = \{a_1, a_2, a_3, a_4\}$. $R \in \mathcal{F}(U \times W)$ is a fuzzy relation from U to W and R can be seen in Table 2:

For a fuzzy attribute set

$$A = 0.8/a_1 + 0.3/a_2 + 1/a_3 + 0.9/a_4 \in \mathcal{F}(W),$$

Table 1: A fuzzy approximation space

	a_1	a_2	a_3	a_4
x_1	1	0.4	0	0.1
x_2	0.3	0.9	0.7	0.6
x_3	0.9	0.2	1	0

if we take a strictly decreasing function as

$$g(x) = 1 - x \ (x \in [0,1]),$$

then the pseudo-lower approximation $\underline{R}_{\oplus}(A)$ and the pseudo-upper approximation $\overline{R}_{\oplus}(A)$ of A can be computed as follows:

$\underline{R}_{\oplus}(A)(x_1) = \min\{1-g^{-1}(0+0.8), 1-g^{-1}(0.6+0.3), 1-g^{-1}(1+1), 1-g^{-1}(0.9+0.9)\} = 0.8;$
$\underline{R}_{\oplus}(A)(x_2) = \min\{1-g^{-1}(0.7+0.8), 1-g^{-1}(0.1+0.3), 1-g^{-1}(0.3+1), 1-g^{-1}(0.4+0.9)\} = 0.4;$
$\underline{R}_{\oplus}(A)(x_3) = \min\{1-g^{-1}(0.1+0.8), 1-g^{-1}(0.8+0.3), 1-g^{-1}(0+1), 1-g^{-1}(1+0.9)\} = 0.9;$
$\overline{R}_{\oplus}(A)(x_1) = \max\{g^{-1}(0+0.2), g^{-1}(0.6+0.7), g^{-1}(1+0), g^{-1}(0.9+0.1)\} = 0.8;$
$\overline{R}_{\oplus}(A)(x_2) = \max\{g^{-1}(0.7+0.2), g^{-1}(0.1+0.7), g^{-1}(0.3+0), g^{-1}(0.4+0.1)\} = 0.7;$
$\overline{R}_{\oplus}(A)(x_3) = \max\{g^{-1}(0.1+0.2), g^{-1}(0.8+0.7), g^{-1}(0+0), g^{-1}(1+0.1)\} = 1.$

That is,

$$\underline{R}_{\oplus}(A) = 0.8/x_1 + 0.4/x_2 + 0.9/x_3,$$
$$\overline{R}_{\oplus}(A) = 0.8/x_1 + 0.7/x_2 + 1/x_3.$$

Remark 3.1 If R is a crisp binary relation from U to W, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [36]. That is, for every $A \in \mathcal{F}(W),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y)|y \in R_s(x)\},\quad \underline{R}_{\oplus}(A)(x) = \inf\{A(y)|y \in R_s(x)\},$$

where $R_s(x) = \{y \in W|(x,y) \in R\}$.

In fact,

$$\begin{aligned}
&\overline{R}_{\oplus}(A)(x)\\
&= \bigvee_{y\in W}\{g^{-1}(g(R(x,y)) + g(A(y)))\}\\
&= \sup\{g^{-1}(g(1) + g(A(y)))|y \in R_s(x)\} \bigvee \sup\{g^{-1}(g(0) + g(A(y)))|y \notin R_s(x)\}\\
&= \sup\{g^{-1}(g(1) + g(A(y)))|y \in R_s(x)\}\\
&= \sup\{g^{-1}(0 + g(A(y)))|y \in R_s(x)\}\\
&= \sup\{A(y)|y \in R_s(x)\},\\
&\underline{R}_{\oplus}(A)(x)\\
&= \bigwedge_{y\in W}\{1 - g^{-1}(g(R(x,y)) + g(1 - A(y)))\}\\
&= \inf\{1-g^{-1}(g(1)+g(1-A(y)))|y \in R_s(x)\} \bigwedge \inf\{1-g^{-1}(g(0)+g(1-A(y)))|y \notin R_s(x)\}\\
&= \inf\{1 - g^{-1}(g(1) + g(1 - A(y)))|y \in R_s(x)\}\\
&= \inf\{1 - g^{-1}(0 + g(1 - A(y)))|y \in R_s(x)\}\\
&= \inf\{A(y)|y \in R_s(x)\}.
\end{aligned}$$

Remark 3.2 If R is a crisp binary relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [7]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U | R_s(x) \cap A \neq \phi\}, \ \ \underline{R}_{\oplus}(A) = \{x \in U | R_s(x) \subseteq A\}.$$

where $R_s(x) = \{y \in U | (x, y) \in R\}$.

In fact, by Remark 3.2, we know that if $A \in P(U)$ then for any $x \in U$,

$x \in \overline{R}_{\oplus}(A) \Leftrightarrow \overline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow \exists\ y \in R_s(x)$ such that $A(y) = 1$, i.e., $y \in A \Leftrightarrow R_s(x) \cap A \neq \phi$,

$x \in \underline{R}_{\oplus}(A) \Leftrightarrow \underline{R}_{\oplus}(A)(x) = 1 \Leftrightarrow A(y) = 1$ for every $y \in R_s(x)$, i.e., $y \in A \Leftrightarrow R_s(x) \subseteq A$.

Remark 3.3 If R is a crisp equivalence relation on U and A is a fuzzy set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [27]. That is, for every $A \in \mathcal{F}(U),\ x \in U$,

$$\overline{R}_{\oplus}(A)(x) = \sup\{A(y) | y \in [x]_R\}, \ \ \underline{R}_{\oplus}(A)(x) = \inf\{A(y) | y \in [x]_R\}.$$

In fact, if R is a crisp equivalence relation on U, then $R_s(x) = [x]_R$.

Remark 3.4 If R is a crisp equivalence relation on U and A is a crisp set on U, then the pseudo fuzzy rough approximation operators defined in Definition 3.1 are degenerated into the approximation operators defined in [1]. That is, for any $A \in P(U),\ x \in U$,

$$\overline{R}_{\oplus}(A) = \{x \in U | [x]_R \cap A \neq \phi\}, \ \ \underline{R}_{\oplus}(A) = \{x \in U | [x]_R \subseteq A\}.$$

Example 3.2 Let $U = \{x_1, x_2, x_3\}$ be the universe of discourse, $R = \begin{pmatrix} 0.8 & 0.9 & 0.6 \\ 0.7 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0.8 \end{pmatrix}$ be a fuzzy relation on U. Suppose that $A, B, C \in \mathcal{F}(U)$, and

$$A = 0.4/x_1 + 0.5/x_2 + 0.8/x_3;$$

$$B = 0.6/x_1 + 0.7/x_2 + 0.2/x_3;$$

$$C = 0.6/x_1 + 0.8/x_2 + 0.9/x_3.$$

Let $g : [0, 1] \to [0, +\infty)$ given by $g(x) = 1 - x$ be a generating function for pseudo-addition $\oplus$, then we can compute that

$$\underline{R}_{\oplus}(A) = 0.6/x_1 + 0.6/x_2 + 0.6/x_3;$$

$$\overline{R}_{\oplus}(A) = 0.4/x_1 + 0.4/x_2 + 0.6/x_3;$$

$$\underline{R}_{\oplus}(B) = 0.6/x_1 + 0.8/x_2 + 0.4/x_3;$$

$$\overline{R}_{\oplus}(B) = 0.6/x_1 + 0.6/x_2 + 0.4/x_3;$$

$$\underline{R}_{\oplus}(C) = 0.9/x_1 + 1/x_2 + 0.9/x_3;$$

$$\overline{R}_{\oplus}(C) = 0.7/x_1 + 0.7/x_2 + 0.7/x_3.$$

From computation above, we can find $A \subseteq C$ implies that $\underline{R}_{\oplus}(A) \subseteq \underline{R}_{\oplus}(C)$ and $\overline{R}_{\oplus}(A) \subseteq \overline{R}_{\oplus}(C)$. Furthermore,

$$A \cap B = 0.4/x_1 + 0.5/x_2 + 0.2/x_3,$$

$$A \cup B = 0.6/x_1 + 0.7/x_2 + 0.8/x_3.$$

And

$$\underline{R}_{\oplus}(A \cap B) = 0.6/x_1 + 0.6/x_2 + 0.4/x_3;$$
$$\overline{R}_{\oplus}(A \cap B) = 0.4/x_1 + 0.4/x_2 + 0.2/x_3;$$
$$\underline{R}_{\oplus}(A \cup B) = 0.8/x_1 + 0.8/x_2 + 0.8/x_3;$$
$$\overline{R}_{\oplus}(A \cup B) = 0.6/x_1 + 0.6/x_2 + 0.6/x_3.$$

Thus, we notice that

$$\underline{R}_{\oplus}(A \cap B) = \underline{R}_{\oplus}(A) \cap \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cup B) = \overline{R}_{\oplus}(A) \cup \overline{R}_{\oplus}(B);$$

$$\underline{R}_{\oplus}(A \cup B) \supseteq \underline{R}_{\oplus}(A) \cup \underline{R}_{\oplus}(B),\ \overline{R}_{\oplus}(A \cap B) \subseteq \overline{R}_{\oplus}(A) \cap \overline{R}_{\oplus}(B).$$

4. Measuring fuzziness of pseudo-generalized fuzzy rough sets

Let (U, W, R) be a fuzzy approximation space, where U and W are two nonempty sets, R is a fuzzy relation from U to W. For any $A \in \mathcal{F}(W)$, the pseudo-generalized fuzzy rough set of A is $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$. Thus in the fuzzy approximation space (U, W, R), A is approximated by two fuzzy sets, one called the pseudo-lower approximation of A, and another called the pseudo-upper approximation of A. In this section, we suppose that $U = W$ and give an approach to measuring the fuzziness of pseudo-generalized fuzzy rough sets.

Definition 4.1 Let U be a universe of discourse, R be a fuzzy relation on U. For any $x \in U$ and $A \in \mathcal{F}(U)$, the degree of rough membership of x in A is defined by

$$r(A)(x) = \frac{\sum_{y \in U}[R(x, y) \oplus A(y)]}{\sum_{y \in U} R(x, y)}.$$

From Definition 4.1, we note that the fuzzy set A and fuzzy relation R on U can induce a new fuzzy set $r(A)$ of U.

Theorem 4.1 For any fuzzy sets $A, B \in \mathcal{F}(U)$,

(1) if $A \subseteq B$, then $r(A) \subseteq r(B)$;

(2) $r(A \cap B) \subseteq r(A) \cap r(B)$, $r(A \cup B) \supseteq r(A) \cup r(B)$.

Proof

(1) Since for any $x \in U$, $A(x) \leq B(x)$. By Definition 4.1, we have

$$r(A)(x) = \frac{\sum_{y \in U}[R(x, y) \oplus A(y)]}{\sum_{y \in U} R(x, y)} \leq \frac{\sum_{y \in U}[R(x, y) \oplus B(y)]}{\sum_{y \in U} R(x, y)} = r(B)(x).$$

So $r(A) \subseteq r(B)$.

(2) For any $A, B \in \mathcal{F}(U)$, we have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. It implies that

$$r(A \cap B) \subseteq r(A), \quad r(A \cap B) \subseteq r(B).$$

Thus, $r(A \cap B) \subseteq r(A) \cap r(B)$.

$r(A \cup B) \supseteq r(A) \cup r(B)$ can be proved in a similar way. $\square$

Definition 4.2 Let U be a universe of discourse, R be a fuzzy relation on U, $A \in \mathcal{F}(U)$. The fuzziness of pseudo-generalized fuzzy rough set $(\underline{R}_{\oplus}(A), \overline{R}_{\oplus}(A))$ is defined by

$$FR(A) = -\frac{1}{|U|}\sum_{x\in U} r(A)(x)\cdot \log_2 r(A)(x).$$

Example 4.1 (Continue the Example 3.2)

In Example 3.2, fuzzy relation $R = \begin{pmatrix} 0.8 & 0.9 & 0.6 \\ 0.7 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0.8 \end{pmatrix}$, three fuzzy sets A, B, C are denoted as follows, respectively:

$$A = 0.4/x_1 + 0.5/x_2 + 0.8/x_3;$$

$$B = 0.6/x_1 + 0.7/x_2 + 0.2/x_3;$$

$$C = 0.6/x_1 + 0.8/x_2 + 0.9/x_3.$$

Meanwhile, $g(x) = 1 - x$ $(x \in [0, 1])$. Thus, we can compute that

$$\begin{aligned} r(A)(x_1) &= \frac{g^{-1}(0.2+0.6) + g^{-1}(0.1+0.5) + g^{-1}(0.4+0.2)}{0.8+0.9+0.6} \\ &= \frac{0.2+0.4+0.4}{0.8+0.9+0.6} \\ &= 0.435. \end{aligned}$$

In a similar way, we get

$$r(A)(x_2) = \frac{0.1+0.4+0}{0.7+0.9+0.1} = 0.294,$$

$$r(A)(x_3) = \frac{0.2+0+0.6}{0.8+0.2+0.8} = 0.444.$$

That is,

$$r(A) = 0.435/x_1 + 0.294/x_2 + 0.444/x_3.$$

In addition, we can obtain that

$$r(B) = 0.435/x_1 + 0.529/x_2 + 0.222/x_3,$$

$$r(C) = 0.783/x_1 + 0.588/x_2 + 0.555/x_3,$$

$$r(A\cap B) = 0.261/x_1 + 0.294/x_2 + 0.111/x_3,$$

$$r(A\cup B) = 0.609/x_1 + 0.529/x_2 + 0.555/x_3.$$

From computation above, we note that
$A \subseteq C \Rightarrow r(A) \subseteq r(C)$, $r(A\cap B) \subseteq r(A)\cap r(B)$ and $r(A\cup B) \supseteq r(A)\cup r(B)$ hold.
Furthermore, we have

$$FR(A) = -\frac{1}{3}(0.435\times\log_2 0.435 + 0.294\times\log_2 0.294 + 0.444\times\log_2 0.444) \approx 0.521;$$

$$FR(B) = -\frac{1}{3}(0.435\times\log_2 0.435 + 0.529\times\log_2 0.529 + 0.222\times\log_2 0.222) \approx 0.497;$$

$$FR(C) = -\frac{1}{3}(0.783 \times \log_2 0.783 + 0.588 \times \log_2 0.588 + 0.555 \times \log_2 0.555) \approx 0.399;$$

$$FR(A \cap B) = -\frac{1}{3}(0.261 \times \log_2 0.261 + 0.294 \times \log_2 0.294 + 0.111 \times \log_2 0.111) \approx 0.459;$$

$$FR(A \cup B) = -\frac{1}{3}(0.609 \times \log_2 0.609 + 0.529 \times \log_2 0.529 + 0.555 \times \log_2 0.555) \approx 0.464.$$

From the results of Example 4.1, we note that $FR(A) \geq FR(C)$ whenever $A \subseteq C$, but for $A \cap B \subseteq A$, $FR(A \cap B) \leq FR(A)$.

It can be shown that for any $A, B \in \mathcal{F}(U)$, if $A \subseteq B$, $FR(A) \leq FR(B)$ or $FR(A) \geq FR(B)$ does not hold.

5. Conclusions

At present, there are many researchers about pseudo-analysis. Pseudo-analysis has been applied in different fields. It is interesting to combine pseudo-operations and rough set in order to expand the application domain of pseudo-analysis and rough set. In this paper, we presented a generalized fuzzy rough set model based on pseudo-operation, constructed pseudo fuzzy rough approximation operations. Some properties of the proposed generalized fuzzy rough approximation operators also investigated. At the same time, the fuzziness of pseudo-generalized fuzzy rough sets is given.

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References

[1] Z. Pawlak, Rough Sets, International Journal of Computer and Information Sciences 11 (5) (1982) 341-356.

[2] Z.T. Gong, B.Z. Sun, D.G. Chen, Rough set theory for the interval-valued fuzzy information systems, Information Sciences 178 (2008) 1968-1985.

[3] T.Y. Lin, Neighborhood systems and relational database, In Proceedings of 1988 ACM sixteenth annual computer science conference, February (1998) 23-25.

[4] Z.H. Shi, Z.T. Gong, The further investigation of covering-based rough sets: uncertainty characterization, similarity measure and generalized models, Information Sciences 180 (2010) 3745-3763.

[5] B.Z. Sun, Z.T. Gong, D.G. Chen, Fuzzy-rough set theory for the interval-valued fuzzy information systems, Information Sciences 178 (2008) 2794-2815.

[6] W.Z. Wu, W.X. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, Information Sciences 159 (2004) 233-254.

[7] Y.Y. Yao, Constructive and algebraic method of rough sets, Information Sciences 109 (1998) 21-47.

[8] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fundamenta Informaticae 27 (1996) 245-253.

[9] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Transactions on Knowledge and Data Engineering 2 (2000) 331-336.

[10] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Information Sciences 111 (1998) 239-259.

[11] G.L. Liu, Generalized rough sets over fuzzy lattices, Information Sciences 178 (6) (2008) 1651-1662.

[12] Z. Pawlak, Skowron A, Rough sets: Some extension, Information Sciences 177 (2006) 28-40.

[13] W. Zhu, Generalized rough sets based on relations, Information Sciences 177 (2007) 4997-5011.

[14] D.G. Chen, C.Z. Wang, Q.H. Hu, A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets, Information Sciences 177 (2007) 3500-3518.

[15] K. Qin, Y. Gao, Z. Pei, On covering rough sets, in: The Second International Conference on Rough Sets and Knowledge Technology (RSKT 2007), Lecture Notes in Computer Science 4481 (2007) 34-41.

[16] W.H. Xu, W.X. Zhang, Measuring roughness of generalized rough sets induced by a covering, Fuzzy Sets and Systems 158 (2007) 2443-2455.

[17] W. Zakowski, Approximations in the space (U, Π), Demonstratio Mathematica 16 (1983) 761-769.

[18] Y.L. Zhang, J.J. Li, W.Z. Wu, On axiomatic characterizations of three pairs of covering based approximation operators, Information Sciences 180 (2010) 274-287.

[19] W. Zhu, F.Y. Wang, Reduction and axiomization of covering generalized rough sets, Information Sciences 152 (2003) 217-230.

[20] W. Zhu, Topological approaches to covering rough sets, Information Sciences 177 (6) (2007) 1499-1508.

[21] T.Q. Deng, Y.M. Chen, W.L. Xu, Q.H. Dai, A novel approach to fuzzy rough sets based on a fuzzy covering, Information Sciences 177 (2007) 2308-2326.

[22] D. Dubois, H. Prade, Putting fuzzy sets and rough sets together, in: R. Slowinski (Ed.), Intelligence Decision Support, Kluwer Academic, Dordrecht, 1992, pp. 203-232.

[23] L.I. Kuncheva, Fuzzy rough set: application to feature selection, Fuzzy Sets and Systems 51 (1992) 147-153.

[24] T.J. Li, Y. Leung, W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, International Journal of Approximate Reasoning 48 (2008) 836-856.

[25] J.S. Mi, W.X. Zhang, An axiomatic characterization of a fuzzy generalization of rough sets, Information Sciences 160 (2004) 235- 249.

[26] S. Nanda, S. Majumda, Fuzzy rough set, Fuzzy Sets and Systems 45 (1992) 157-160.

[27] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of Gerneral Systems 17 (1990) 191-208.

[28] N.N. Morsi, M.M. Yakout, Axiomatics for fuzzy rough sets, Fuzzy Sets and Systems 100 (1998) 327-342.

[29] A.M. Radzikowska, E.E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2002) 137-155.

[30] J.S. Mi, Y. Leung, H.Y. Zhao, T. Feng, Generalized fuzzy rough sets determined by a triangular norm, Information Sciences 178 (2008) 3203-3213.

[31] Y. Ouyang, Z.D. Wang, H.P. Zhang, On fuzzy rough sets based on tolerance relations, Information Sciences 180 (2010) 532-542.

[32] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, Berlin, 1994, pp. 256-260.

[33] Y.Y. Yao, A comparative study of fuzzy sets and rough sets, Information Sciences 109 (1998) 227-242.

[34] G.L. Liu, Axiomatic systems for rough sets and fuzzy rough sets, International Journal of Approximation Reasoning 48 (2008) 857-867.

[35] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, Computers and Mathematics with Applications 58 (2009) 189-201.

[36] W.Z. Wu, J.S. Mi, W.X. Zhang, Generalized fuzzy rough sets, Information Sciences 151 (2003) 263-282.

[37] W.Z. Wu, Y. Leung, J.S. Mi, On characterizations of (I,T)-fuzzy rough approximation operators, Fuzzy Sets and Systems 154 (1) (2005) 76-102.

[38] H. Ichihashi, M. Tanaka, K. Asai, Fuzzy integrals based on pseudo-additions and multiplications, Journal of Mathematical Analysis and Applications 130 (1988) 354-364.

[39] J. Li, M. Radko, S. Peter, Pseudo-optimal measures, Information Sciences 180 (2010) 4015-4021.

[40] E. Pap, Pseudo-additive measures and their applications, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, North-Holland, Amsterdam, 2002, 1237-1260.

[41] S.P. Ivana, G. Tatjana, D. Martina, Riemann-Stieltjes type integral based on generated pseudo-operations, Novi Sad Journal of Mathematics 36 (2) (2006) 111-124.

[42] K. Lendelova, On the pseudo-Lebesgue-Stieltjes integral, Novi Sad Journal of Mathematics 36 (2) (2006) 125-136.

[43] E. Pap, N. Ralevic, Pseudo-Laplace transform, Nonlinear Analysis 33 (1998) 533-550.

[44] E. Pap, I. Stajner, Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory, Fuzzy Sets and Systems 102 (1999) 393-415.

[45] E. Pap, Generalization of the Jensen inequality for pseudo-integral, Information Sciences 180 (2010) 543-548.

[46] E. Pap, Generalized real analysis and its applications, International Journal of Approximate Reasoning 47 (2008) 368-386.

[47] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, Journal of Mathematical Analysis and Applications 122 (1987) 197-222.

[48] J. Aczel, Lectures on Functional Equations and their Applications, Academic Press, New York, 1966.

[49] L.A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338-353.

[50] W.X. Zhang, Y. Liang, P. Xu, Uncertainty reasoning based on inclusion degree, Tsinghua University Press, Beijing, 2007.

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Global analysis for delay virus infection model with multitarget cells

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Abstract

This paper investigates the qualitative behavior of viral infection model with multitarget cells in vivo. The infection rate is given by Crowley-Martin functional response. By assuming that the virus attack n classes of uninfected target cells, we study a viral infection model of dimension $2n+1$ with distributed delay. To describe the latent period for the contacted target cells with viruses to begin producing viruses, two types of distributed delay are incorporated into the model. The basic reproduction number R_0 of the model is defined which determines the dynamical behavior of the model. Utilizing Lyapunov functionals and LaSalle's invariance principle, we have proven that if $R_0 \leq 1$ then the uninfected steady state is globally asymptotically stable, and if $R_0 > 1$ then the infected steady state is globally asymptotically stable.

Keywords: Viral infection; Global stability; Delay; Crowley-Martin functional response.
AMS subject classifications. 92D25, 34D20, 34D23:

1 Introduction

Mathematical models have proven their importance in understanding the dynamical behaviors of various viruses such as human immunodeficiency virus (HIV), hepatitis B virus (HBV), hepatitis C virus (HCV), etc. [1]. The interatcion of the virus and target cells has been formulated as ordinary differential equations in several works (see e.g. [2], [3], [4], [12], [11], [5] and [6]). The basic mathematical model describing the dynamics of viral infection can be written in a general form as [6]:

$$\dot{x} = \lambda - dx - h(x,v), \tag{1}$$
$$\dot{y} = h(x,v) - \delta y, \tag{2}$$
$$\dot{v} = ky - rv, \tag{3}$$

where x, y and v represent the populations of the uninfected target cells, infected cells and free virus particles, respectively. The uninfected cells are generated from sources within the body at rate λ. The parameter d is the death rate constant of the uninfected target cells. Eq. (2) describes the population dynamics of the infected cells and shows that they die with rate constant δ. The virus particles are produced by the infected cells with rate constant k, and are cleared from plasma with rate constant r. The function $h(x,v)$ represents the incidence rate of infection and it has been considered in the viral infection models by different forms:

- Bilinear incidence rate [2], [3]: $h(x,v) = \beta xv$.
- Saturated incidence rate [30]: $h(x,v) = \frac{\beta xv}{1+bv}$.
- Holling type II functional response [34]: $h(x,v) = \frac{\beta xv}{1+ax}$.
- Beddington-DeAngelis infection rate [28]: $h(x,v) = \frac{\beta xv}{1+ax+bv}$.
- Crowley-Martin functional response [31], [32]: $h(x,v) = \frac{\beta xv}{(1+ax)(1+bv)}$,where $a, b \geq 0$ and β is the rate constant characterizing infections of the cells. The Crowley-Martin type of functional response was first introduced by Crowley and Martin [33].

Model (1)-(3) is based on the assumption that, once the virus contacts a target cell, the cell begins producing new virus particles. More realistic models incorporate the delay between the time of viral entry into the target cell and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations. Many researchers have devoted their effort in developing various mathematical models of viral infections with discrete or distributed delays and studying their qualitative behaviors (see e.g. [8], [10], [9], [27], [29], [24], [26], [22], [21], [34]).

In the literature, most of the proposed mathematical models for viral infection assume that the virus has one class of target cells, (e.g. CD4$^+$ T cells in case of HIV or hepatic cells in case of HCV and HBV) (see e.g. [2], [3] and the book Nowak and May [1]). In [7], [25], [13], [15], [18], [19], and [16], some HIV models with two classes of target cells, CD4$^+$ T cells and macrophages have been proposed. The global stability of these models has been investigated in ([13], [15] and [16]). Because the interactions of some types of viruses in *vivo* is complex and is not known clearly, we would suppose that the virus may attack n classes of target cells where $n \geq 1$ [14], [17]. In [17], models with discrete-time delays and saturated incidence rate have been studied. Elaiw [14] studied a class of virus infection models with multitarget cells without time delay.

The purpose of this paper is to propose a viral infection model with multitarget cells and Crowley-Martin functional response and investigate its qualitative behavior. We incorporate distributed delay into the model which represents an intracellular latent period for the contacted uninfected target cells with virus to begin producing new virus particles. The global stability of this model is established using Lyapunov functionals and LaSalle's invariance principle. We prove that the global dynamics of this model is determined by the basic reproduction number R_0. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS) and if $R_0 > 1$, then the infected steady state exists and is GAS.

2 Model with distributed time delays

In this section we propose a virus dynamics model with multitarget cells and multiple distributed intracellular delays.

$$\dot{x}_i(t) = \lambda_i - d_i x_i - \frac{\beta_i x_i(t) v(t)}{(1 + a_i x_i(t))(1 + b_i v(t))}, \qquad i = 1, ..., n \tag{4}$$

$$\dot{y}_i(t) = \beta_i \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau) v(t-\tau)}{(1 + a_i x_i(t-\tau))(1 + b_i v(t-\tau))} d\tau - \delta_i y_i(t), \qquad i = 1, ..., n \tag{5}$$

$$\dot{v}(t) = \sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau - r v(t), \tag{6}$$

where x_i and y_i represent the populations of the uninfected target cells and infected cells of class i, respectively, v is the population of the virus particles. To account for the time lag between viral contacting a target cell and the production of new virus particles, two distributed intracellular delays are introduced. It is assumed that the target cells of class i are contacted by the virus particles at time $t - \tau$ become infected cells at time t, where τ is a random variable with a probability distribution $f_i(\tau)$ over the interval $[0, \tau_i]$ and τ_i is limit superior of this delay. The factor $e^{-m_i \tau}$ accounts for the loss of target cells during delay period where m_i is positive constant. On the other hand, it is assumed that, a cell infected at time $t - \tau$ starts to yield new infectious virus at time t where τ is distributed according to a probability distribution $g_i(\tau)$ over the interval $[0, \mu_i]$ and μ_i is limit superior of this delay. The factor $e^{-n_i \tau}$ account for the cells loss during this delay period where n_i is positive constant. All the other parameters of the model have the same biological meaning as given in model (1)-(3).

The probability distribution functions $f_i(\tau) : [0, \tau_i] \to \mathbb{R}_+$ and $g_i(\tau) : [0, \mu_i] \to \mathbb{R}_+$ are integral functions with $\int_0^{\tau_i} f_i(\tau) d\tau = \int_0^{\mu_i} g_i(\tau) d\tau = 1, \ i = 1, ..., n$. Define $F_i = \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} d\tau$ and $G_i = \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} d\tau$, $m_i \geq 0$, $n_i \geq 0$. It is clear that $0 < F_i \leq 1$ and $0 < G_i \leq 1$, $i = 1, ..., n$.

The initial conditions for system (4)-(6) take the form

$$\begin{aligned} &x_j(\theta) = \varphi_j(\theta), \ y_j(\theta) = \varphi_{j+n}(\theta), \quad j = 1, ..., n, \ v(\theta) = \varphi_{2n+1}(\theta), \\ &\varphi_j(\theta) \geq 0, \ \theta \in [-\ell, 0), \ \varphi_j(0) > 0, \ j = 1, ..., 2n+1, \end{aligned} \tag{7}$$

where $\ell = \max\{\tau_1, ..., \tau_n, \mu_1, ..., \mu_n\}$, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_{2n+1}(\theta)) \in C$ and $C = C([-\ell, 0], \mathbb{R}_+^{2n+1})$ is the Banach space of continuous functions mapping the interval $[-\ell, 0]$ into $\mathbb{R}_+^{2n+1}$. By the fundamental theory of functional differential equations [20], system (4)-(6) has a unique solution satisfying initial conditions (7).

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (4)-(6) with initial conditions (7). Let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ and $\mathbf{y} = (y_1, y_2, ..., y_n)^T$.

Proposition 2. Let $(\mathbf{x}(t), \mathbf{y}(t), v(t))$ be any solution of (4)-(6) satisfying the initial conditions (7), then $\mathbf{x}(t), \mathbf{y}(t)$ and $v(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, ..., n$, for all $t \geq 0$. Assume that $x_i(t)$ lose its non-negativity on some local existence interval $[0, \omega]$ for some constant ω and let $t_1 \in [0, \omega]$ be such that $x_i(t_1) = 0$. From Eq. (4) we have $\dot{x}_i(t_1) = \lambda_i > 0$. Hence $x_i(t) < 0$ for some $t \in (t_1 - \varepsilon, t_1)$, where $\varepsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Further, from Eqs. (5) and (6) we have

$$y_i(t) = y_i(0)e^{-\delta_i t} + \beta_i \int_0^t e^{-\delta_i(t-\eta)} \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} \frac{x_i(\eta-\tau)v(\eta-\tau)}{(1+a_i x_i(\eta-\tau))(1+b_i v(\eta-\tau))} d\tau d\eta,$$

$$v(t) = v(0)e^{-rt} + \sum_{i=1}^{n} k_i \int_0^t e^{-r(t-\eta)} \int_0^{\mu_i} g_i(\tau)e^{-n_i\tau} y_i(\eta-\tau) d\tau d\eta,$$

confiming that $y_i(t) \geq 0$, $i = 1, ..., n$, and $v(t) \geq 0$ for all $t \in [0, \ell]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $i = 1, ..., n$, and $v(t) \geq 0$ for all $t \geq 0$.

Now we show the boundedness of the solutions of (4)-(6). Eqs. (4) imply that $\limsup_{t\to\infty} x_i(t) \leq x_i^0$, where $x_i^0 = \lambda_i/d_i$, and thus $x_i(t)$ is ultimately bounded. If follows that $\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}x_i(t-\tau)d\tau \leq F_i x_i^0$. Let $X_i(t) = \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}x_i(t-\tau)d\tau + y_i(t)$, $i = 1, ..., n$, then

$$\begin{aligned}\dot{X}_i(t) &= \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} \left(\lambda_i - d_i x_i(t-\tau) - \frac{\beta_i x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right) d\tau \\ &+ \int_0^{\tau_i} f_i(\tau)e^{-m_i\tau} \frac{\beta_i x_i(t-\tau)v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \delta_i y_i(t) \leq F_i\lambda_i - \sigma_i X_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, \delta_i\}$. Hence $\limsup_{t\to\infty} X_i(t) \leq L_i$, where $L_i = \lambda_i F_i/\sigma_i$. Since $\int_0^{\tau_i} f_i(\tau)e^{-m_i\tau}x_i(t-\tau)d\tau > 0$, we get $\limsup_{t\to\infty} y_i(t) \leq L_i$. On the other hand,

$$\dot{v}(t) \leq \sum_{i=1}^{n} k_i L_i \int_0^{\mu_i} g_i(\tau)e^{-n_i\tau} d\tau - rv = \sum_{i=1}^{n} k_i L_i G_i - rv,$$

then $\limsup_{t\to\infty} v(t) \leq L^*$, where $L^* = \sum_{i=1}^{n} \frac{k_i L_i G_i}{r}$. Therefore, $\mathbf{x}(t), \mathbf{y}(t)$ and $v(t)$ are ultimately bounded.□

2.2 Steady states

System (4)-(6) has an uninfected steady state $E_0 = (\mathbf{x}^0, \mathbf{y}^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i = 1, ..., n$ and $v^0 = 0$. In addition to E_0, the system can has a positive infected steady state $E_1(\mathbf{x}^*, \mathbf{y}^*, v^*)$. The coordinates of the infected steady state, if they exist, satisfy the equalities:

$$\lambda_i = d_i x_i^* + \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)}, \qquad i = 1, ..., n, \tag{8}$$

$$\delta_i y_i^* = F_i \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)}, \qquad i = 1, ..., n, \tag{9}$$

$$rv^* = \sum_{i=1}^{n} G_i k_i y_i^*. \tag{10}$$

The basic reproduction number of system (4)-(6) is given by

$$R_0 = \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \frac{F_i G_i \beta_i k_i x_i^0}{\delta_i r(1+a_i x_i^0)}, \tag{11}$$

where R_i is the basic reproduction number for the dynamics of the interaction of the virus only with the target cells of class i.

Lemma 1. If $R_0 > 1$, then there exists a positive steady state E_1.

Proof. To compute the steady states of model (4)-(6), we let the right-hand sides of Eqs. (4)-(6) equal zero,

$$\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)} = 0, \qquad i = 1, ..., n, \tag{12}$$

$$\frac{F_i \beta_i x_i v}{(1+a_i x_i)(1+b_i v)} - \delta_i y_i = 0, \qquad i = 1, ..., n, \tag{13}$$

$$\sum_{i=1}^{n} G_i k_i y_i - rv = 0. \tag{14}$$

Solving Eq. (12) with respect to x_i, we get x_i as a function of v as:

$$x_i^+ = \frac{a_i x_i^0 (1+b_i v) - (1+\phi_i v) + \sqrt{\left[(1+\phi_i v) - a_i x_i^0 (1+b_i v)\right]^2 + 4 a_i x_i^0 (1+b_i v)^2}}{2 a_i (1+b_i v)}, \tag{15}$$

$$x_i^- = \frac{a_i x_i^0 (1+b_i v) - (1+\phi_i v) - \sqrt{\left[(1+\phi_i v) - a_i x_i^0 (1+b_i v)\right]^2 + 4 a_i x_i^0 (1+b_i v)^2}}{2 a_i (1+b_i v)}, \tag{16}$$

where, $\phi_i = b_i + \frac{\beta_i}{d_i}$.

It is clear that if $v > 0$ then $x_i^+ > 0$ and $x_i^- < 0$. Let us choose $x_i = x_i^+$. From Eqs. (12)-(14) we have

$$\sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} (\lambda_i - d_i x_i) - rv = 0. \tag{17}$$

Since x_i is a function of v, then we can define a function $S_1(v)$ as:

$$S_1(v) = \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} (\lambda_i - d_i x_i) - rv = 0.$$

It is clear that when $v = 0$, then $x_i = x_i^0$ and $S_1(0) = 0$ and when $v = \overline{v} = \sum_{i=1}^{n} \frac{F_i G_i k_i \lambda_i}{\delta_i r} > 0$, then substituting it in Eq. (15) we obtain $\overline{x}_i > 0$ and

$$S_1(\overline{v}) = -\sum_{i=1}^{n} \frac{k_i d_i F_i G_i}{\delta_i} \overline{x}_i < 0.$$

Since $S_1(v)$ is continuous for all $v \geq 0$, we have that

$$S_1'(0) = \sum_{i=1}^{n} \frac{k_i \beta_i x_i^0 F_i G_i}{\delta_i \left(1 + a_i x_i^0\right)} - r = r(R_0 - 1).$$

Therefore, if $R_0 > 1$, then $S_1'(0) > 0$. It follows that there exists $v^* \in (0, \overline{v})$ such that $S_1(v^*) = 0$. From Eq. (15), we obtain $x_i^* > 0$, $i = 1, ..., n$. Moreover, from Eq, (13) we get $y_i^* > 0$, $i = 1, ..., n$. $\square$

2.3 Global stability

In this section, we prove the global stability of the uninfected and infected steady states of system (4)-(6) employing the method of Lyapunov functional which is used in [23] for SIR epidemic model with distributed delay.

Next we shall use the following notation: $z = z(t)$, for any $z \in \{x_i, y_i, v, \ i = 1, ..., n\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as $H(z) = z - 1 - \ln z$. It is clear that $H(z) \geq 0$ for any $z > 0$ and H has the global minimum $H(1) = 0$.

Theorem 1. (i) If $R_0 \leq 1$, then E_0 is GAS.

(ii) If $R_0 > 1$, then E_1 is GAS.

Proof. (i) Define a Lyapunov functional W_1 as:

$$W_1 = \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{x_i^0}{1 + a_i x_i^0} H\left(\frac{x_i}{x_i^0}\right) + \frac{1}{F_i} y_i + \frac{\delta_i}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \int_0^{\tau} y_i(t-\theta) d\theta d\tau \right.$$
$$\left. + \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \int_0^{\tau} \frac{x_i(t-\theta) v(t-\theta)}{(1 + a_i x_i(t-\theta))(1 + b_i v(t-\theta))} d\theta d\tau \right] + v.$$

The time derivative of W_1 along the trajectories of (4)-(6) satisfies

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{1}{1+a_i x_i^0} \left(1 - \frac{x_i^0}{x_i}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)}\right) \right. \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau) v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \frac{\delta_i}{F_i} y_i \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \left(\frac{x_i v}{(1+a_i x_i)(1+b_i v)} - \frac{x_i(t-\tau) v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} \right) d\tau \\
&\left. + \frac{\delta_i}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} (y_i - y_i(t-\tau)) d\tau \right] + \sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau - rv \\
&= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{\lambda_i}{1+a_i x_i^0} \left(2 - \frac{x_i}{x_i^0} - \frac{x_i^0}{x_i}\right) - \frac{\beta_i x_i v}{(1+a_i x_i^0)(1+a_i x_i)(1+b_i v)} \right. \\
&\left. + \frac{\beta_i x_i^0 v}{(1+a_i x_i^0)(1+a_i x_i)(1+b_i v)} + \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)} \right] - rv \\
&= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\frac{-\lambda_i}{x_i x_i^0 (1+a_i x_i^0)} \left(x_i - x_i^0\right)^2 + \frac{\beta_i x_i^0 v}{(1+a_i x_i^0)(1+b_i v)} \right] - rv \\
&= -\sum_{i=1}^{n} \frac{k_i F_i G_i d_i \left(x_i - x_i^0\right)^2}{\delta_i x_i (1+a_i x_i^0)} + r \sum_{i=1}^{n} \frac{F_i G_i k_i \beta_i x_i^0 v}{\delta_i r (1+a_i x_i^0)(1+b_i v)} - rv \\
&= -\sum_{i=1}^{n} \left(\frac{k_i F_i G_i d_i \left(x_i - x_i^0\right)^2}{\delta_i x_i (1+a_i x_i^0)} + \frac{r b_i R_i v^2}{1+b_i v} \right) + (R_0 - 1) rv. \qquad (18)
\end{aligned}$$

If $R_0 \leq 1$, then $\frac{dW_1}{dt} \leq 0$ for all $x_i, v > 0$. By Theorem 5.3.1 in [20], the solutions of system (4)-(6) limit to M, the largest invariant subset of $\left\{\frac{dW_1}{dt} = 0\right\}$. Clearly, it follows from (18) that $\frac{dW_1}{dt} = 0$ if and only if $x_i = x_i^0$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (6) we drive that $0 = \dot{v} = \sum_{i=1}^{n} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} k_i y_i(t-\tau) d\tau$. This yields $y_i = 0$ and hence $\frac{dW_1}{dt} = 0$ if and only if $x_i = x_i^0$, $y_i = 0$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

(ii) We construct the following Lyapunov functional

$$\begin{aligned}
W_2 &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[x_i - x_i^* - \int_{x_i^*}^{x_i} \frac{x_i^* (1+a_i \eta)}{\eta (1+a_i x_i^*)} d\eta + \frac{1}{F_i} y_i^* H\left(\frac{y_i}{y_i^*}\right) \right. \\
&+ \frac{1}{F_i} \frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \int_0^{\tau} H\left(\frac{x_i(t-\theta) v(t-\theta)(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* (1+a_i x_i(t-\theta))(1+b_i v(t-\theta))} \right) d\theta d\tau \\
&\left. + \frac{\delta_i y_i^*}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \int_0^{\tau} H\left(\frac{y_i(t-\theta)}{y_i^*}\right) d\theta d\tau \right] + v^* H\left(\frac{v}{v^*}\right).
\end{aligned}$$

Differentiating with respect to time yields

$$\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i} \left[\left(1 - \frac{x_i^* (1+a_i x_i)}{x_i (1+a_i x_i^*)}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+a_i x_i)(1+b_i v)}\right) \right. \\
&+ \frac{1}{F_i} \left(1 - \frac{y_i^*}{y_i}\right) \left(\beta_i \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau) v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau - \delta_i y_i \right) \\
&+ \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \left\{ \frac{x_i v}{(1+a_i x_i)(1+b_i v)} - \frac{x_i(t-\tau) v(t-\tau)}{(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} \right.
\end{aligned}$$

5

$$
\begin{aligned}
&+\frac{x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)} \ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right)\Bigg\} d\tau \\
&+\frac{\delta_i}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \left(y_i - y_i(t-\tau) + y_i^* \ln\left(\frac{y_i(t-\tau)}{y_i}\right)\right) d\tau \Bigg] \\
&+\left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau - rv\right).
\end{aligned} \tag{19}
$$

Collecting terms of (19) we obtain

$$
\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i}\Bigg[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(\lambda_i - d_i x_i) \\
&+\frac{\beta_i v x_i^*}{(1+a_i x_i^*)(1+b_i v)} - \frac{\beta_i}{F_i} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{y_i^* x_i(t-\tau) v(t-\tau)}{y_i(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau + \frac{\delta_i}{F_i} y_i^* \\
&+\frac{1}{F_i}\frac{\beta_i x_i^* v^*}{(1+a_i x_i^*)(1+b_i v^*)} \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right) d\tau \\
&+\frac{\delta_i y_i^*}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \ln\left(\frac{y_i(t-\tau)}{y_i}\right) d\tau\Bigg] - rv - \frac{v^*}{v}\sum_{i=1}^{n} k_i \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} y_i(t-\tau) d\tau + rv^*.
\end{aligned}
$$

Using the infected steady state conditions (8)-(10), we obtain

$$
\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i}\Bigg[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(d_i x_i^* - d_i x_i) - \frac{\delta_i}{F_i} y_i^* \frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)} + \frac{\delta_i}{F_i} y_i^* \frac{v(1+b_i v^*)}{v^*(1+b_i v)} \\
&-\frac{\delta_i}{F_i^2} y_i^* \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \frac{x_i(t-\tau)v(t-\tau) y_i^*(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* y_i\,(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))} d\tau \\
&+3\frac{\delta_i}{F_i} y_i^* + \frac{\delta_i}{F_i^2} y_i^* \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} \ln\left(\frac{x_i(t-\tau)v(t-\tau)(1+a_i x_i)(1+b_i v)}{x_i v(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right) d\tau \\
&+\frac{\delta_i y_i^*}{F_i G_i} \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \ln\left(\frac{y_i(t-\tau)}{y_i}\right) d\tau - \frac{\delta_i}{F_i} y_i^* \frac{v}{v^*} - \frac{\delta_i}{F_i G_i} y_i^* \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} \frac{v^* y_i(t-\tau)}{v y_i^*} d\tau\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{k_i F_i G_i}{\delta_i}\Bigg[\left(1-\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right)(d_i x_i^* - d_i x_i) + \frac{\delta_i}{F_i} y_i^* \left(-1 - \frac{v}{v^*} + \frac{v(1+b_i v^*)}{v^*(1+b_i v)} + \frac{1+b_i v}{1+b_i v^*}\right) \\
&-\frac{\delta_i}{F_i} y_i^* H\left(\frac{x_i^*(1+a_i x_i)}{x_i(1+a_i x_i^*)}\right) - \frac{\delta_i}{F_i} y_i^* H\left(\frac{1+b_i v}{1+b_i v^*}\right) \\
&-\frac{\delta_i}{F_i^2} y_i^* \int_0^{\tau_i} f_i(\tau) e^{-m_i \tau} H\left(\frac{x_i(t-\tau)v(t-\tau) y_i^*(1+a_i x_i^*)(1+b_i v^*)}{x_i^* v^* y_i\,(1+a_i x_i(t-\tau))(1+b_i v(t-\tau))}\right) d\tau \\
&-\frac{\delta_i}{F_i G_i} y_i^* \int_0^{\mu_i} g_i(\tau) e^{-n_i \tau} H\left(\frac{v^* y_i(t-\tau)}{v y_i^*}\right) d\tau\Bigg]
\end{aligned}
$$

$$= -\sum_{i=1}^{n}\frac{k_iF_iG_i}{\delta_i}\left[\frac{d_i\left(x_i-x_i^*\right)^2}{x_i\left(1+a_ix_i^*\right)}+\frac{\delta_iy_i^*b_i\left(v-v^*\right)^2}{F_iv^*\left(1+b_iv\right)\left(1+b_iv^*\right)}+\frac{\delta_i}{F_i}y_i^*H\left(\frac{x_i^*(1+a_ix_i)}{x_i(1+a_ix_i^*)}\right)\right.$$

$$+\frac{\delta_i}{F_i}y_i^*H\left(\frac{1+b_iv}{1+b_iv^*}\right)+\frac{\delta_i}{F_i^2}y_i^*\int_0^{\tau_i}f_i(\tau)e^{-m_i\tau}H\left(\frac{x_i(t-\tau)v(t-\tau)y_i^*(1+a_ix_i^*)(1+b_iv^*)}{x_i^*v^*y_i\left(1+a_ix_i(t-\tau)\right)(1+b_iv(t-\tau))}\right)d\tau$$

$$\left.+\frac{\delta_i}{F_iG_i}y_i^*\int_0^{\mu_i}g_i(\tau)e^{-n_i\tau}H\left(\frac{v^*y_i(t-\tau)}{vy_i^*}\right)d\tau\right].$$

It is easy to see that if $x_i^*, y_i^*, v^* > 0$, $i = 1, ..., n$, then $\frac{dW_2}{dt} \le 0$. By Theorem 5.3.1 in [20], the solutions of system (4)-(6) limit to M, the largest invariant subset of $\left\{\frac{dW_2}{dt} = 0\right\}$. It can be seen that $\frac{dW_2}{dt} = 0$ if and only if $x_i = x_i^*$, $v = v^*$, and $H = 0$ i.e.

$$\frac{x_i(t-\tau)v(t-\tau)y_i^*(1+a_ix_i^*)(1+b_iv^*)}{x_i^*v^*y_i\left(1+a_ix_i(t-\tau)\right)(1+b_iv(t-\tau))}=\frac{v^*y_i(t-\tau)}{vy_i^*}=1 \text{ for all } \tau\in[0,\ell]. \tag{20}$$

If $v = v^*$, then from Eq. (20) we have $y_i = y_i^*$, and hence $\frac{dW_2}{dt}$ equal to zero at E_1. LaSalle's invariance principle implies global stability of E_1.□

3 Conclusion

In this paper, we have investigated mathematical model of virus dynamics with distributed delay. We have assumed that the virus attack n classes of target cells. The infection rate is given by Crowley-Martin functional response. By defining the delay-dependent basic reproduction number R_0, we have discussed the existence of the steady states. The global stability of the uninfected and infected steady states of the model has been established using suitable Lyapunov functionals and LaSalle's invariant principle. We have proven that, if $R_0 < 1$, then the uninfected steady state is GAS and if $R_0 > 1$, then infected steady state is GAS.

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References

[1] M.A. Nowak, and R.M. May, "Virus dynamics: Mathematical Principles of Immunology and Virology," Oxford Uni., Oxford, 2000.

[2] M.A. Nowak, C.R.M. Bangham, Population dynamics of immune responses to persistent viruses, Science, 272 (1996), 74-79.

[3] M. A. Nowak, S. Bonhoeffer, A. M. Hill, Viral Dynamics in Hepatitis B Virus Infection, Proc. Natl. Acad. Sci. USA, 93 (1993), 4398-4402.

[4] L. Wang, M. Y. Li, and D. Kirschner, Mathematical analysis of the global dynamics of a model for HTLV-I infection and ATL progression, Math. Biosc., 179 (2002), 207-217.

[5] A. Korobeinikov, Global properties of basic virus dynamics models, Bull. Math. Biol., 66 (2004), 879-883.

[6] A. Korobeinikov, Global asymptotic properties of virus dynamics models with dose-dependent parasite reproduction and virulence, and non-linear incidence rate, Math. Med. Biol., 26 (2009), 225-239.

[7] D. S. Callaway, and A. S. Perelson, HIV-1 infection and low steady state viral loads, Bull. Math. Biol., 64 (2002), 29-64.

[8] R.V. Culshaw, and S. Ruan, A delay-differential equation model of HIV infection of CD4$^+$ T-cells, Math. Biosci., 165 (2000), 27-39.

[9] P. W. Nelson, J. Murray, and A. Perelson, A model of HIV-1 pathogenesis that includes an intracellular delay, Math. Biosci., 163 (2000), 201-215.

[10] N. M. Dixit, and A. S. Perelson, Complex patterns of viral load decay under antiretroviral therapy: Influence of pharmacokinetics and intracellular delay, J. Theoret. Biol., 226 (2004), 95-109.

[11] E. Gyurkovics, and A. M. Elaiw, A Stabilizing sampled-data $\ell-$step receding horizon control with application to a HIV/AIDS model, Differ. Eqs. Dynamic. Systems 14 (3-4) (2006), 323-352.

[12] A. M. Elaiw, K. Kiss, M.A.L. Caetano, Stabilization of HIV/AIDS model by receding horizon control, J. of Appl. Math. Computing 18 (1-2) (2005), 95-112.

[13] A. M. Elaiw, Global properties of a class of HIV models, Nonlinear Anal. Real World Appl., 11 (2010), 2191-3286.

[14] A. M. Elaiw, Global properties of a class of virus infection models with multitarget cells, Nonlinear Dynam., 69 (2012) 423-435.

[15] A. M. Elaiw, and X. Xia, HIV dynamics: Analysis and robust multirate MPC-based treatment schedules, J. Math. Anal. Appl., 356 (2009), 285-301.

[16] A.M. Elaiw, I. A. Hassanien, and S. A. Azoz, Global stability of HIV infection models with intracellular delays, J. Korean Math. Soc., 49(4) (2012), 779-794.

[17] A. M. Elaiw and M. A. Alghamdi, Global properties of virus dynamics models with multitarget cells and discrete-time delays, Discrete Dynamics in Nature and Society, 2011, Article ID 201274.

[18] A.M. Elaiw, Global dynamics of an HIV infection model with two classes of target cells and distributed delays, Discrete Dynamics in Nature and Society, vol. 2012, Article ID 253703, 13 pages, 2012.

[19] A.M. Elaiw and S.A. Azoz, Global properties of a class of HIV infection models with Beddington-DeAngelis functional response, Mathematical Methods in the Applied Sciences, 36 (2013), 383-394.

[20] J.K. Hale, and S. Verduyn Lunel, "Introduction to functional differential equations," Springer-Verlag, New York, 1993.

[21] G. Huang, Y. Takeuchi, and W. Ma, Lyapunov functionals for delay differential equations model of viral infection, SIAM J. Appl. Math., 70 (2010), 2693-2708.

[22] M. Y. Li, and H. Shu, Impact of intracellular delays and target-cell dynamics on in vivo infection, SIAM J. Appl. Math., 70 (2010), 2434-2448.

[23] C. C. McCluskey, Complete global stability for an SIR epidemic model with delay-distributed or discrete, Nonlinear Anal. Real World Appl., 11 (2010), 55-59.

[24] P. W. Nelson, and A. S. Perelson, Mathematical analysis of delay differential equation models of HIV-1 infection, Math. Biosci., 179 (2002), 73-94.

[25] A. S. Perelson, and P. W. Nelson, Mathematical analysis of HIV-1 dynamics in vivo, SIAM Rev., 41 (1999), 3-44.

[26] R. Xu, Global dynamics of an HIV-1 infection model with distributed intracellular delay, Comput. Math. Appl., 61 (2011), 2799-2805.

[27] S. Liu and L. Wang, Global stability of an HIV-1 model with distributed intracellular delays and a compination therapy, Math. Biosc. Eng., 7 (2010) 675-685.

[28] G. Huang, W. Ma, Y. Takeuchi, Global properties for virus dynamics model with Beddington-DeAngelis functional response, Appl. Math. Lett., 22 (2009), 1690-1694.

[29] G. Huang, W. Ma, Y. Takeuchi, Global analysis for delay virus dynamics model with Beddington-DeAngelis functional response, Appl. Math. Lett., 24, no. 7 (2011), 1199–1203.

[30] X. Song, A.U. Neumann, Global stability and periodic solution of the viral dynamics, J. Math. Anal. Appl., 329 (2007), 281-297.

[31] S. Xu, Global stability of the virus dynamics model with Crowley-Martin functional response, Electronic Journal of Qualitative Theory of Differential Equations, 2012 no. 9, (2012), 1-9.

[32] X. Y. Zhou, J. A. Cui, Global stability of the viral dynamics with Crowley-Martin functional response. Bull. Korean. Math. Soc., 48 (2011), 555-574.

[33] P. H. Crowley, E. K. Martin, Functional responses and interference within and between year classes of a dragonfly population. J. North. Am. Benth. Soc., 8 (1989), 211-221.

[34] G. Huang, H. Yokoi, Y. Takeuchi, T. Kajiwara, T. Sasaki, Impact of intracellular delay, immune activation delay and nonlinear incidence on viral dynamics, Japan J. Indust. Appl. Math., 28 (2011), 383-411.

The parameter reduction of soft sets and its algorithm *

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Abstract: Soft set theory is a new mathematical tool to deal with uncertain problems. In this paper, we prove the fact that there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems". Base on this fact, we investigate the parameter reduction of soft sets by means of the knowledge reduction in rough set theory and give an algorithm. Parameters of soft sets are classified and the core of soft sets are obtained.

Keywords: Soft sets; Rough sets; Information systems; One-to-one correspondences; Parameter reductions; Cores.

1 Introduction

In 1999, Molodtsov [6] proposed soft set theory as a new mathematical tool for dealing with uncertainties which is free from the difficulties affecting existing method. As reported in [6, 7], a wide range of applications of soft sets have been developed in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory.

Presently, works on theory of soft sets are progressing rapidly. Maji et al. [8, 9] further studied the theory of soft sets, used this theory to solve some decision making problems. Jiang et al. [4] extended soft sets with description logics. Ge et al. [3] discussed relationships between soft sets and topological spaces.

Rough set theory was initiated by [10] for dealing with vagueness and granularity in information systems. This theory handles the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning,

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intelligent information systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert information systems and many other fields (see [11]).

Soft set itself has classification ability. The parameter reduction of soft sets means reducing the number of parameters for a soft set to the minimum without distorting its original classification ability. Thus, the parameter reduction of soft sets is a very important problem in soft set theory. Maji et al. [9] introduce parameter reduction of soft sets. Unfortunately some errors in [9] were pointed out by Chen et al. [2]. They present a new definition of parameterization reduction in soft sets. In [5], Kong et al. pointed out some odd situations which may occur when method of reduction of parameters in case of soft sets given in [2] is applied. So they introduced the concept of reduction of normal parameters.

In [1], it has been seen that there is a very close relationship between soft sets and rough sets. The purpose of this paper is to investigate further the parameter reduction of soft sets with the help of rough set theory. We prove the fact that there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems". Base on this fact, we can do consider the parameter reduction of soft sets by means of the knowledge reduction in rough set theory.

2 Preliminaries

2.1 Soft sets

Definition 2.1 ([6]). *Let U be an initial universe and let A be a set of parameters. A pair (f, A) is called a soft set over U, if f is a mapping given by $f : A \to 2^U$ where 2^U is the power set of U. We denote (f, A) by f_A.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of the soft set f_A.

Example 2.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe consisting of five stores. Let $A = \{a_1, a_2, a_3, a_4,$
$a_5, a_6, a_7\}$ be is a set of status of stores where a_1, a_2, a_3, a_4, a_5, a_6 and a_7 represent respectively the parameters "high empowerment of sales personnel", "medium empowerment of sales personnel", "low empowerment of sales personnel", "good perceived quality of merchandise", "average perceived quality of merchandise", "high traffic location" and "low traffic location", respectively. We define f_A as follows*

$$f(a_1) = \{h_1\}, \ f(a_2) = \{h_2, h_3, h_5\}, \ f(a_3) = \{h_4\}, \ f(a_4) = \{h_1, h_2, h_3\},$$
$$f(a_5) = \{h_4, h_5\}, \ f(a_6) = \{h_1, h_2, h_3\}, \ f(a_7) = \{h_4, h_5\}.$$

Soft sets f_A can be described as the following Table 1. If $h_i \in f(a_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where h_{ij} are the entries in Table 1.

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Table 1: Tabular representation of the soft set f_A

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
h_1	1	0	0	1	0	1	0
h_2	0	1	0	1	0	1	0
h_3	0	1	0	1	0	1	0
h_4	0	0	1	0	1	0	1
h_5	0	1	0	0	1	0	1

Definition 2.3. *Let f_A be a soft set over U. f_A is called non-trivial, if for any $a \in A$, $f(a) \neq \emptyset$ and $f(a) \neq U$.*

In this paper, we only consider non-trivial soft sets.

2.2 Information systems

Definition 2.4 ([11, 12])**.** *Let U be a finite set of objects and let A be a finite set of attributes. The pair (U, A, V, g) is called an information system (a knowledge representation system), if g is an information function from $U \times A$ to $V = \bigcup\limits_{a\in A} V_a$ where every $V_a = \{g(x, a) : a \in A \text{ and } x \in U\}$ is the values of the attribute a.*

Definition 2.5. *An information system (U, A, V, g) is called 2-value, if $V = \{0, 1\}$.*

Example 2.6. *Let $U = \{h_1, h_2, h_3, h_4\}$ be a universe consisting of four patients, and let $A = \{a_1, a_2, a_3\}$ be a set of attributes where a_1, a_2 and a_3 represent respectively the attributes " headache", " muscle pain" and " fever".*

Now, we consider an information system (U, A, V, g), which describes the " symptoms of patients". For instance, "$g(h_1, a_1) = yes$" means "h_1 suffers from headache" and its functional value is yes; "$g(h_3, a_2) = no$ "means "h_3 has no muscle pain" and its functional value is no; "$g(h_3, a_3) = no$" means "h_3 doesn't have a fever" and its functional value is no.

We define
$g(h_1, a_1) =$*yes*, $g(h_1, a_2) =$*yes*, $g(h_1, a_3) =$*no;*
$g(h_2, a_1) =$*yes*, $g(h_2, a_2) =$*yes*, $g(h_2, a_3) =$*yes;*
$g(h_3, a_1) =$*yes*, $g(h_3, a_2) =$*yes*, $g(h_3, a_3) =$*no;*
$g(h_4, a_1) =$*no*, $g(h_4, a_2) =$*yes*, $g(h_4, a_3) =$*no.*

Let h_{ij} be the entries. If $g(h_i, a_j) =$yes, then $h_{ij} = 1$; if $g(h_i, a_j) =$no, then $h_{ij} = 0$. A 2-value information system (U, A, V, g) can be described as the following Table 2.

In Table 2, $V_{a_1} = \{0, 1\}$, $V_{a_2} = \{0, 1\}$, $V_{a_3} = \{0, 1\}$, $V = \bigcup_{a\in A} V_a = \{0, 1\}$.

Let (U, A, V, g) be an information system and let $P \subseteq A$. We denote

$$ind(P) = \{(x, y) \in U \times U : g(x, a) = g(y, a) \text{ for any } a \in P\}.$$

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Table 2: The 2-value information system (U, A, V, g)

	a_1	a_2	a_3
h_1	1	1	0
h_2	1	1	1
h_3	1	1	0
h_4	0	1	0

Obviously, $ind(P)$ is an equivalence relation on U, which is called the equivalence relation induced by P. Sometimes, we replace respectively $ind(P)$ and $U/ind(P)$ by $\mathbf{P}$ and $U/\mathbf{P}$ where

$$U/ind(P) = \{[x]_{ind(P)} : x \in U\}.$$

Specially, we replace $ind(\{a\})$ by $\mathbf{a}$ for $a \in A$.

Theorem 2.7. *Let $S = f_A$ be a soft set over U and let $I_S = (U, A, V, g_s)$ be a 2-value information system induced by S. Then for any $a \in A$, $U/\mathbf{a} = \{f(a), U - f(a)\}$.*

Proof. Since

$$\mathbf{a} = \{(x, y) \in U \times U : g_s(x, a) = g_s(y, a)\},$$

$$g_s(x, a) = g_s(y, a) = 1 \textit{ or } g_s(x, a) = g_s(y, a) = 0.$$

This implies that $\{x, y\} \subset f(a)$ *or* $\{x, y\} \subset U - f(a)$. Thus $U/\mathbf{a} = \{f(a), U - f(a)\}$. $\square$

2.3 The relationship between soft sets and information systems

Definition 2.8. *Let $S = f_A$ be a soft set over U. Then $I_S = (U, A, V, g_s)$ is called a 2-value information system induced by S where $g_s : U \times A \to V$.*

For any $x \in U$ and $a \in A$,

$$g_s(x, a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

Definition 2.9. *Let $I = (U, A, V, g)$ be a 2-value information system. Then $S_I = (f_I, A)$ is called a soft set over U induced by I where $f_I : A \to 2^U$ and for any $x \in U$ and $a \in A$, $f_I(a) = \{x \in U : g(x, a) = 1\}$.*

Lemma 2.10. *Let $S = f_A$ be a soft set over U, let $I_S = (U, A, V, g_s)$ be a 2-value information system induced by S over U and let $S_{I_S} = (f_{I_S}, A)$ be a soft set over U induced by I_S. Then $S = S_{I_S}$.*

Proof. By Definition 2.9, for any $a \in A$, $f_{I_S}(a) = \{x \in U : g_s(x,a) = 1\}$.
By Definition 2.8, for any $x \in U$ and $a \in A$,

$$g_s(x,a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

This implies that $g_s(x,a) = 1 \Leftrightarrow x \in f(a)$. So, for $\forall x \in U, a \in A$, $f(a) = f_{I_S}(a)$. Hence $f_A = (f_{I_S}, A)$. This implies that $S = S_{I_S}$. □

Lemma 2.11. *Let $I = (U, A, V, g)$ be a 2-value information system, Let $S_I = (f_{I,A})$ be a soft set over U induced by I and let $I_{S_I} = (U, A, V, g_{s_I})$ be a 2-value information system induced by S_I. Then $I = I_{S_I}$.*

Proof. By Definition 2.8, for any $x \in U$ and $a \in A$,

$$g_{s_I}(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

For any $x \in U$ and $a \in A$, by Definition 2.9, $f_I(a) = \{x \in U : g(x,a) = 1\}$. Since $I = (U, A, V, g)$ is a 2-value information system, $g(x,a) = 0$ for $x \notin f_I(a)$, This implies that

$$g(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

So for any $x \in U$ and $a \in A$, $g_{s_I}(x,a) = g(x,a)$. Hence $g_{s_I} = g$. This implies that that $I = I_{S_I}$. □

Theorem 2.12. *Let*

$$\Sigma = \{S : \ S = f_A \ is \ a \ soft \ set \ over \ U\}$$

and

$$\Gamma = \{I : I = (U, A, V, g) isa2 - valueinformationsystem\}.$$

Then there exists a one-to-one correspondence between Σ and Γ.

Proof. Two mappings $F : \Sigma \to \Gamma$ and $G : \Gamma \to \Sigma$ are defined as follows:

$$F(S) = I_S \ for \ \forall S \in \Sigma; \quad G(I) = S_I \ for \ \forall I \in \Gamma.$$

By Lemma 2.9, $G \circ F = i_\Sigma$ where $G \circ F$ is the composition of F and G, and i_Σ is the identity mapping on Σ.

By Lemma 2.10, $F \circ G = i_\Gamma$ where $G \circ F$ is the composition of G and F, and i_Γ is the identity mapping on Γ.

Hence F and G are both a one-to-one correspondence. This prove that there exists a one-to-one correspondence between Σ and Γ. □

3 The parameter reduction of soft sets

Soft sets and rough sets are two different concepts to deal with uncertainty. Both of these concepts help in decision-making problems. Soft set itself has classification ability. The parameter reduction of soft sets means reducing the number of parameters for a soft set to the minimum without distorting its original classification ability. Specific approach is first classifying the parameter according to the importance of parameters and then finding the minimum set of parameters (ie., the core for a soft set) without distorting the original classification ability of soft sets.

Reduction of parameters of soft sets plays a vital role in decision-making problems. Reduction of parameters can save expensive tests and time.

Since there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems" (see Theorem 2.12), we can do the parameter reduction of soft sets with the help of the knowledge reduction in rough set theory.

Definition 3.1. *Let f_A be a soft set over U.*

(1) *$A^* \subseteq A$ is called a parameter reduction of f_A (brief. a f_A-parameter reduction), if $ind(A) = ind(A^*)$ and $ind(A) \neq ind(B)$ for any $B \subsetneq A^*$.*

(2) *The intersection set of all f_A-parameter reductions is called the core of f_A. We denote it by core(f_A).*

In this paper, we denote the set of all f_A-parameter reductions by $pr(f_A)$.

Proposition 3.2. *Let f_A be a soft set over U. Then $pr(f_A) \neq \emptyset$.*

Proof. (1) If $ind(A) \neq ind(A - \{a\})$ for any $a \in A$, then A itself is a f_A-parameter reduction.

(2) If $ind(A) = ind(A - \{a\})$ for some $a \in A$, then we consider $B_1 = A - \{a\}$. If $ind(A) \neq ind(B_1 - \{b_1\})$ for any $b_1 \in B_1$, B_1 is a f_A-parameter reduction. Otherwise, we consider $B_1 - \{b_1\}$ again and repeat the above mentioned process. Since A is a finite set, we can find a f_A-parameter reduction.

Thus, $pr(f_A) \neq \emptyset$. □

Definition 3.3. *Let f_A be a soft set over U and let $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. Then*

(1) *$a \in A$ is called core, if $a \in \bigcap_{i=1}^{n} C_i = core(f_A)$.*

(2) *$a \in A$ is called relative indispensable, if $a \in \bigcup_{i=1}^{n} C_i - core(f_A)$.*

(3) *$a \in A$ is called absolutely dispensable, if $a \in A - \bigcup_{i=1}^{n} C_i$.*

(4) *$a \in A$ is called dispensable, if $a \in A - core(f_A)$.*

Obviously, $a \in A$ is dispensable if and only if a is relative indispensable or absolutely dispensable.

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Definition 3.4. *Let $\mathcal{A}, \mathcal{B} \subset 2^U$. $\mathcal{A}$ is called a refinement of $\mathcal{B}$, if for any $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$. We denote it by $\mathcal{A} \leq \mathcal{B}$.*

Lemma 3.5. *Let R and ρ be two equivalence relations on U. If $R \subseteq \rho$, then $U/R \leq U/\rho$.*

Proof. Suppose that $A \in X/R$. Since R is an equivalence relation on X, there exists $x \in X$, such that $A = [x]_R$.

Suppose that $y \in [x]_R$. Then xRy. This implies that $(x, y) \in R$. Since $R \subseteq \rho$, $(x, y) \in \rho$. This implies that $y \in [x]_\rho$. Then $[x]_R \subseteq [x]_\rho$.

Pick $B = [x]_\rho$. Then $A \subseteq B$ and so $X/R \leq X/\rho$. □

The following Theorem 3.6 and Corollary 3.7 give the parameter reduction of soft sets.

Theorem 3.6. *Let f_A be a soft set over U. Then*
(1) $|pr(f_A)| = 1$ if and only if $core(f_A) \in pr(f_A)$.
(2) $a \in core(f_A)$ if and only if $U/ind(A) \neq U/ind(A - \{a\})$.
(3) $a \in A$ is dispensable if and only if $U/ind(A) = U/ind(A - \{a\})$.

Proof. (1) Sufficiency. Let $core(f_A) \in pr(f_A)$. Note that $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. We only need to prove $n = 1$.

1) Suppose $n = 2$. Then there are only two different f_A-parameter reductions C_1 and C_2.

a) If $C_1 \subsetneq C_2$. Since $C_2 \in pr(f_A)$, $ind(A) \neq ind(C_1)$. Then $C_1 \notin pr(f_A)$. This is a contradiction.

b) If $C_2 \subsetneq C_1$. We can similarly prove that this implies a contradiction.

c) If $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$. Obviously, $core(f_A) = C_1 \cap C_2$ and $core(f_A) \subsetneq C_1$. Since $C_1 \in pr(f_A)$, $ind(A) \neq ind(core(f_A))$. Then $core(f_A) \notin pr(f_A)$. This is also a contradiction.

2) Suppose $n \geq 3$. This is similar to the proof of 1).

Thus $|pr(f_A)| = 1$.

Necessity. This is obvious.

(2) Sufficiency. Suppose that $U/ind(A) \neq U/ind(A - \{a\})$. We claim that $a \in C_i$ for any $1 \leq i \leq n$. Otherwise. $a \notin C_{i_0}$ for some C_{i_0}. This implies that $U/ind(A) = U/ind(C_{i_0})$. Since $ind(C_{i_0}) \supseteq ind(A - \{a\}) \supseteq ind(A)$, by Lemma 3.5, $U/ind(C_{i_0}) \geq U/ind(A - \{a\}) \geq U/ind(A)$. So $U/ind(A) = U/ind(A - \{a\})$, a contradiction.

This implies that $a \in core(f_A)$.

Necessity. Suppose that $U/ind(A) = U/ind(A - \{a\})$. Since $pr(f_A) \neq \emptyset$, there exists $B_1' \subseteq A - \{a\}$ such that $B_1' \in pr(f_A)$. So $a \notin core(f_A)$. This is a contradiction.

Thus $U/ind(A) \neq U/ind(A - \{a\})$.

(3) Sufficiency. Suppose that $U/ind(A) = U/ind(A - \{a\})$. Since $A - \{a\}$ is a finite set, there exists $B_2 \subseteq A - \{a\}$ such that $B_2 \in pr(f_A)$. So $a \notin core(f_A)$. This implies that $a \in A - core(f_A)$.

Thus a is a dispensable parameter.

Necessity. Suppose that $U/ind(A) \neq U/ind(A-\{a\})$. Similar to the proof of (2), we have $a \in core(f_A)$. Then $a \notin A-core(f_A)$. Note that a is a dispensable parameter. Then $a \in A - core(f_A)$. This implies a contradiction.

Thus $U/ind(A) = U/ind(A - \{a\})$. □

Corollary 3.7. $core(f_A) = \{a \in A : U/ind(A) \neq U/ind(A - \{a\})\}$.

4 Algorithms

Algorithms 4.1. *Let f_A be a soft set over U. The algorithm of parameter reduction is shown as follows:*

Input: A soft set f_A.

Output: $pr(f_A)$ and $core(f_A)$.

Step 1. *Calculate $U/ind(A)$ and $U/ind(A - \{a\})$ for any $a \in A$;*

Step 2. *If $U/ind(A) \neq U/ind(A - \{a\})$ for any $a \in A$, then $pr(f_A) = \{A\}$ and $core(f_A) = A$;*

Step 3. *If $U/ind(A) = U/ind(A - \{a\})$ for some $a \in A$, then we consider $B_1 = A - \{a\}$. If $U/ind(A) \neq U/ind(B_1 - \{b_1\})$ for any $b_1 \in B_1$, then $B_1 \in pr(f_A)$; Otherwise, we consider $B_1 - \{b_1\}$ again;*

Step 4. *Output $pr(f_A)$ and $core(f_A)$.*

Example 4.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let f_A be a soft set over U, defined as follows*

$$f(a_1) = \{h_1, h_2, h_5\}, \quad f(a_2) = \emptyset, \ f(a_3) = \{h_3\}, \ f(a_4) = \{h_3, h_4\}.$$

By Theorem 2.7, we have $U/\mathbf{a_1} = \{f(a_1), U - f(a_1)\} = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\}$,

$U/\mathbf{a_2} = \{f(a_2), U - f(a_2)\} = \{\{h_1, h_2, h_3, h_4, h_5\}\}$,

$U/\mathbf{a_3} = \{f(a_3), U - f(a_3)\} = \{\{h_3\}, \{h_1, h_2, h_4, h_5\}\}$,

$U/\mathbf{a_4} = \{f(a_4), U - f(a_4)\} = \{\{h_3, h_4\}, \{h_1, h_2, h_5\}\}$. *And*

$U/\mathbf{A} = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\}$. $U/ind(A-\{a_1\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

$U/ind(A - \{a_2\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

$U/ind(A - \{a_3\}) = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\} \neq U/ind(A)$.

$U/ind(A - \{a_4\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.

This implies that

$U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_2, a_3\}) = U/ind(A)$.

Since $U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_3, a_4\})$, $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_3\})$ *and* $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_4\})$, $\{a_3, a_4\}$ *is a f_A-parameter reduction.*

Since $U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_3\})$, $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_1\})$ *and* $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_4\})$, $\{a_1, a_3\}$ *also is a f_A-parameter reduction.*

Obviously,

$$pr(f_A) = \{\{a_3, a_4\}, \{a_1, a_3\}\},$$
$$core(f_A) = \{a_3, a_4\} \cap \{a_1, a_3\} = \{a_3\}.$$

Example 4.3. *In Example 4.2, we have*
(1) a_3 *is core.* (2) a_1 *and* a_4 *are relative indispensable.*
(3) a_2 *is absolutely dispensable.* (4) a_1, a_2 *and* a_4 *are dispensable.*

References

[1] M.Irfan Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Appl. Soft Comput., 13(2011), 3329-3332.

[2] D.Chen, E.C.C.Tsang, D.S.Yeung, X.Wang, The parameterization reduction of soft sets and its applications, Comput. Math. Appl., 49(2005), 757-763.

[3] X.Ge, Z.Li, Y.Ge, Topological spaces and soft sets, J. Comput. Anal. Appl., 13(2011), 881-885.

[4] Y.Jiang, Y.Tang, Q.Chen, J.Wang, S.Tang, Extending soft sets with description logics, Comput. Math. Appl., 59(2010), 2087-2096.

[5] Z.Kong, L.Gao, L.Wang, S.Li, The normal parameter reduction of soft sets and its algorithm, Comput. Math. Appl., 56(2008), 3029-3037.

[6] D.Molodtsov, Soft set theory-first result, Comput. Math. Appl., 37(1999), 19-31.

[7] D. Molodtsov, The theory of soft sets, URSS Publishers, Moscow, 2004.

[8] P.K.Maji, R.Biswas, A.R.Roy, Soft set theory, Comput. Math. Appl., 45(2003), 555-562.

[9] P.K.Maji, A.R.Roy, An application of soft sets in a decision making problem, Comput. Math. Appl., 44(2002), 1077-1083.

[10] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci., 11(1982), 341-356.

[11] Z. Pawlak, Rough Sets: Theoretical aspects of reasoning about data, Kluwer Academic Publishers., Dordrecht, 1991.

[12] W.Zhang, W.Wu, J.Liang, D.Li, Rough sets theorey and methods, Chinese Scientific Publishers, Beijing, 2001.

FUNCTIONAL INEQUALITIES ASSOCIATED WITH BI-CAUCHY ADDITIVE FUNCTIONAL EQUATIONS

GANG LU, CHOONKIL PARK, AND DONG YUN SHIN*

Abstract. In this paper, we prove the Hyers-Ulam stability for the following functional inequalities:

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \le \|f(x_1 + x_2 + x_3, y_1 + y_2 + y_3)\|, \tag{1}$$

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \le \left\|2f\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right)\right\|, \tag{2}$$

$$\|f(x_1, y_1) + f(x_2, y_2) + 2f(x_3, y_3)\| \le \left\|2f\left(\frac{x_1 + x_2}{2} + x_3, \frac{y_1 + y_2}{2} + y_3\right)\right\| \tag{3}$$

in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the question of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equation is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$ and some $\epsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathcal{R}$ for each fixed $x \in X$. Th.M. Rassias

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[3] introduced the following inequality, that we call *Cauchy-Rassias inequality* : Assume that there exist constants $\lambda \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \lambda(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th.M. Rassias [3] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\lambda}{2 - 2^p}\|x\|^p$$

for all $x \in X$. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [4] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replace by an Abelian group. In [7], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [8] generalized the stability result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found in [9]–[28].

In this paper, let X be a vector space and Y a Banach space. A mapping $f : X \to Y$ is called a Cauchy additive mapping if f satisfies the functional equation $f(x+y) = f(x)+f(y)$. For a given mapping $f : X \times X \to Y$, we define

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2) \tag{1.1}$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. A mapping $f : X \times X \to Y$ is called a bi-Cauchy mapping if f satisfies the functional equation (1.1). We investigate the functional inequalities (1), (2) and (3) and prove the Hyers-Ulam stability of the functional inequalities (1), (2) and (3).

2. Hyers-Ulam stability of the functional inequality (1)

Proposition 2.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \leq \|f(x_1 + x_2 + x_3, y_1 + y_2 + y_3)\| \tag{2.1}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then the mapping $f : X \to Y$ is bi-Cauchy additive.

Proof. Letting $(x_1, y_1) = (x_2, y_2) = (x_3, y_3) = (0, 0)$ in (2.1), we have

$$\|3f(0, 0)\| \leq \|f(0, 0)\|$$

and so $f(0, 0) = 0$.

Letting $x_1 = x, x_2 = -x, x_3 = 0, y_1 = y, y_2 = -y, y_3 = 0$ in (2.1), we get

$$\|f(x, y) + f(-x, -y)\| \leq 0$$

and so $f(x, y) = -f(-x, -y)$ for all $(x, y) \in X \times X$.

Next, we show that f is a bi-Cauchy additive mapping.

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)-f(x_1+x_2,y_1+y_2)\|\\&=\|f(x_1,y_2)+f(x_2,y_2)+f(-x_1-x_2,-y_1-y_2)\|\\&\le\|f(0,0)\|=0\end{aligned}$$

and so $f(x_1+x_2,y_1+y_2)=f(x_1,y_1)+f(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2)\in X\times X$, as desired. □

Theorem 2.2. *Assume that a mapping $f:X\times X\to Y$ satisfies the inequality*

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)+f(x_3,y_3)\|\\&\le\|f\left(x_1+x_2+x_3,y_1+y_2+y_3\right)\|+\phi((x_1,y_1),(x_2,y_2),(x_3,y_3)),\end{aligned}\tag{2.2}$$

where $\phi:(X\times X)^3\to[0,\infty)$ satisfies

$$\widetilde{\phi}((x_1,y_1),(x_2,y_2),(x_3,y_3)):=\sum_{j=1}^{\infty}2^j\phi\left(\left(\frac{x_1}{2^j},\frac{y_1}{2^j}\right),\left(\frac{x_2}{2^j},\frac{y_2}{2^j}\right),\left(\frac{x_3}{2^j},\frac{y_3}{2^j}\right)\right)<\infty\tag{2.3}$$

for all $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times X$. Then there exists a unique bi-Cauchy additive mapping $A:X\times X\to Y$ such that

$$\|A(x,y)-f(x,y)\|\le\widetilde{\phi}\left(\left(\frac{x}{2},\frac{y}{2}\right),\left(\frac{x}{2},\frac{y}{2}\right),(-x,-y)\right)+\widetilde{\phi}((x,y),(-x,-y),(0,0))\tag{2.4}$$

for all $(x,y)\in X\times X$.

Proof. Letting $x_1=x_2=x_3=0$ and $y_1=y_2=y_3=0$ in (2.2), we get $f(0,0)=0$.

Letting $x_1=x_2=x,y_1=y_2=y$ and $x_3=-2x,y_3=-2y$ in (2.2), we get

$$\|2f(x,y)+f(-2x,-2y)\|\le\phi((x,y),(x,y),(-2x,-2y))$$

for all $(x,y)\in X\times X$.

Letting $x_1=2x,x_2=-2x,x_3=0$ and $y_1=2y,y_2=-2y,y_3=0$ in (2.2), we obtain

$$\|f(2x,2y)+f(-2x,-2y)\|\le\phi((2x,0),(-2x,-2y),(0,0))$$

for all $(x,y)\in X\times X$.

Thus

$$\begin{aligned}&\left\|f(x,y)-2f\left(\frac{1}{2}x,\frac{1}{2}y\right)\right\|\\&\le\left[\phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),(-x,-y)\right)+\phi((x,y),(-x,-y),(0,0))\right]\end{aligned}$$

and so

$$\begin{aligned}&\left\|2^lf\left(\frac{x}{2^l},\frac{y}{2^l}\right)-2^mf\left(\frac{x}{2^m},\frac{y}{2^m}\right)\right\|\\&\le\sum_{j=l}^{m-1}2^j\left[\phi\left(\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}\right),\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}\right),\left(-\frac{x}{2^j},-\frac{y}{2^j}\right)\right)\right.\\&\left.+\phi\left(\left(\frac{x}{2^j},\frac{y}{2^j}\right),\left(-\frac{x}{2^j},-\frac{y}{2^j}\right),(0,0)\right)\right]\end{aligned}\tag{2.5}$$

for all nonnegative integers m and l with $m>l$ and all $(x,y)\in X\times X$. It follows from (2.3) and (2.5) that the sequence $\{2^kf(\frac{x}{2^k},\frac{y}{2^k})\}$ is a Cauchy sequence for all $(x,y)\in X\times X$. Since Y is

complete, the sequence $\{2^k f(\frac{x}{2^k}, \frac{y}{2^k})\}$ converges. So we can define the mapping $A : X \times X \to Y$ by

$$A(x,y) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, \frac{y}{2^k}\right)$$

for all $(x,y) \in X \times X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.4).

Now, we show that $A(x,y)$ is a bi-Cauchy additive mapping.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x,y) + A(-x,-y)\| \\
&= \lim_{k\to\infty} 2^k \left\| f\left(\frac{x}{2^k}, \frac{y}{2^k}\right) + f\left(\frac{-x}{2^k}, \frac{-y}{2^k}\right) + f(0,0)\right\| \\
&\le \lim_{k\to\infty} 2^k \left[\left\| f\left(\frac{x}{2^k} + \frac{-x}{2^k} + 0, \frac{y}{2^k} + \frac{-y}{2^k} + 0\right)\right\| + \phi\left(\left(\frac{x}{2^k}, \frac{y}{2^k}\right), \left(\frac{-x}{2^k}, \frac{-y}{2^k}\right), (0,0)\right)\right] \\
&= 0
\end{aligned}$$

and so $A(x,y) = -A(-x,-y)$ for any $(x,y) \in X \times X$.

$$\begin{aligned}
&\|A(x_1,y_1) + A(x_2,y_2) - A(x_1+x_2, y_1+y_2)\| \\
&= \|A(x_1,y_1) + A(x_2,y_2) + A(-x_1-x_2, -y_1-y_2)\| \\
&= \lim_{k\to\infty} 2^k \left\| f\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right) + f\left(\frac{x_2}{2^k}, \frac{y_2}{2^k}\right) + f\left(\frac{-x_1-x_2}{2^k}, \frac{-y_1-y_2}{2^k}\right)\right\| \\
&\le \lim_{k\to\infty} 2^k \left[\left\| f\left(\frac{x_1}{2^k} + \frac{x_2}{2^k} + \frac{-x_1-x_2}{2^k}, \frac{y_1}{2^k} + \frac{y_2}{2^k} + \frac{-y_1-y_2}{2^k}\right)\right\|\right. \\
&\left. + \phi\left(\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right), \left(\frac{x_2}{2^k}, \frac{y_2}{2^k}\right), \left(\frac{-x_1-x_2}{2^k}, \frac{-y_1-y_2}{2^k}\right)\right)\right] \\
&= 0
\end{aligned}$$

for all $(x_1,y_1), (x_2,y_2) \in X \times X$. Thus the mapping $A : X \times X \to$ is bi-Cauchy additive.

Next, we prove the uniqueness of A. Suppose that $T : X \times X \to Y$ is another additive mapping satisfying (2.4). We may obtain

$$\begin{aligned}
\|A(x,y) - T(x,y)\| &= \lim_{k\to\infty} 2^k \left\| A\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - T\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\
&\le \lim_{k\to\infty} 2^k \left\| A\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - f\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\
&+ \lim_{k\to\infty} 2^k \left\| T\left(\frac{x}{2^k}, \frac{y}{2^k}\right) - f\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\
&\le \lim_{k\to\infty} 2\left[\widetilde{\phi}\left(\left(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}\right), \left(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}\right), \left(-\frac{x}{2^k}, -\frac{y}{2^k}\right)\right)\right. \\
&+ \left.\widetilde{\phi}\left(\left(\frac{x}{2^k}, \frac{y}{2^k}\right), \left(-\frac{x}{2^k}, -\frac{y}{2^k}\right), (0,0)\right)\right] \\
&= 0
\end{aligned}$$

for all $(x,y) \in X \times X$. Thus we can conclude that $A(x,y) = T(x,y)$ for all $(x,y) \in X \times X$. This complete the proof. $\square$

3. Hyers-Ulam stability of the functional inequality (2)

Proposition 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \leq \left\| 2f\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right)\right\| \tag{3.1}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then the mapping $f : X \times X \to Y$ is bi-Cauchy additive.

Proof. Letting $x_1 = x_2 = x_3 = 0, y_1 = y_2 = y_3 = 0$ in (3.1), we get

$$\|3f(0, 0)\| \leq \|2f(0, 0)\|.$$

So $f(0, 0) = 0$.

Letting $x_1 = x, y_1 = y, x_2 = -x, y_2 = -y$ and $x_3 = y_3 = 0$ in (3.1), we get

$$\|f(x, y) + f(-x, -y) + f(0, 0)\| \leq \|2f(0, 0)\| = 0$$

for all $(x, y) \in X \times X$. So $f(-x, -y) = -f(x, y)$ for all $(x, y) \in X \times X$.

Letting $x_3 = -x_1 - x_2, y_3 = -y_1 - y_2$ in (3.1), we obtain

$$\begin{aligned}&\|f(x_1, y_1) + f(x_2, y_2) - f(x_1 + x_2, y_1 + y_2)\| \\ &= \|f(x_1, y_1) + f(x_2, y_2) + f(-x_1 - x_2, -y_1 - y_2)\| \leq \|2f(0, 0)\| = 0\end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Thus

$$f(x_1, y_1) + f(x_2, y_2) = f(x_1 + x_2, y_1 + y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$, as desired. □

Theorem 3.2. *Assume that a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\begin{aligned}\|f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3)\| \leq &\left\| 2f\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right)\right\| \\ &+ \phi((x_1, y_1), (x_2, y_2), (x_3, y_3))\end{aligned} \tag{3.2}$$

where $\phi : (X \times X)^3 \to [0, \infty)$ satisfies

$$\widetilde{\phi}((x_1, y_1), (x_2, y_2), (x_3, y_3)) := \sum_{j=1}^{\infty} 2^j \phi\left(\left(\frac{x_1}{2^j}, \frac{y_1}{2^j}\right), \left(\frac{x_2}{2^j}, \frac{y_2}{2^j}\right), \left(\frac{x_3}{2^j}, \frac{y_3}{2^j}\right)\right) < \infty$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then there exists a unique bi-Cauchy additive mapping $A : X \times X \to Y$ such that

$$\|A(x, y) - f(x, y)\| \leq \widetilde{\phi}\left(\left(\frac{x}{2}, \frac{y}{2}\right), \left(\frac{x}{2}, \frac{y}{2}\right), (-x, -y)\right) + \widetilde{\phi}((x, y), (-x, -y), (0, 0))$$

for all $(x, y) \in X \times X$.

Proof. Letting $x_1 = x_2 = x_3 = 0$ and $y_1 = y_2 = y_3 = 0$ in (3.2), we get $f(0, 0) = 0$.

Letting $x_1 = x_2 = x, y_1 = y_2 = y$ and $x_3 = -2x, y_3 = -2y$ in (3.2), we get

$$\|2f(x, y) + f(-2x, -2y)\| \leq \phi((x, y), (x, y), (-2x, -2y))$$

for all $(x, y) \in X \times X$.

Letting $x_1 = 2x, x_2 = -2x, x_3 = 0$ and $y_1 = 2y, y_2 = -2y, y_3 = 0$ in (3.2), we obtain

$$\|f(2x, 2y) + f(-2x, -2y)\| \leq \phi((2x, 0), (-2x, -2y), (0, 0))$$

for all $(x, y) \in X \times X$.

Thus we get

$$\begin{aligned}&\left\|f(x,y)-2f\left(\frac{1}{2}x,\frac{1}{2}y\right)\right\|\\&\le\left[\phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),(-x,-y)\right)+\phi((x,y),(-x,-y),(0,0))\right]\end{aligned}$$

for any $(x,y)\in X\times X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

4. Hyers-Ulam stability of the functional inequality (3)

Proposition 4.1. *Let $f:X\times X\to Y$ be a mapping such that*

$$\|f(x_1,y_1)+f(x_2,y_2)+2f(x_3,y_3)\|\le\left\|2f\left(\frac{x_1+x_2}{2}+x_3,\frac{y_1+y_2}{2}+y_3\right)\right\| \tag{4.1}$$

for all $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times X$. Then the mapping $f:X\times X\to Y$ is bi-Cauchy additive.

Proof. Letting $x_1=x_2=x_3=0, y_1=y_2=y_3=0$ in (4.1), we get

$$\|4f(0,0)\|\le\|2f(0,0)\|.$$

So $f(0,0)=0$.

Letting $x_1=x, x_2=-x, x_3=0$ and $y_1=y, y_2=-y, y_3=0$ in (4.1), we obtain

$$\|f(x,y)+f(-x,-y)+2f(0,0)\|\le\|2f(0,0)\|=0$$

for all $(x,y)\in X\times X$. Hence $f(-x,-y)=-f(x,y)$ for all $(x,y)\in X\times X$.

Letting $x_1=-2x, x_2=0, x_3=x$ and $y_1=-2y, y_2=0, y_3=y$ in (4.1), we get

$$\|f(-2x,-2y)+f(0,0)+2f(x,y)\|\le\|2f(0,0)\|=0$$

for all $(x,y)\in X\times X$. Hence $f(2x,2y)=2f(x,y)$ for all $(x,y)\in X\times X$.

Replacing $x_3=-\frac{x_1+x_2}{2}$ and $y_3=-\frac{y_1+y_2}{2}$ in (4.1), we have

$$\begin{aligned}&\|f(x_1,y_1)+f(x_2,y_2)-f(x_1+x_2,y_1+y_2)\|\\&=\left\|f(x_1,y_1)+f(x_2,y_2)+2f\left(-\frac{x_1+x_2}{2},-\frac{y_1+y_2}{2}\right)\right\|\\&\le\|2f(0,0)\|=0\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$. Thus

$$f(x_1,y_1)+f(x_2,y_2)=f(x_1+x_2,y_1+y_2)$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$, as desired. □

Theorem 4.2. *Assume that a mapping $f:X\times X\to Y$ satisfies the inequality*

$$\begin{aligned}\|f(x_1,y_1)+f(x_2,y_2)+2f(x_3,y_3)\le&\left\|2f\left(\frac{x_1+x_2}{2}+x_3,\frac{y_1+y_2}{2}+y_3\right)\right\|\\&+\phi((x_1,y_1),(x_2,y_2),(x_3,y_3))\end{aligned} \tag{4.2}$$

where $\phi:(X\times X)^3\to[0,\infty)$ satisfies

$$\widetilde{\phi}((x_1,y_1),(x_2,y_2),(x_3,y_3)):=\sum_{j=1}^{\infty}2^j\phi\left(\left(\frac{x_1}{2^j},\frac{y_1}{2^j}\right),\left(\frac{x_2}{2^j},\frac{y_2}{2^j}\right),\left(\frac{x_3}{2^j},\frac{y_3}{2^j}\right)\right)<\infty \tag{4.3}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then there exists a unique bi-Cauchy additive mapping $A : X \times X \to Y$ such that

$$\begin{aligned}&\|A(x,y) - f(x,y)\| \\ &\le \widetilde{\phi}\left((x,y),(0,0),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right) + \widetilde{\phi}\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right)\end{aligned}$$

for all $(x, y) \in X \times X$.

Proof. It follows from (4.3) that $\phi((0,0),(0,0),(0,0)) = 0$. Letting $x_1 = x_2 = x_3 = 0$ and $y_1 = y_2 = y_3 = 0$ in (4.2), we get $\|4f(0,0)\| \le \|2f(0,0)\| + \phi((0,0),(0,0),(0,0)) = \|2f(0,0)\|$. So $f(0,0) = 0$.

Letting $x_1 = 2x, x_2 = 0, x_3 = -x$ and $y_1 = 2y, y_2 = 0, y_3 = -y$ in (4.2), we get

$$\|f(2x,2y) + f(0,0) + 2f(-x,-y)\| \le \phi((2x,2y),(0,0),(-x,-y))$$

for all $(x, y) \in X \times X$.

Letting $x_1 = x, x_2 = x, x_3 = -x$ and $y_1 = y, y_2 = y, y_3 = -y$ in (4.2), we get

$$\|2f(x,y) + 2f(-x,-y)\| \le \phi((x,y),(x,y),(-x,-y))$$

for any $(x, y) \in X \times X$.

Thus we get

$$\|f(2x,2y) - 2f(x,y)\| \le \phi((2x,2y),(0,0),(-x,-y)) + \phi((x,y),(x,y),(-x,-y))$$

for all $(x, y) \in X \times X$. So

$$\begin{aligned}&\left\|f(x,y) - 2f\left(\frac{x}{2},\frac{y}{2}\right)\right\| \\ &\le \phi\left((x,y),(0,0),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right) + \phi\left(\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(\frac{1}{2}x,\frac{1}{2}y\right),\left(-\frac{1}{2}x,-\frac{1}{2}y\right)\right)\end{aligned}$$

for all $(x, y) \in X \times X$.

The rest of the proof is the same as in the proof of Theorem 2.2. □

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References

[1] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

[2] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[3] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431-436.

[5] F. Skof, *Proprità locali e approssimazion di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[6] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.

[7] S. Czerwik, *On the stability of the quatradic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
[8] C. Borelli and G.L. Forti, *On a general Hyers-Ulam stability result*, Internat. J. Math. Math. Sci. **18** (1995), 229–236.
[9] S. Czerwik, *The stability of the quadratic functional equation*, in: Stability of Mappings of Hyers-Ulam Type' (edited by Th.M. Rassias and J. Tabor), Hadronic Press, Palm Harbor, Florida, 1994, pp. 81–91.
[10] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type* , Hardronic Press, Palm Harbor, Florida, 2003.
[11] G.L. Forti, *Comments on the core of the direct method for provinh Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl. **295** (2004), 127–133.
[12] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[13] D.H. Hyers and Th.M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.
[14] K. Jun and H. Kim, *Ulam stability problem for quadratic mappings of Eulaer-Lagrange*, Nonlinear Anal.–TMA **61** (2005), 1093–1104.
[15] K. Jun and H. Kim, *On the generalized A-quadratic mappings associated with the variance of a discrete-type distribution*, Nonlinear Anal.–TMA **62** (2005), 975–987.
[16] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.
[17] C. Park, *Generalized quadratic mappings in several variables*, Nonlinear anal.–TMA **57** (2004), 713–722.
[18] C. Park, *Cauchy-Rassias stability of a generalized Trif's mapping associated in Banach modulules and its applications*, Nonlinear Anal.–TMA **62** (2005), 595–613.
[19] Th.M. Rassias, *On the stability of the quadratic functional and its applications*, Studia Univ. Babes-Bolyai **XLIII** (1998), 89–124.
[20] Th.M. Rassias, *The prolem of S.M.Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
[21] Th.M. Rassias, *On the stability of functional equations in Banach space*, J. Math. Anal. Appl. **251** (2000), 264–284.
[22] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
[23] Th.M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
[24] Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.
[25] P. Šemrl, *On quadratic functionals*, Bull. Austral. Math. Soc. **37** (1987), 27–28.
[26] J. Tabor, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math. **83** (2004), 243–255.
[27] C. Park, K. Jun and G. Lu, *On the quadratic mapping in generalized quasi-Banach spaces*, J. Chungcheong Math. Soc. **19** (2006), 263–274.
[28] C. Park, S. Hong and M. Kim, *Jensen type quadratic-quadratic mapping in Banach space*, Bull. Korean Math. Soc. **43** (2006), 703–709.

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Robust CVaR-based portfolio optimization under a genal affine data perturbation uncertainty set *

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Abstract: Under a genal affine data perturbation uncertainty set, we propose a computationally tractable robust optimization method for minimizing the CVaR of a portfolio. Using L_1 norm, the robust counterpart problem can be a linear programming problem. Moreover, it is less conservative than the Quaranta and Zaffaroni's method which is under box uncertainty set. We present some numerical experiments with real market data to illustrate the behavior of robust optimization model.

Keywords: Conditional value at risk(CVaR), robust optimization, line programming(LP), second-order cone programming(SOCP).

1. Introduction

Portfolio selection optimization is one of the best known and most widely used methods in financial optimization. The mean-variance (MV) portfolio selection model, proposed by Markowitz [1], provides a fundamental basis for portfolio selection in both theoretical and practical applications.

Since the middle of 1990s, Value-at-Risk (VaR, [2]), a new measure of downside risk, has become popular in financial risk management. It has even been recommended as a standard on banking supervision by the Basel Committee. However, VaR has been criticized for its theoretical deficiency[3]. Conditional value at risk(CVaR), defined as the mean of the tail distribution exceeding VaR, has attracted much attention in recent years. CVaR is known to have nice properties such as the coherence [3, 4] and the consistency with the second-order stochastic dominance [5]. Also, Rockafellar and Uryasev [6, 7] show that the minimization of CVaR results in a tractable optimization problem.

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However, as pointed out by Black and Litterman [8], in the classical mean-variance model, the portfolio decision is very sensitive to the mean and the covariance matrix, especially to the mean. Chopra and Ziemba [9] showed that small changes in the input parameters can result in large changes in the optimal portfolio allocation. Thus, the modeling risk arises due to the uncertainty of the underlying probability distribution. Being aware of the importance of robustness, researchers have paid increasing attention to the robust version of portfolio selection problems, for example, (Lobo and Boyd [14], Goldfarb and Iyengar [15], El Ghaoui, Oks and Oustry [16], Zhu and Fukushima [17]).

By utilizing the Soyster's approach [18], under the assumption that the expected returns lie in a box uncertainty set, Quaranta and Zaffaroni [10] considered robust optimization of conditional value at risk and portfolio selection problem. Although they succeeded in obtaining a linear robust copy of the bicriteria minimization model proposed by Rockafellar and Uryasev, the associated consequences are that the resulting robust portfolios can be too conservative. Under the assumption that the expected returns lie in an ellipsoidal uncertainty set, An and Luo (2010) [11] considered robust optimization of conditional value at risk and portfolio selection problem. They showed that the robust optimization problem can be reformulated as a second order cone programming (SOCP), however, a practical drawback of such an approach, is that it leads to nonlinear, although convex, models, which are more demanding computationally than the earlier linear models by Quaranta and Zaffaroni [10].

In robust portfolio selection problems, one try to find portfolios with the worst-case return under a given uncertainty set, in which asset returns can be realized. A too large uncertainty set will lead to a too conservative robust portfolio. However, if the given uncertainty set is not large enough, the realized returns of resulting portfolios will be outside of the uncertainty set when an extreme event such as market crash or a large shock of asset returns occurs. Motivated by the works in [20], under an affine data perturbation uncertainty set, we provide a computationally tractable robust optimization method for minimizing the CVaR of a portfolio which is less conservative than box uncertainty set. specifically the robust optimization problem retains its original structure,i.e., the robust counterpart problem is still a linear programming problem.

The rest of this paper is organized as follows: In the next section, we introduce the concept of CVaR and the mean-CVaR portfolio optimization model. In Section 3, we review the main ideas behind the robust optimization methodology, and present the computationally tractable robust optimization method for minimizing the CVaR of a portfolio. In Section 4, we report some numerical results to test the proposed methods.

2. Conditional value-at-risk measure

Conditional VaR (CVaR) is a popular example of such a coherent risk measure and is discussed in Rockafellar and Uryasev [6, 7]. The CVaR measure can be

written as

$$CVaR_\beta(\boldsymbol{x}) = (1-\beta)^{-1} \int_{f(\boldsymbol{x},\boldsymbol{y}) \geq VaR_\beta(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y}) p(\boldsymbol{y}) d\boldsymbol{y} \tag{2.1}$$

where

$$VaR_\beta(\boldsymbol{x}) = \min\{\alpha \in R : \Psi(\boldsymbol{x},\alpha) \geq \beta\} \tag{2.2}$$

and

$$\Psi(\boldsymbol{x},\alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y}) \leq \alpha} p(\boldsymbol{y}) d\boldsymbol{y} \tag{2.3}$$

is the probability of $f(\boldsymbol{x},\boldsymbol{y})$ not exceeding a threshold α.

In practice, the probability density function $p(\boldsymbol{y})$ is often not available, or is very difficult to estimate. Instead, we might have T different scenarios $Y = (\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_T)$ that are sampled from the probability distribution or that have been obtained from computer simulations. Evaluating the auxiliary function $\tilde{F}_\beta(\boldsymbol{x},\alpha)$ using the scenarios Y, we have

$$\tilde{F}_\beta(\boldsymbol{x},\alpha) = \alpha + (1-\beta)^{-1} \sum_{j=1}^{T} \pi_j [f(\boldsymbol{x}, \boldsymbol{y}_{[j]}) - \alpha]^+ \tag{2.4}$$

where $\boldsymbol{y}_{[j]}$ denotes the jth sample (the subscript $[j]$ is used to distinguish a vector from a scalar) generated by simple random sampling with respect to $\mathbf{y}$ according to its density function $p(.)$, and T denotes the number of samples. where π_j are probabilities of scenarios $\boldsymbol{y}_{[j]}$. If π_j is equal to T^{-1} for all j, then (2.4) reduces to

$$\tilde{F}_\alpha(\boldsymbol{x},\alpha) = \alpha + \frac{1}{T(1-\beta)} \sum_{j=1}^{T} [f(\boldsymbol{x}, \boldsymbol{y}_{[j]}) - \alpha]^+ \tag{2.5}$$

Obviously, $\tilde{F}_\alpha(\boldsymbol{x},\alpha)$ is convex and piecewise linear with respect to α. Further, $\tilde{F}_\alpha(\boldsymbol{x},\alpha)$ is convex for $(\mathbf{x},\alpha)$, if $f(\boldsymbol{x},\boldsymbol{y})$ is convex (see Theorem 2 in [6]).

Replacing $[f(\boldsymbol{x}, \boldsymbol{y}_{[j]}) - \alpha]^+$ by the auxiliary variables z_j along with appropriate constraints, we obtain the equivalent optimization problem

$$\begin{array}{lll} \min\limits_{\boldsymbol{x},\alpha} & \alpha + \dfrac{1}{T(1-\beta)} \displaystyle\sum_{i=1}^{T} z_j & \\ s.t. & z_j \geq f(\boldsymbol{x}, \boldsymbol{y}_{[j]}) - \alpha, & j = 1, \ldots, T \\ & z_j \geq 0 & \end{array} \tag{2.6}$$

Generally, the loss and return functions of portfolio allocation are chosen by:

$$f(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{y}, \quad R_p(\boldsymbol{x}) = E_p[\boldsymbol{x}^T\boldsymbol{y}] = \boldsymbol{x}^T E_p[\boldsymbol{y}] = \boldsymbol{x}^T\mathbf{r} \tag{2.7}$$

in which $\mathbf{y}$ is the vector of the assets' return, $\boldsymbol{x}^T\boldsymbol{r}$ is the mean return of the portfolio.

2.1. portfolio optimization with CVaR measure

Portfolio optimization tries to find an optimal trade-off between the risk and the return according to the investor's preference. Thus, the portfolio selection problem using CVaR as a risk measure can be represented as

$$\min_{\boldsymbol{x}\in\mathcal{X}} \quad CVaR_\beta(\boldsymbol{x})$$

where $\mathcal{X}$ denotes the constraint on the portfolio position, which usually includes the budget constraint and no short sales constraint

$$\boldsymbol{x}^T\mathbf{1} = 1, \qquad \mathbf{x} \geq 0. \tag{2.8}$$

Let μ be the worst-case minimum mean return required by the investor. From (2.7), this can be represented as

$$\min \quad \boldsymbol{x}^T\boldsymbol{r} \geq \mu \tag{2.9}$$

Hence, adding an auxiliary variable $\theta \in R$, the mean-CVaR Portfolio optimization can be be written as the following linear program

$$\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \alpha + \frac{1}{T(1-\beta)}\sum_{j=1}^{T} z_j \leq \theta && (2.10)\\
& z_j \geq -\boldsymbol{x}^T\boldsymbol{y}_{[j]} - \alpha, \qquad j = 1,\dots,T && (2.11)\\
& z_j \geq 0 && (2.12)\\
& \boldsymbol{x}^T\boldsymbol{r} \geq \mu && (2.13)\\
& \boldsymbol{x}^T\mathbf{1} = 1, \qquad \mathbf{x} \geq 0. && (2.14)
\end{aligned}$$

The expected return $\boldsymbol{r}$ in constraint (2.9) is assumed to be exactly known. In fact, the expected assets' return $\boldsymbol{r}$ is uncertain. On the other hand, it is difficult to estimate the mean return vector, and the solution to the problem is sensitive to the mean return vector. One way to address this issue is to consider a robust version of the portfolio problem. More specifically, we propose the following robust version of the constraint on the expected portfolio return

$$\min \quad \boldsymbol{x}^T\boldsymbol{r} \geq \mu_w. \tag{2.15}$$

where μ_w denotes the worst-case required expected return specified by the investor. In next section, we will investigate a tractable robust formulation of the constraint on the expected return which belongs to a genal affine data perturbation uncertainty set.

3. Robust portfolio optimization

Robust optimization is emerging as a leading methodology to address optimization problems under uncertainty. In this section, we will discuss different robust optimization methods which can be applied to deal with the uncertain minimal constraint (2.15).

3.1. Genal Affine Data Perturbation uncertainty

In this paper, We consider instead the following uncertainty set, suggested by Chen et al. (2007) [20] in the context of stochastic programming applications:

$$V_\Omega = \Big\{ \boldsymbol{r} : \boldsymbol{r} = \boldsymbol{r}_0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}_j z_j, \boldsymbol{z} \in A_\Omega(\boldsymbol{z}) \Big\}, \tag{3.1}$$

where

$$A_\Omega(\boldsymbol{z}) = \Big\{ \boldsymbol{z} : \exists \boldsymbol{v}, \boldsymbol{w} \in R_+^N, \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \|\mathbf{P}^{-1}\boldsymbol{v} + \mathbf{Q}^{-1}\boldsymbol{w}\| \leq \Omega \Big\}, \tag{3.2}$$

and $\mathbf{P} = \text{diag}(p_1, \dots, p_N)$, $\mathbf{Q} = \text{diag}(q_1, \dots, q_N)$. The parameters $p_j > 0$ and $q_j > 0$ are the "forward"and the "backward" deviations of random variable $z_j, j = 1, \dots, N$, respectively.

For the stochastic linear constraint (2.15), the **worst-case** convex support of the uncertain parameter can be specified as follows,

$$W = \Big\{ \boldsymbol{r} : \exists z \in R^N, \boldsymbol{r} = \boldsymbol{r}^0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}^j \tilde{z}_j, -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \overline{\boldsymbol{z}} \Big\} \tag{3.3}$$

Therefore, under affine data perturbation, the worse-case uncertainty set is a parallelotope in which the feasible solution is characterized by Soyster [18], which, of course, is a very conservative.

To derive a less conservative approximation, we need to choose the budget of uncertainty, Ω, appropriately. The natural uncertainty set to consider is the intersection of a norm uncertainty set, V_Ω and the worst-case support set, W as follows.

$$S_\Omega = \Big\{ \boldsymbol{r} : \exists z \in R^N, \boldsymbol{r} = \boldsymbol{r}^0 + \sum_{j=1}^{N} \Delta \boldsymbol{r}^j \tilde{z}_j, \boldsymbol{z} \in A_\Omega(\boldsymbol{z}), -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \overline{\boldsymbol{z}} \Big\} \tag{3.4}$$

As the budget of uncertainty Ω increases, the norm uncertainty set, V_Ω expands radially from the point $\boldsymbol{r}^0$ until it engulfs the set W. In this case, the uncertainty set $S_\Omega = W$. Hence, for any choice of Ω, the uncertainty set S_Ω is always less conservative than the worst-case uncertainty set W. We call the uncertainty S_Ω as genal affine data perturbation uncertainty.

We will show an equivalent formulation of the corresponding robust counterpart of (2.15) under the generalized uncertainty set, S_Ω. The dual norm $\|u\|^*$ is defined as

$$\|u\|^* = \max_{\{\|\boldsymbol{x}\| \leq 1\}} \boldsymbol{u}'\boldsymbol{x} \tag{3.5}$$

Theorem 3.1 *The robust counterpart of (2.15) in which $U_\Omega = S_\Omega$ is equivalent to*

$$\begin{cases} \exists \boldsymbol{u}, \lambda, \boldsymbol{s} \in R^N, h \in R \\ -\boldsymbol{r}_0'\boldsymbol{x} + \Omega h + \lambda'\overline{z} + s'\underline{z} \leq -\mu \\ \|u\|^* \leq h, \\ u_j \geq -p_j(\Delta r_j'\boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\ u_j \geq q_j(\Delta r_j'\boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\ \boldsymbol{u}, \lambda, \boldsymbol{s} \geq 0. \end{cases} \tag{3.6}$$

Proof: From (2.15), we have

$$max \quad -\mathbf{x}^T\mathbf{r} \leq -\mu \tag{3.7}$$

Under the condition $U_\Omega = S_\Omega$, the robust counterpart of (3.7) is as follows,

$$-\boldsymbol{r}_0^T\boldsymbol{x} + \underset{\{z \in C\}}{max}\ z'\boldsymbol{y} \leq -\mu \tag{3.8}$$

where

$$C = \left\{(\boldsymbol{v}, \boldsymbol{w}) : \|\mathbf{P}^{-1}\boldsymbol{v} + \mathbf{Q}^{-1}\boldsymbol{w}\| \leq \Omega, -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \overline{\boldsymbol{z}}, \boldsymbol{v}, \boldsymbol{w} \geq 0\right\}$$

and $y_j = -\Delta r^{j'}\boldsymbol{x}$. Since C is a compact convex set with nonempty interior, we can use strong duality to obtain the equivalent representation. Observe that

$$\begin{aligned} & \max_{\{(\boldsymbol{v},\boldsymbol{w}):\|\mathbf{P}^{-1}\boldsymbol{v}+\mathbf{Q}^{-1}\boldsymbol{w}\|\leq\Omega,-\underline{\boldsymbol{z}}\leq\boldsymbol{z}\leq\overline{\boldsymbol{z}},\boldsymbol{v},\boldsymbol{w}\geq 0\}} (\boldsymbol{v}-\boldsymbol{w})'\boldsymbol{y} \\ = & \min_{\mathbf{r},\ \mathbf{s}\geq 0} \left\{ \underset{\{(v,w):\|\mathbf{P}^{-1}v+\mathbf{Q}^{-1}w\|\leq\Omega,v,w\geq 0\}}{max} (v-w)'y + \mathbf{r}'(\overline{z} - v + w) + \mathbf{s}'(\underline{z} + v - w) \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\geq 0} \left\{ \underset{\{(v,w):\|\mathbf{P}^{-1}v+\mathbf{Q}^{-1}w\|\leq\Omega,v,w\geq 0\}}{max} (y - r + s)'v - (y - r + s)'w + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\geq 0} \left\{ \underset{\{(v,w):\|v+w\|\leq\Omega,v,w\geq 0\}}{max} \mathbf{P}(y - r + s)'v - \mathbf{Q}(y - r + s)'w + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \right\} \\ = & \min_{\mathbf{r},\ \mathbf{s}\geq 0} \Omega\|\mathbf{u}\|^* + \mathbf{r}'\overline{\boldsymbol{z}} + \mathbf{s}'\underline{\boldsymbol{z}} \end{aligned}$$

where

$$\begin{aligned} u_j &= max\{p_j(y_j - r_j + s_j), -q_j(y_j - r_j + s_j)\} \\ &= max\{-p_j(\Delta r^{j'}\boldsymbol{x} + r_j - s_j), q_j(\Delta r^{j'}\boldsymbol{x} + r_j - s_j)\}. \end{aligned}$$

Hence the robust counterpart is the same as

$$-\boldsymbol{r}_0^T\boldsymbol{x} + \Omega\|\mathbf{u}\|^* + \mathbf{r}'\overline{z} + \mathbf{s}'\underline{z} \leq -\mu \tag{3.9}$$

Adding an auxiliary variable $h \in R$, we can easily obtain the equivalent formulation of (3.9), that is (3.6).

The complete formulation and complexity class of the robust counterpart depends on the representation of the dual norm constraint, $\|u\|^* \leq h$. In this paper, we select the l_1 norm. So the $\|u\|^* \leq h$ is equivalent to

$$u_j \leq h, \forall j \in N. \tag{3.10}$$

By (2.10)-(2.14) and (3.10), the robust portfolio selection problem can be written as the following linear programming problem with variables $(\mathbf{x}, \mathbf{z}, \mathbf{u}, \lambda, \mathbf{s}, \mathbf{v}, \theta, \alpha, h) \in R^n \times R^T \times R^N \times R^N \times R^N \times R^N \times R \times R \times R$:

$$\begin{array}{ll}
\min & \theta \\
s.t. & \alpha + \dfrac{1}{T(1-\beta)} \displaystyle\sum_{i=1}^{J} z_j \leq \theta \\
& z_j \geq -\mathbf{x}^T \mathbf{y}_{[j]} - \alpha, \qquad j = 1, \ldots, T \\
& \mathbf{x}^T \mathbf{1} = 1, \qquad \mathbf{x} \geq 0 \\
& -\boldsymbol{r}_0' \boldsymbol{x} + \Omega h + \lambda' \overline{z} + s' \underline{z} \leq -\mu \\
& u_j \leq h, \forall j \in N \\
& u_j \geq -p_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\
& u_j \geq q_j(\Delta r_j' \boldsymbol{x} + \lambda_j - s_j), \forall j = \{1, \ldots, N\} \\
& \mathbf{z}, \mathbf{u}, \lambda, \mathbf{s} \geq 0, v \in R_+^N, p \in R_+
\end{array} \tag{3.11}$$

4. Empirical Results

In this section, we apply the robust portfolio optimization methods discussed in the previous sections to real market data and compare the behavior of the solutions obtained by the robust optimization technique.

In all tables and figure, the methods have the following meanings:

- "CVaR" stands for the initial CVaR method in [7].
- "BCVAR" stands for the robust mean-CVaR Portfolio optimization under box uncertainty set in [10].
- "ECVaR" stands for the robust mean-CVaR Portfolio optimization under ellipsoidal uncertainty set in [11].
- "ACVaR" stands for the robust mean-CVaR Portfolio optimization (3.11) under a genal affine data perturbation uncertainty set.

We utilize MatLab7.0 for solving models CVaR, BCVAR, and ACVaR which are linear programming problems. The model ECVaR is an SOCP and solved by SeDuMi1.02 [21].

We consider a portfolio of 10 small cap stocks from 5 different industry categories of the S&P 600 index(Table 2), and use historical returns from May,

1998 to June, 2006. There are a total of 2,000 observations for each stock.

Table 2: List of Stocks and Corresponding Industries

Industry discretionary	Company name (ticker)
Consumer discretionary	Aztar Corp. (AZR), Hancock Fabrics Inc. (HKF)
Financials	Downey S & L Assn. (DSL), HARB
Industrials	AAR Corp. (AIR), CDI Corp. (CDI)
Information technology	FEI Company (FEIC), Exar Corp. (EXAR)
Healthcare	BioLase Technology (BLTI), BDR

In our first experiment, using the data presented above, we generated the classical and robust efficient frontiers. The parameters for all optimization models are set as follows:

- For the CVaR formulation, mean return r is given by the sample mean.
- For the BCVaR formulation, we assume that mean return r^0 is given by the sample mean, and that $\overline{r}_i$ is determined by the standard deviation of the stock i' sample return.
- According to the ECVaR formulation,we assume that mean return r^0 is given by the sample mean. For simplicity, the scaling matrix of the ellipsoid P is assumed to be a diagonal matrix ρI, where ρ is a nonnegative parameter.
- For the ACVaR formulation, we assume that mean return r^0 is given by the sample mean, and that Δr_i is determined by the standard deviation of the stock i' sample return, and assume they are also stochastically independent (N=10). We set $\Omega = 0.8$, $\overline{z} = \underline{z} = 1$ and $p_j = 1.5, q_j = 2$ in our Numerical experiments.

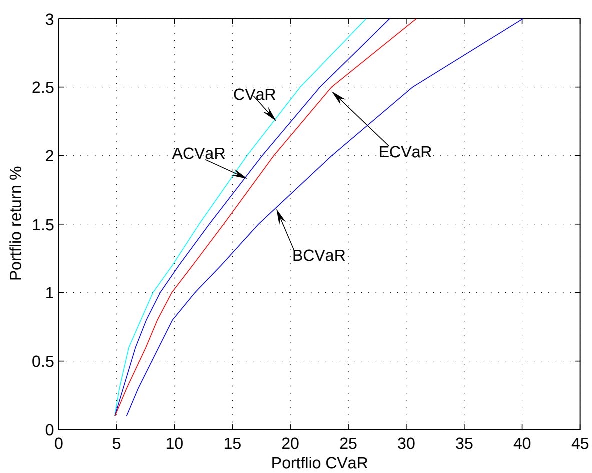

Figure 1-portfolio efficient frontiers for the different optimization formulations with $\beta = 1\%$.

As shown in Figure 1, it is apparent that ACVaR outperforms both ECVaR and BCVaR in terms of realized CVaR. As expected, CVaR is dominated by ACVaR. We also can see that the robust optimal portfolios are somewhat conservative in comparison to that of the CVaR model. But in the next experiment, we will see the robust optimal portfolios can result in more stable portfolio returns.

In our second experiment, we study the cumulative portfolio wealth if a portfolio manager employs a simple buy-and-hold strategy. The entire data sequence is divided into investment periods of length $T = 200$ days. In all there are $p = 10$ time periods. For each period p, first, we consider moving windows of $n = 10$ days and compute the parameters for all optimization models as experiment 1.

Once all the parameters are set, the portfolio $x^p_{CVaR}, x^p_{BCVaR}, x^p_{ECVaR}, x^p_{ACVaR}$ for period p can be computed by solving the portfolio selection model C-VaR, BCVaR, ECVaR, and ACVaR respectively. The portfolio $x^p_{CVaR}, x^p_{BCVaR}, x^p_{ECVaR}, x^p_{ACVaR}$ are held constant for the period p and then rebalanced to the portfolio $x^{p+1}_{CVaR}, x^{p+1}_{BCVaR}, x^{p+1}_{ECVaR}, x^{p+1}_{ACVaR}$ for period $p+1$.

Let $W^p_{CVaR}, W^p_{BCVaR}, W^p_{ECVaR}, W^p_{ACVaR}$ denote the wealth at the end of period p of an investor with initial wealth $w_0 = 1$. Because these strategies require a block of data of length $T = 200$ to estimate all of parameters, the first investment period $p = 1$ starts from the time instant $T + 1$. Therefore, 10 time periods of length "$T = 200$" only have 9 investment periods.

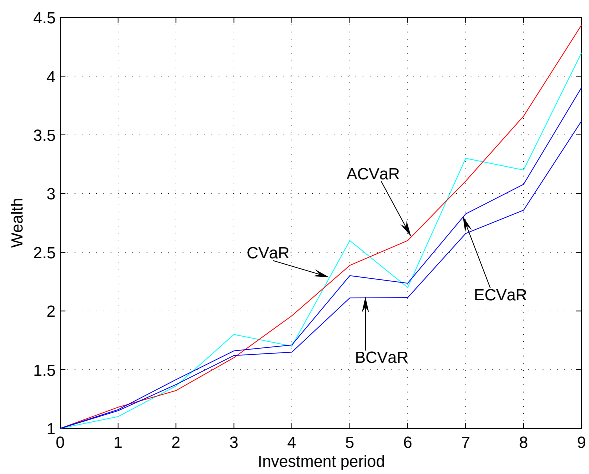

Figure 2-The wealth resulting from the four strategies with window n = 10 at each investment period.

It is clear that the wealth generated by the ACVaR model is much better than other models at the end of investment period. But in Figure 2 the wealth generated by the ACVaR model is a litter lower than other models at early investment periods. Therefore, it is not guaranteed that the ACVaR model always has an advantage of other three models. On the other hand, the optimal

portfolio allocation based on the ACVaR approach tends to result in stable returns, whereas, for example, the behavior of the optimal portfolio obtained with the CVaR approach is erratic.

5. Conclusion

under a genal affine data perturbation uncertainty set, we propose a computationally tractable robust optimization method for minimizing the CVaR of a portfolio. The remarkable characteristic of the new method is that, using L_1 norm, the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. This fact has important theoretical and practical implications. Since the computational complexity of an LP is simplest in all of program problems, it follows that robust portfolio optimization is able to provide protection against parameter fluctuations at light computational cost. Moreover, the LP problem is maybe the best known and the most frequently solved optimization problem in the real world. The numerical experiments presented in this paper suggest that the behavior of portfolios can be improved by using the robust CVaR model under a genal affine data perturbation uncertainty set. And the robustness is achieved at relatively high performance and low cost.

References

[1] H.M. Markowitz, Portfolio selection. J. Finance. 1952, 7: 77-91.

[2] T.J. Linsmeier, N. D. Pearson, Risk Measurement: An introduction to value-at-risk. Technical report 96-04, OFOR, University of Illinois, Urbana-Champaign, IL, 1996.

[3] P. Artzner, F. Delbaen, J. M. Eber, D. Heath, Coherence measures of risk. Math. Finance. 1999, 9: 203-228.

[4] Dowd, K.: Measuring Market Risk. JohnWiley and Sons, NewYork 2002.

[5] Ogryczak W, Ruszczýnski A. Dual stochastic dominance and related meanCrisk models. SIAM Journal on Optimization. 2002, 13: 60-78.

[6] R.T. Rockafellar, S. Uryasev, Optimization of conditional Valueat-Risk. J. Risk. 2000, 2: 21-41.

[7] R.T. Rockafellar, S. Uryasev, Conditional Value-at-Risk for general loss distributions. J. Banking and Finance. 2002, 26: 1443-1471.

[8] F. Black, R. Litterman, Global portfolio optimization. Financial Analysts J. 1992, 48: 28-43.

[9] V.K. Chopra, W.T. Ziemba, The effect of errors in means, variance and covariances on optimal portfolio choice. J. Portfolio Manag. winter, 1993, 6-11.

[10] A.G. Quaranta, A. Zaffaroni: Robust optimization of conditional value at risk and portfolio selection. J. Banking and Finance. 2008, 32: 2046-2056.

[11] X.M. An and G.M. Luo, Robust Portfolio Selection under Ellipsoidal Uncertainty, Journal of Hunan University(Natural Sciences), 2010, 37: 89-92.

[12] F.H. Wen, X.G. Yang, Skewness of Return Distribution and Coefficient of Risk Premium, Journal of Systems Science and Complexity. 2009, 22: 360-371.

[13] F.H. Wen, Z.F. Liu, A Copula-based Correlation Measure and Its Application in Chinese Stock Market, International Journal of Information Technology & Decision Making. 2009, 8: 1-15.

[14] M.S. Lobo and S. Boyd. The worst-case risk of a portfolio, Technical Report, http://faculty.fuqua.duke.edu/∼ mlobo/bio/researchfiles/rsk-bnd.pdf, 2000.

[15] D. Goldfarb, G. Iyengar, Robust portfolio selection problems. Mathematics of Operations Research. 2003, 28: 1-38.

[16] El Ghaoui, L., Oks, M., Oustry, F.: Worst-Case Value-at-Risk and Robust Portfolio Optimization: A Conic Programming Approach. Operations Research, 2003, 51: 543-556

[17] S.S. Zhu, M. Fukushima, Worst-Case Conditional Value-at-Risk with application to robust portfolio management. Operations Research. 2009, 57: 1155-1168.

[18] A.L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming. Operations Research. 1973, 21: 1154-1157.

[19] Z.F. Dai, D.H. Li, F.H. Wen. Robust Conditional value-at-risk optimization for Asymmetrically Distributed Asset Returns. Pacific Journal of Optimization, 2012, 8(3): 429-445.

[20] X. Chen, M. Sim and P. Sun, A robust optimization perspective of stochastic programming, Operations Research, 2007, 55: 1058-1077.

[21] J. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optim. Methods and Software. 1999, 11: 625-653.

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RANDOM DERIVATIONS ON RANDOM NORMED ALGEBRAS

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of random derivations in random normed algebras associated with the Cauchy functional equation.

1. Introduction

Fuzzy set theory is a powerful tool set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics [4], chaos control [13], computer programming [15], etc. Recently, the fuzzy topology has proved to be a very useful tool to deal with such situations where the use of classical theories breaks down.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [7, 25, 26, 31, 32]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([31]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1, 0)$ (the Lukasiewicz t-norm). Recall (see [16, 17]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \ge 2$. $T_{i=n}^\infty x_i$ is defined as

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$T_{i=1}^{\infty}x_{n+i-1}$. It is known ([17]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}x_{n+i-1}=1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)<\infty.$$

Definition 1.2. ([32]) A *random normed space* (briefly, RN-space) is a triple (X,μ,T_M), where X is a vector space and μ is a mapping from X into D^+ such that the following conditions hold:
(RN_1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;
(RN_2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for all $x\in X$, $\alpha\neq 0$;
(RN_3) $\mu_{x+y}(t+s)\geq T_M(\mu_x(t),\mu_y(s))$ for all $x,y\in X$ and all $t,s>0$.

Every normed space $(X,\|\cdot\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t)=\frac{t}{t+\|x\|}$$

for all $t>0$. This space is called the induced random normed space.

Definition 1.3. Let (X,μ,T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon)>1-\lambda$ whenever $n\geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon>0$ and $\lambda>0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\epsilon)>1-\lambda$ whenever $n\geq m\geq N$.
(3) An RN-space (X,μ,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([31]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n\to x$, then $\lim_{n\to\infty}\mu_{x_n}(t)=\mu_x(t)$ almost everywhere.*

Definition 1.5. A *random normed algebra* is a random normed space with algebraic structure such that (RN_4) $\mu_{xy}(ts)\geq\mu_x(t)\mu_y(s)$ for all $x,y\in X$ and all $t,s>0$.

Example 1.6. *Every normed algebra $(X,\|.\|)$ defines a random normed algebra (X,μ,T_M), where*

$$\mu_x(t)=\frac{t}{t+\|x\|}$$

for all $t>0$. This space is called the induced random normed algebra.

Definition 1.7. Let (X,μ,T_M) be a random normed algebra. An $\mathbb{R}$-linear mapping $f:X\to X$ is called a *random derivation* if $f(xy)=f(x)y+xf(y)$ for all $x,y\in X$.

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in X$.
We recall a fundamental result in fixed point theory.

Theorem 1.8. ([6, 9]) *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [30] has provided a lot of influence in the development of what we call *Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 3, 5, 8, 10, 11, 19, 21, 22, 23, 29]).

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [25, 27, 28]).

The Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [24, 26].

Using the fixed point method, we prove the Hyers-Ulam stability of random derivations in random normed algebras, associated with the Cauchy functional equation

$$f(x+y) = f(x) + f(y).$$

Throughout this paper, assume that (X, μ, T_M) is a complete random normed algebra.

2. Hyers-Ulam stability of random derivations in random normed algebras

Using the fixed point method, we prove the Hyers-Ulam stability of random derivations associated with the Cauchy functional equation.

Theorem 2.1. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists a constant* $0 < L < \frac{1}{2}$ *with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. *Let* $f : X \to X$ *be a mapping satisfying*

$$\mu_{f(rx+ry)-rf(x)-rf(y)}(t) \geq \frac{t}{t + \varphi(x, y)}, \tag{2.1}$$

$$\mu_{f(xy)-f(x)y-xf(y)}(t) \geq \frac{t}{t + \varphi(x, y)} \tag{2.2}$$

for all $r \in \mathbb{R}$, *all* $x, y \in X$ *and all* $t > 0$. *Then*

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for each $x \in X$ and defines a random derivation $D : X \to X$ such that

$$\mu_{f(x)-D(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ and $r = 1$ in (2.1), we get

$$\mu_{f(2x)-2f(x)}(t) \geq \frac{t}{t + \varphi(x,x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}(t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{2t}{2t + L\varphi(x,x)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \to X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \geq \frac{t}{t + \varphi(x,x)}, \ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}\left(\frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.8, there exists a mapping $D : X \to X$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{2}\right) = \frac{1}{2}D(x) \tag{2.6}$$

for all $x \in X$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.6) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu_{f(x)-D(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, D) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = D(x)$$

for all $x \in X$;

(3) $d(f,D) \leq \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,D) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.1),

$$\mu_{2^n f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)}(2^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{D(x+y)-D(x)-D(y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $D : X \to X$ is Cauchy additive.

Let $y = 0$ in (2.1). By (2.1),

$$\mu_{2^n f\left(\frac{rx}{2^n}\right)-2^n r f\left(\frac{x}{2^n}\right)}(2^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},0\right)}$$

for all $r \in \mathbb{R}$, all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n f\left(\frac{rx}{2^n}\right)-2^n r f\left(\frac{x}{2^n}\right)}(t) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,0)}$$

for all $r \in \mathbb{R}$, all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,0)} = 1$ for all $x \in X$ and all $t > 0$,

$$\mu_{H(rx)-rH(x)}(t) = 1$$

for all $r \in \mathbb{R}$, all $x \in X$ and all $t > 0$. Thus the additive mapping $D : X \to X$ is $\mathbb{R}$-linear.

By (2.2),

$$\mu_{4^n f\left(\frac{x}{2^n}\cdot\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)\cdot y - x\cdot 2^n f\left(\frac{y}{2^n}\right)}(4^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{4^n f\left(\frac{x}{2^n}\cdot\frac{y}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)\cdot y - x\cdot 2^n f\left(\frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{D(xy)-D(x)y-xD(y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $D : X \to X$ satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in X$.

Therefore, there exists a unique random derivation $D : X \to X$ satisfying (2.3). $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $0 < L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to X$ be a mapping satisfying (2.1) and (2.2). Then

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

exists for each $x \in X$ and defines a random derivation $D : X \to X$ such that

$$\mu_{f(x)-D(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$\mu_{f(x)-\frac{1}{2}f(2x)}\left(\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] E. Baktash, Y. Cho, M. Jalili, R. Saadati and S.M. Vaezpour, *On the stability of cubic mappings and quartic mappings in random normed spaces*, J. Inequal. Appl. **2008**, Art. ID 902187 (2008).
[3] M. Bavand Savadkouhi, M. Eshaghi Gordji, J.M. Rassias and N. Ghobadipour, *Approximate ternary Jordan derivations on Banach ternary algebras*, J. Math. Phys. **50** (2009), Article ID 042303, 9 pages.
[4] L.C. Barros, R.C. Bassanezi and P.A. Tonelli, *Fuzzy modelling in population dynamics*, Ecol. Model. **128** (2000), 27–33.
[5] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.
[6] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, Art. ID 4 (2003).
[7] S.S. Chang, Y. Cho and S. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers Inc. New York, 2001.
[8] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
[9] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[10] A. Ebadian, N. Ghobadipour and M. Eshaghi Gordji, *A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras*, J. Math. Phys. **51** (2010), Article ID 103508, 10 pages.
[11] M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-Archimdean spaces*, Appl. Math. Lett. **23** (2010), 1198–1202.
[12] M. Eshaghi Gordji and N. Ghobadipour, *Stability of (α,β,γ)-derivations on Lie C^*-algebras*, International Journal of Geometric Methods in Modern Physics **7** (2010), 1093–1102.
[13] A.L. Fradkov and R.J. Evans, *Control of chaos: Methods and applications in engineering*, Chaos, Solitons and Fractals **29** (2005), 33–56.
[14] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[15] R. Giles, *A computer program for fuzzy reasoning*, Fuzzy Sets and Systems **4** (1980), 221–234.
[16] O. Hadžić and E. Pap, *Fixed Point Theory in PM Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
[17] O. Hadžić, E. Pap and M. Budincević, *Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces*, Kybernetica **38** (2002), 363–381.
[18] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[19] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[20] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[21] S. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17** (2004), 1135–1140.

[22] S. Jung, *Hyers-Ulam stability of linear differential equations of first order II*, Appl. Math. Lett. **19** (2006), 854–858.

[23] S. Jung, *Hyers-Ulam stability of linear partial differential equations of first order*, Appl. Math. Lett. **22** (2009), 70–74.

[24] D. Miheţ, *The probabilistic stability for a functional equation in a single variable*, Acta Math. Hungar. **123** (2009), 249–256.

[25] D. Miheţ, *The fixed point method for fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **160** (2009), 1663–1667.

[26] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[27] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.

[28] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[29] J.M. Rassias and H. Kim, *Approximate homomorphisms and derivations between C^*-ternary algebras*, J. Math. Phys. **49** 063507 (2008).

[30] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[31] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, 1983.

[32] A.N. Sherstnev, *On the notion of a random normed space*, Dokl. Akad. Nauk SSSR **149** (1963), 280–283 (in Russian).

[33] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

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A note on the q-extension of second kind Euler numbers and polynomials

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Abstract : In this paper, by using the p-adic integral on $\mathbb{Z}_p$, we construct a new type of the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. From these numbers and polynomials, we establish some interesting identities and relations. By using the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$, the q-Euler zeta function and Hurwitz-type q-Euler zeta functions are defined.

Key words : the second kind Euler numbers and polynomials, the q-extension of the second kind Euler numbers and polynomials

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \text{ cf. } [1, 2, 3, 4, 5, 6] .$$

For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim[1, 2] defined the p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \leq x < p^N} g(x)(-1)^x. \tag{1.1}$$

From (1.1), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \text{ (see [1-3])}. \tag{1.2}$$

where $g_n(x) = g(x+n)$.

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$(see [4]). The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2e^t}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.4}$$

2. q-extension of the second kind Euler numbers and polynomials

In this section, we introduce the q-extension of the second kind Euler Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, q-extension of the second kind Euler numbers $E_{n,q}$ are defined by

$$E_{n,q} = \int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x). \tag{2.1}$$

By using p-adic integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} q^x [2x+1]_q^n (-1)^x \\
&= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l+1}} \\
&= 2\sum_{m=0}^{\infty} (-1)^m q^m [2m+1]_q^n.
\end{aligned} \tag{2.2}$$

By (2.1), we have the following theorem.

Theorem 1. For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, we have

$$\begin{aligned}
E_{n,q} &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l+1}} \\
&= 2\sum_{m=0}^{\infty} (-1)^m q^m [2m+1]_q^n.
\end{aligned}$$

We set

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$\begin{aligned}
F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} &= 2\sum_{n=0}^{\infty}\left(\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l+1}}\right)\frac{t^n}{n!} \\
&= 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+1]_q t}.
\end{aligned} \tag{2.3}$$

Thus, q-extension of the second kind Euler numbers, $E_{n,q}$ are defined by means of the generating function

$$F_q(t) = 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+1]_q t}. \tag{2.4}$$

By using (2.1), we have

$$\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^x [2x+1]_q^n d\mu_{-1}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^x e^{[2x+1]_q t} d\mu_{-1}(x). \tag{2.5}$$

By (2.3), (2.5), we have

$$\int_{\mathbb{Z}_p} q^x e^{[2x+1]_q t} d\mu_{-1}(x) = 2\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+1]_q t}.$$

Next, we introduce q-extension of the second kind Euler polynomials $E_{n,q}(x)$. The q-extension of the second kind Euler polynomials $E_{n,q}(x)$ are defined by

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^y [x+2y+1]_q^n d\mu_{-1}(y). \tag{2.6}$$

By using p-adic integral, we obtain

$$E_{n,q}(x) = 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(x+1)l}\frac{1}{1+q^{2l+1}}. \tag{2.7}$$

We set

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2.8}$$

By using (2.7) and (2.8), we obtain

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+1+x]_q t}. \tag{2.9}$$

Since $[x+2y+1]_q = [x]_q + q^x[2y+1]_q$, we easily see that

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} q^y [x+2y+1]_q^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l} q^{xl} E_{l,q} \\ &= 2\sum_{m=0}^{\infty}(-1)^m q^m [x+2m+1]_q^n, \end{aligned} \tag{2.10}$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

By (1.3), (1.4), (2.3), and (2.10), we have the following remark.

Remark 1. Note that

$$\begin{aligned} &(1)\ E_{n,q}(0) = E_{n,q}, \\ &(2)\ \text{If } q \to 1,\ \text{then } E_{n,q}(x) = E_n(x), \quad E_{n,q} = E_n, \\ &(3)\ \text{If } q \to 1,\ \text{then } F_q(x,t) = F(x,t), \quad F_q(t) = F(t). \end{aligned}$$

By (2.7), we obtain the following theorem.

Theorem 2(Property of complement).

$$E_{n,q^{-1}}(-x) = (-1)^n q^{n+1} E_{n,q}(x)$$

By (2.7), we have the following distribution relation:

Theorem 3. For any positive integer m(=odd), we have

$$E_{n,q}(x) = [m]_q^n \sum_{a=0}^{m-1}(-1)^a q^a E_{n,q^m}\left(\frac{2a+x+1-m}{m}\right), n \in \mathbb{Z}_+.$$

By (1.2), (2.1), and (2.6), we easily see that

$$q^n E_{m,q}(2n) + (-1)^{n-1} E_{m,q} = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} q^l [2l+1]_q^m.$$

Hence, we obtain the following theorem.

Theorem 4. Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod 2$, then

$$q^n E_{m,q}(2n) - E_{m,q} = 2\sum_{l=0}^{n-1} (-1)^{l+1} q^l [2l+1]_q^m.$$

If $n \equiv 1 \pmod 2$, then

$$q^n E_{m,q}(2n) + E_{m,q} = 2\sum_{l=0}^{n-1} (-1)^{l} q^l [2l+1]_q^m.$$

From (1.2), we note that

$$\begin{aligned}
2e^t &= q\int_{\mathbb{Z}_p} e^{[2x+3]_q t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{[2x+1]_q t} d\mu_{-1}(x) \\
&= \sum_{n=0}^{\infty} \left(q\int_{\mathbb{Z}_p} [2x+3]_q^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} [2x+1]_q^n d\mu_{-1}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(qE_{n,q}(2) + E_{n,q} \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$qE_{n,q}(2) + E_{n,q} = 2.$$

By Theorem 5 and (2.10), we have the following corollary.

Corollary 6. For $n \in \mathbb{Z}_+$, we have

$$q(q^2 E_q + [2]_q)^n + E_{n,q} = 2,$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

3. The analogue of the Euler zeta function

By using q-extension of second kind Euler numbers and polynomials, q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. These functions interpolate the q-extension of second kind Euler numbers $E_{n,q}$, and polynomials $E_{n,q}(x)$, respectively. Let q be a complex number with $|q| < 1$. From (2.4), we note that

$$\begin{aligned}
\left.\frac{d^k}{dt^k} F_q(t)\right|_{t=0} &= 2\sum_{m=0}^{\infty} (-1)^n q^m [2m+1]_q^k \\
&= E_{k,q}, (k \in \mathbb{N}).
\end{aligned}$$

By using the above equation, we are now ready to define q-Euler zeta functions.

Definition 7. Let $s \in \mathbb{C}$ with $Res > 1$.

$$\zeta_q(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[2n+1]_q^s}. \tag{3.1}$$

Note that $\zeta_q(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_q(s) = \zeta(s)$ which is the Euler zeta functions(see [6]). Relation between $\zeta_q(s)$ and $E_{k,q}$ is given by the following theorem.

Theorem 8. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k) = E_{k,q}.$$

Observe that $\zeta_q(s)$ function interpolates $E_{k,q}$ numbers at non-negative integers. By using (2.9), we note that

$$\left.\frac{d^k}{dt^k} F_q(t,x)\right|_{t=0} = 2\sum_{m=0}^{\infty} (-1)^m q^m [2x+1+m]_q^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left.\left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 9. Let $s \in \mathbb{C}$ with $Res > 1$.

$$\zeta_q(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+2x+1]_q^s}. \tag{3.4}$$

Note that $\zeta_q(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q(s,x) = \zeta(s,x)$ which is the Hurwitz Euler zeta functions(see [6]). Relation between $\zeta_q(s,x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 10. For $k \in \mathbb{N}$, we have

$$\zeta_q(-k,x) = E_{k,q}(x).$$

Observe that $\zeta_q(-k,x)$ function interpolates $E_{k,q}(x)$ numbers at non-negative integers.

REFERENCES

1. Kim, T.(2007). q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys., v. 14, pp. 15-27.

2. Kim, T.(2002). q-Volkenborn integration, *Russ. J. Math. Phys.*, v. 9, pp. 288-299.

3. Kim, T., Jang, L. C., Pak, H. K.(2001). A note on q-Euler and Genocchi numbers , Proc. Japan Acad. , v.77 A, pp. 139-141.

4. Ryoo, C.S. (2010). Calculating zeros of the second kind Euler polynomials, Journal of Computational Analysis and Applications, v.12, pp. 828-833.

5. Ryoo, C.S.(2009). Calculating zeros of the q-Euler polynomials, Proceeding of the Jangjeon Mathematical Society, v.12, pp. 253-259.

6. Ryoo, C.S. (2011). A note on the q-Hurwitz Euler zeta functions, Journal of Computational Analysis and Applications, v.13, pp. 1012-1018.

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SOME PROPERTIES OF BAZILEVIC FUNCTIONS RELATED WITH CONIC DOMAINS

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Abstract. The aim of this paper is to study the Bazilevic functions associated with conic domains. Some properties of analytic functions related with Bazilevic functions by using the concept of convolution are examined. We investigate some results concerned with integral preserving property and radius problems which generalize the already proved results.

1. Introduction

Let A be the class of analytic functions

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$. For any two analytic functions f and g with

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} c_n z^n, \ z \in E,$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^n, \ z \in E.$$

A function $f \in A$ is starlike univalent function of order ρ, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho, \quad 0 \leq \rho < 1, \ z \in E.$$

This class of functions is denoted by $S^*(\rho)$. Kanas and Wisnowska [7] studied $k-UCV$, the class of k-uniformly convex and $k-ST$, the corresponding class of k-starlike functions. A function $f \in A$ is said to be in the class $k-UCV$ of k-uniformly convex function, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k \left|\frac{zf''(z)}{f'(z)}\right|, \quad k \geq 0, \quad z \in E. \tag{1.2}$$

Similarly a function $f \in A$ is said to be in the class denoted by $k-ST$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq k \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \geq 0, \quad z \in E. \tag{1.3}$$

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Geometric interpretation. The function $f \in k-UCV$ and $f \in k-ST$, if and only if $\frac{zf''(z)}{f'(z)}+1$ and $\frac{zf'(z)}{f(z)}$, respectively, take all values in the conic domain Ω_k, which is included in the right half plane such that

$$\Omega_k = \{u+iv : u > k\sqrt{(u-1)^2+v^2}\},$$

with $p(z)=\frac{zf''(z)}{f'(z)}+1$ or $p(z)=\frac{zf'(z)}{f(z)}$ and considering the functions which map E onto the conic domain Ω_k such that $1 \in \Omega_k$, we may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) \prec q_k(z).$$

The domain $\Omega_{k,\rho}$ is such that

$$\Omega_{k,\rho} = (1-\rho)\,\Omega_k + \rho, \quad 0 \le \rho < 1.$$

The function $q_{k,\rho}$ plays the role of extremal for these classes and is given by

$$q_{k,\rho}(z) = \begin{cases} \frac{1+(1-\rho)z}{1-z}, \quad k=0, \\ 1+\frac{2\gamma(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad k=1, \\ 1+\frac{2(1-\rho)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], \quad 0<k<1, \\ 1+\frac{(1-\rho)}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{(1-\rho)}{k^2-1}, \quad k>1, \end{cases} \tag{1.4}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}},\ t\in(0,1),\ z\in E$ and t is chosen such that $k=cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. By virtue of (1.4) and the properties of the domains $\Omega_{k,\rho}$, we have $p \prec q_{k,\rho}$ implies

$$\operatorname{Re} p(z) > \operatorname{Re} q_{k,\rho}(z) > \frac{k+\rho}{k+1}.$$

A function p, analytic in E with $p(0)=1$, is said to be in the class $k-P(\rho) \subset P$, if it is subordinate to $q_{k,\rho}$ in E. That is $p \in k-P(\rho)$, if and only if $p \prec q_{k,\rho}$, where $q_{k,\rho}$ is given by (1.4) and $p(E) \subset q_{k,\rho}(E)$.

It is noted that $0-P(0)=P$, the class of analytic functions with positive real part and $p \in 0-P(\rho)=P(\rho)$ implies that $Rep(z)>\rho,\ z\in E$.

Recently Noor [11] has extended the class $k-P(\rho)$ and defined the following subclass of caratheodory class P.

Definition 1.1. Let p be analytic in E with $p(0)=1$. Then $p \in k-P_m(\rho)$, if and only if

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \quad p_1(z),\ p_2(z)\in k-P(\rho),$$

for $m\ge 2,\ 0\le\rho<1,\ k\in[0,\infty),\ z\in E$. We note that $k-P_2(\rho)=k-P(\rho)$ and $0-P_m(0)=P_m$, the well-known class defined in [12].

Definition 1.2. [11] Let $f \in A$. Then $f \in k - UR_m(\rho),\ 0 \leq \rho < 1,\ k \in [0, \infty)$ and $m \geq 2$, if and only if

$$\frac{zf'(z)}{f(z)} \in k - P_m(\rho),\ z \in E.$$

Noor called $k - UR_m(\rho)$, the class of functions of k-uniform bounded boundary rotation m with order ρ. It can easily be seen that $0 - UR_m(0) = R_m$, the class of functions of bounded boundary rotation. It is also noted that $1 - UR_2(0) = UST$, the class of unifromly starlike functions.

Now using the concepts of class $k - P_m$ and the class of uniformly starlike functions, we define the following:

Definition 1.3. Let $F \in A,\ \alpha, \beta \in \mathbb{R},\ \alpha > 0,\ f \in UST$. Then $F \in k - UB_m(\alpha, \beta)$, if and only if

$$\left\{ \frac{zF'(z)\, F^{\alpha + i\beta - 1}(z)}{z^{i\beta} f^{\alpha}(z)} \right\} \in k - P_m, \quad z \in E. \tag{1.5}$$

Remark 1.4. From (1.5) it can easily be seen that $F \in k - UB_m(\alpha, \beta)$ can be represented by the following integral representation

$$F(z) = \left[(\alpha + i\beta) \int_0^z h(t) f^{\alpha}(t) t^{i\beta - 1} dt \right]^{\frac{1}{\alpha + i\beta}} \quad h \in k - P_m,\ \ f \in UST,\ z \in E. \tag{1.6}$$

We note that, with $m = 2,\ k = 0$, the class $k - UB_m(\alpha, \beta)$ reduces to the class of Bazilevic functions introduced in [3], where he showed that a Bazilevic function is univalent in E and has the integral representation given by (1.6).

For recent work of the above mentioned classes, we refer [1,2,6,9,13].

We need the following lemmas which will be used in our main results.

2. Preliminary Results

Lemma 2.1. *Let $g \in UST$. Then $z\left(\frac{g(z)}{z}\right)^{\alpha}$, where $\alpha > 0$ also belongs to UST in E.*

Proof. Let

$$G_1(z) = z\left(\frac{g(z)}{z}\right)^{\alpha}.$$

Taking logarithmic differentiation of both sides we have

$$\begin{aligned} \frac{zG_1'(z)}{G_1(z)} &= \alpha \frac{zg'(z)}{g(z)} + (1 - \alpha) \\ &= \alpha h_0(z) + (1 - \alpha). \end{aligned}$$

Since $h_0 \in 1 - P$, $p_0(z) = 1 \in 1 - P$ and $1 - P$ is convex set, see [11], therefore $G_1(z) \in 1 - P$. ∎

Remark 2.2. This result can easily be extended to the class $k - UR_m$ using the fact that $k - P_m$ is convex set, see [11].

Lemma 2.3. *[15] Let $f \in C$ and $g \in S^*$. Then for any analytic function F with $F(0) = 1$ in E*

$$\frac{f * Fg}{f * g}(E) \subset \overline{co}F(E),$$

where $\overline{co}F(E)$ denotes the convex hull of $F(E)$ (the smallest convex set which contains $F(E)$).

Lemma 2.4. *[10] Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex valued function satisfying the conditions:*
(i) *$\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*
(ii) *$(1, 0) \in D$ and $Re\psi(1, 0) > 0$,*
(iii) *$Re\psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*
If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $Re\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $Reh(z) > 0$ in E.

3. Main Results

Theorem 3.1. *Let $\alpha \in \mathbb{R}, \alpha > 0$ and $c \in \mathbb{C}$, $\operatorname{Re} c \geq 0$ and let $f \in UST$. Then*

$$g(z) = \left[(c+1) z^{-c} \int_0^z t^{c-1} f^{\alpha}(t) dt\right]^{\frac{1}{\alpha}} \in UST, \ z \in E. \tag{3.1}$$

Proof. We can write (3.1) by using convolution as

$$g(z) = z\left[\left(\frac{f(z)}{z}\right)^{\alpha} * \frac{\phi_{\alpha+c}(z)}{z}\right]^{\frac{1}{\alpha}}, \tag{3.2}$$

where $\phi_{\alpha+c}(z) = \sum_{n=1}^{\infty} \frac{\alpha+c}{\alpha+c+n-1} z^n$ is convex in E, see [14]. Now from (3.2), we get

$$\frac{zg'(z)}{g(z)} = \frac{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha} \frac{zf'(z)}{f(z)}}{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha}}.$$

Since, by Lemma 2.1, $z\left(\frac{f(z)}{z}\right)^{\alpha} \in UST \subset S^*\ \frac{1}{2}) \subset S^*$, $\phi_{\alpha+c}(z)$ is convex, it follows from Lemma 2.3 that

$$\frac{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha} H(z)}{\phi_{\alpha+c}(z) * z\left(\frac{f(z)}{z}\right)^{\alpha}}(E) \subset \overline{co}H(E), \quad H(z) = \frac{zf'(z)}{f(z)}.$$

This proves that $\frac{zg'(z)}{g(z)} \in 1 - P$ and thus $g \in UST$. ∎

Theorem 3.2. *Let $F \in k - UB_m(\alpha, \beta)$, $\phi \in C$. Then*

$$G(z) = z\left[\left(\frac{F(z)}{z}\right)^{\alpha+i\beta} * \frac{\phi(z)}{z}\right]^{\frac{1}{\alpha+i\beta}} \in k - UB_m(\alpha, \beta) \tag{3.3}$$

in E.

Proof. Since $F \in k-UB_m(\alpha,\beta)$, there exists $f \in UST$ such that

$$\left\{\frac{zF'(z)\,F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)}\right\} \in k-P_m,\ z\in E. \tag{3.4}$$

Define

$$g(z) = z\left[\left(\frac{f(z)}{z}\right)^{\alpha} * \frac{\phi(z)}{z}\right]^{\frac{1}{\alpha}}. \tag{3.5}$$

Then $g \in UST$ by Theorem 3.1. Therefore from (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned}\frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)} &= \frac{\phi(z)*z\left(\frac{f(z)}{z}\right)^{\alpha}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)}}{\phi(z)*z\left(\frac{f(z)}{z}\right)^{\alpha}}\\ &= \frac{\phi(z)*f_1(z)H_0(z)}{\phi(z)*f_1(z)},\end{aligned}$$

where $H_0=\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in k-P_m$, $f_1(z)=z\left(\frac{f(z)}{z}\right)^{\alpha}\in S^*$. Now using Lemma 2.3, we obtain the desired result. ∎

Applications of Theorem 2.3

The class $k-UB_m(\alpha,\beta)$ is invariant under the following integral representation

$$F_1(z)=\left[(\alpha+i\beta+c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}},$$

where $\operatorname{Re} c\geq 0$ and $F(z)\in k-UB_m(\alpha,\beta)$. In fact we can write

$$F_1(z)=z\left[\left(\frac{F(z)}{z}\right)^{\alpha+i\beta} * \frac{\phi_{\alpha+i\beta+c}(z)}{z}\right]^{\frac{1}{\alpha+i\beta}},$$

and since $\phi_{\alpha+i\beta+c}$ is convex in E, the result is immediate from Theorem 3.2.

We note the following special cases.

(i) For $\beta=0,\ \alpha=1$, we have

$$F_1(z)=(1+c)z^{-c}\int_0^z t^{c-1}F(t)dt,\quad \operatorname{Re} c\geq 0, \tag{3.6}$$

and this is the generalized Bernadi integral operator [4]. When $k=0, m=2$, we have the result for the class K of close-to-convex functions [4].

(ii) In (3.6), by taking $c=1$, we obtain Libera operator [8] and $c=0$ leads us to the well-known Alexander operator.

(iii) When $\beta=c=0$ implies that

$$[F_1(z)]^{\alpha}=\alpha\int_0^z\left(\frac{F(t)}{t}\right)^{\alpha}dt,\quad \alpha>0, \tag{3.7}$$

a generalized form of Alexander operator.

With $m = 2,\ k = 0$, we note that the class of Bazilevic functions of type α (see, [16]) is invariant under the integral operator defined in (3.6).

Class $B_m(\alpha, \beta, \gamma)$

We now deal the case $k = 0$, when $F \in A,\ \alpha,\ \beta \in \mathbb{R},\ \alpha > 0,\ f \in S^*$. Then $F \in B_m(\alpha, \beta, \gamma)$, if and only if

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in P_m(\gamma),\ 0 \leq \gamma < 1.$$

Theorem 3.3. *Let $F \in B_m(\alpha, \beta, \gamma)$ and set*

$$G(z) = \left[(\alpha + i\beta + c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}},\ c \geq 0. \tag{3.8}$$

Then $G \in B_m(\alpha, \beta, \gamma_1)$, where

$$\gamma_1 = \frac{2\gamma C_1 + c + \alpha h_1}{2C_1 + c + \alpha h_1},\ h_1 = \operatorname{Re}\frac{zg'(z)}{g(z)},\ C_1 = \left|\alpha\frac{zg'(z)}{g(z)} + c + i\beta\right|^2,$$

and g is integral representation of $f \in S^*$ and is starlike.

Proof. Since $F \in B_m(\alpha, \beta, \gamma)$ so there exists $f \in S^*$ such that

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \in P_m(\gamma),\ z \in E. \tag{3.9}$$

Set

$$\frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)} = h(z), \tag{3.10}$$

where

$$g(z) = \left[\frac{\alpha + i\beta + c}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}f^{\alpha}(t)dt\right]^{\frac{1}{\alpha}} \in S^* \tag{3.11}$$

by Theorem 3.1. Since $g \in S^*$ we set $\frac{zg'(z)}{g(z)} = h_0(z) = h_1 + ih_2,\ h_0 \in P$ in E. Now from (3.8) – (3.11) , we obtain after some computations

$$\left\{h(z) + \frac{zh'(z)}{\alpha h_0(z) + c + i\beta}\right\} \in P_m(\gamma). \tag{3.12}$$

Writing

$$h(z) = (1 - \gamma_1)p(z) + \gamma_1.$$

It follows from (3.12) that for $i = 1, 2$,

$$\left\{(1 - \gamma_1)\,p_i(z) + \frac{(1 - \gamma_1)\,zp_i'(z)}{\alpha h_0(z) + c + i\beta} + \gamma_1 - \gamma\right\} \in P,\ z \in E.$$

We construct the functional $\psi(u, v)$ by taking $u = p_i(z),\ v = zp_i'(z)$ as follows:

$$(u, v) = (1 - \gamma_1)u + \frac{(1 - \gamma_1)v}{\alpha h_0(z) + c + i\beta} + (\gamma_1 - \gamma).$$

The first two conditions of Lemma 2.4 can easily be verified. We check condition (iii) as below:

$$\begin{aligned}
\operatorname{Re}\psi(iu_2,v_1) &= (\gamma_1-\gamma)+\frac{(1-\gamma_1)(c+\alpha h_1)v}{\left|\alpha h_0(z)+c+i\beta\right|^2} \\
&\leq (\gamma_1-\gamma)-\frac{(1-\gamma_1)(c+\alpha h_1)(1+u_2^2)}{2\left|\alpha h_0(z)+c+i\beta\right|^2} \\
&= \frac{2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)(1+u_2^2)}{2C_1} \\
&= \frac{2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)+(\gamma_1-1)(c+\alpha h_1)u_2^2}{2C_1} \\
&= \frac{A+Bu_2^2}{2C_1}, \qquad (3.13)
\end{aligned}$$

where $A=2(\gamma_1-\gamma)C_1-(1-\gamma_1)(c+\alpha h_1)$ and $B=(\gamma_1-1)(c+\alpha h_1)$. Since $C_1=\left|\alpha h_0(z)+c+i\beta\right|^2>0$ and the right hand side of (3.13) is less than or equal to zero, if $A\leq 0$ and $B\leq 0$. Now from $B\leq 0$, we have $\gamma_1<1$ and from $A\leq 0$, we obtain the value of γ_1 given by

$$\gamma_1=\frac{2\gamma C_1+c+\alpha h_1}{2C_1+c+\alpha h_1}.$$

Thus all the conditions of Lemma 2.4 are satisfied and $p_i\in P$ which mean $h_i\in P(\gamma_1)$ and hence $h\in P_m(\gamma_1)$. This proves our result. ∎

Theorem 3.4. *Let $G\in B_2(\alpha,\beta,0)$, where*

$$G(z)=\left[(\alpha+i\beta+c)z^{-c}\int_0^z t^{c-1}F^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}}, \quad c\geq 0.$$

Then $F\in B_2(\alpha,\beta,0)$, for $|z|<r_0$ and

$$r_0=\begin{cases}\frac{-(\alpha+1)+\sqrt{c^2+2\alpha+1}}{c-\alpha}, & c>\alpha,\\ \frac{1}{2}, & c=\alpha=1.\end{cases} \qquad (3.14)$$

Proof. Since $G\in B_2(\alpha,\beta,0)$, so there exists $g\in S^*$ such that

$$\frac{zG'(z)G^{\alpha+i\beta-1}(z)}{z^{i\beta}g^{\alpha}(z)}=h(z),$$

where

$$g(z)=\left[\frac{\alpha+i\beta+c}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}f^{\alpha}(t)dt\right]^{\frac{1}{\alpha}}\in S^*$$

by Theorem 3.1. Since $g\in S^*$ we set $\frac{zg'(z)}{g(z)}=h_0(z)\in P$ in E. Now from (3.8), (3.10) and (3.11), we obtain after some computation

$$\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)}=h(z)+\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}.$$

This implies that

$$\begin{aligned}\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} &= \operatorname{Re}\left\{h(z)+\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}\right\}\\ &\geq \operatorname{Re}h(z)-\left|\frac{zh'(z)}{\alpha h_0(z)+c+i\beta}\right|.\end{aligned}$$

Using the well-known distortion results for class P, we have

$$\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} \geq \operatorname{Re}h(z)\left\{1-\frac{2r}{1-r^2}\left|\frac{1}{\alpha h_0(z)+c+i\beta}\right|\right\}.$$

Since $h_0 \in P$, we have

$$\begin{aligned}|\alpha h_0(z)+c+i\beta| &\geq \operatorname{Re}\{\alpha h_0(z)+c+i\beta\}\\ &\geq \alpha\left(\frac{1-r}{1+r}\right)+c\\ &= \frac{\alpha(1-r)+c(1+r)}{1+r}.\end{aligned}$$

It follows easily that

$$\begin{aligned}\operatorname{Re}\frac{zF'(z)F^{\alpha+i\beta-1}(z)}{z^{i\beta}f^{\alpha}(z)} &\geq \operatorname{Re}h(z)\left\{1-\frac{2r}{(1-r)\{\alpha(1-r)+c(1+r)\}}\right\}\\ &= \operatorname{Re}h(z)\left\{\frac{(\alpha+c)-2(\alpha+1)r+(\alpha-c)r^2}{(1-r)\{\alpha(1-r)+c(1+r)\}}\right\}.\end{aligned}$$

Hence $F \in B_2(\alpha,\beta,0)$, for $|z| < r_0$, where r_0 is given in (3.14). This completes the proof. ∎

For $\alpha = 1$ and $\beta = 0$, we have the result proved by Bernardi [5] for Bernardi operator.

Corollary 3.5. *Let $G \in K$, where*

$$G(z) = (1+c)\,z^{-c}\int_0^z t^{c-1}F(t)dt.$$

Then $F \in K$, for $|z| < r_0$ and

$$r_0 = \begin{cases}\frac{-2+\sqrt{c^2+3}}{c-1}, & c > 1,\\ \frac{1}{2}, & c = 1.\end{cases}$$

References

[1] Arif, M., Noor, K. I., Raza, M. On a class of analytic functions related with generalized Bazilevic type functions. Comput. Math. Appl., 61, 2456–2462 (2011).

[2] Arif, M., Raza, M., Noor, K. I., Malik, S. N. On strongly Bazilevic functions associated with generalized Robertson functions. Math. Comput. Mod., 54, 1608-1612 (2011).

[3] Bazilevic, I. E. On a class of integrability in quadratures of the Loewner-Kufarev equation. Mat. Sb., 37, 471-476 (1955).

[4] Bernardi, S. D. Convex and Starlike Univalent Functions. Trans. Amer. Math. Soc., 135, 429–446 (1969).

[5] Bernardi, S. D. The radius of univalence of certain analytic functions. Proc. Amer. Math. Soc., 24, 312–318 (1970) .

[6] Irmak, H., Bulboacă, T., Tuneski, N. Some relations between certain classes consisting of α-convex type and Bazilević type functions. Appl. Math. Lett., 24, 2010–2014 (2011).

[7] Kanas, S., Wisniowska, A. Conic regions and k-uniform convexity. J. Comput. Appl. Math., 105, 327–336 (1999).

[8] Libra, R. J. Some Classes of Regular Univalent Functions. Proc. Amer. Math. Soc., 16, 755–758 (1965).

[9] Malik, S. N., Raza, M., Arif, M., Hussain, S. Coeffecient estimates for some sub classes of analytic functions related with conic domains. An. St. Univ. Ovidius Constanta, (2012) (in press).

[10] Miller, S. S. Differential inequalities and Caratheodory functions. Bull. Amer. Math. Soc., 81, 79–81 (1975).

[11] Noor, K. I. On a generalization of uniformly convex and related functions. Comp. Math. Appl., 61(1), 117-125 (2011).

[12] Pinchuk, B. Functions with bounded boundary rotation. Israel J. Math., 10, 7–16 (1971).

[13] Raza, M., Noor, K. I. A class of Bazilevic type functions defined by convolution operator, J. Math. Inequal., 5(2), 253-261 (2011).

[14] Ruscheweyh, S. Eine Invarianzeigenschaft der Basilevič-funktionen. Math. Z., 134, 215-219 (1973).

[15] Ruscheweyh, S., Sheil-Small, T. Hadmard products of schlicht functions and the polya-Shoenberg conjecture. Comm. Math. Helv., 48, 119-135 (1973).

[16] Singh, R. On Bazilevič functions, Proc. Amer. Math. Soc., 38, 261-271 (1973).

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Some characterizations in some Möbius invariant spaces

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Abstract. We give two characterizations of the Möbius invariant $Q_K(p,q)$ spaces, one in terms of a double integral and the other in terms of the mean oscillation in the Bergman metric. Both characterizations avoid the use of derivatives.

1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$. The Green's function in the unit disk Δ with singularity at $a \in \Delta$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with the center $a \in \Delta$ and radius r. Through this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ if

$$\|f\|^p_{K,p,q} \quad = \quad \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on Δ. It is clear that $Q_K(p,q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q}$ where $p \geq 1$. If $q+2 = p$, $Q_K(p,q)$ is Möbius invariant, i.e., $\|f \circ \varphi_a\| = \|f\|_{K,p,q}$ for all $a \in \Delta$. Now we consider some special cases. If $p = 2$, and $q = 0$, we obtain that $Q_K(p,q) = Q_K$ (cf. [4, 9]). If $K(t) = t^s$, then $Q_K(p,q) = F(p,q,s)$ (cf. [11]) that $F(p,q,s)$ is contained in $\frac{q+2}{p}-$ Bloch space.
The space $Q_{K,0}(p,q)$ consists of analytic function f in Δ with the property that

$$\lim_{|a|\to 1^-} \int_\Delta |f'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z) = 0.$$

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It can be checked that $Q_{K,0}(p,q)$ is a closed subspace in $Q_K(p,q)$. The following identity is easily verified:

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}=(1-|z|^2)|\varphi_a'(z)|.$$

For $a\in\Delta$, the substitution $z=\varphi_a(w)$ results in the Jacobian change in measure given by $dA(w)=|\varphi_a(z)|^2dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)}h(\varphi_a(w))dA(w)=\int_{\Delta(a,r)}h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2dA(z)\ .$$

Note that $\varphi_a(\varphi_a(z))=z$ and thus $\varphi_a^{-1}(z)=\varphi_a(z)$. For $a,z\in\Delta$ and $0<r<1$, the pseudo-hyperbolic disk $\Delta(a,r)$ is defined by $\Delta(a,r)=\{z\in\Delta:|\varphi_a(z)|<r\}$. We will also need to use the so-called Berezin transform. More specifically, for any function $f\in L^1(\Delta,dA)$, we define a function Bf by

$$Bf(z)=\int_\Delta\frac{(1-|z|^2)^2}{|1-zw|^4}f(w)dA(w),\quad z\in\Delta$$

we call Bf the Berezin transform of f. By a change of variables, we can also write

$$Bf(z)=\int_\Delta f\circ\varphi_z(w)dA(w),\quad z\in\Delta$$

see [1, 2, 3, 6] and [12] for basic properties of the Berezin transform.
If the function K is only defined on $(0,1]$, then we extend it to $(0,\infty)$ by setting $K(t)=K(1)$ for $t>1$. We can then define on auxiliary function as follows:

$$\varphi_K(s)=\sup_{0<t\leq 1}\frac{K(st)}{K(t)},\ \ 0<s<\infty.$$

We further assume that K is continuous and nondecreasing on $(0,1]$ This ensures that the function φ_K is continuous and nondecreasing on $(0,\infty)$.
The following estimate is the key to the main results of this paper.

Lemma 1.1 *[10] Let K be any nonnegative and Lebesgue measurable function on $(0,\infty)$ and $f(z)=K(1-|z|^2)$. If*

$$\int_0^\infty\frac{\varphi_K(x)}{(1+x)^3}dx<\infty,\tag{1}$$

then there exists a positive constant C such that $Bf(z)\leq Cf(z)$ for all $z\in\Delta$.

Hereafter, C stands for absolute constants, which may indicate different constants from one occurrence to the next.

2 A double integral characterization in $Q_K(p,q)$ spaces

In this section we characterize the space $Q_K(p,q)$ in terms of a double integral that does not involve the use of derivatives. We begin with the following estimate of Bloch type integrals.

Theorem 2.1 *Suppose that $K(t)\approx t^nK(t);\ 0<t<1,\ n\geq 0$. There exists a constant $C>0$ (independent of K) such that*

$$\int_\Delta\left|f'(z)\right|^p\left(1-|z|^2\right)^{p-2}K(1-|z|^2)dA(z)\leq CI(f)$$

for all analytic functions f in Δ, where

$$I(f)=\int_\Delta\int_\Delta\frac{\left|f(z)-f(w)\right|^p}{\left|1-z\bar{w}\right|^4}\left(1-|z|^2\right)^{p-2}K(1-|z|^2)d(z)dA(w).$$

2

Proof. We write the double integral $I(f)$ as an iterated integral

$$I(f)=\int_{\Delta}\frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}}dA(z)\int_{\Delta}\frac{\left(1-|z|^2\right)^2}{\left|1-z\bar{w}\right|^4}\left|f(z)-f(w)\right|^p dA(w)$$

Making a change of variables in the inner integral, we obtain

$$I(f)=\int_{\Delta}\frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}}dA(z)\int_{\Delta}\left|f(\varphi_z(w))-f(z)\right|^p dA(w). \tag{2}$$

It is well known that

$$\int_{\Delta}|g(w)-g(0)|^p dA(w)\sim\int_{\Delta}|g'(w)|^p(1-|w|^2)^p dA(w), \tag{3}$$

for analytic functions g in Δ. Applying (3) to the inner integral in (2) with the function $g(w)=f(\varphi_z(w))$, we deduce that

$$I(f)\sim\int_{\Delta}\frac{K(1-|z|^2)}{(1-|z|^2)^{4-p}}dA(z)\int_{\Delta}\left|(f\circ\varphi_z)'(w)\right|^p\left(1-|w|^2\right)^p dA(w).$$

Therefore, by the chain rule and a change of variables, we get

$$I(f)\quad\sim\int_{\Delta}\left(1-|z|^2\right)^{p-2}K(1-|z|^2)dA(z)\int_{\Delta}\left|f'(w)\right|^p\frac{(1-|w|^2)^p}{|1-z\bar{w}|^4}dA(w). \tag{4}$$

Fix any positive radius R. Then there exists a constant $C>0$ such that

$$I(f)\geq C\int_{\Delta}\left(1-|z|^2\right)^{p-2}K(1-|z|^2)dA(z)\int_{\Delta(z,R)}\left|f'(w)\right|^p\frac{(1-|w|^2)^p}{|1-z\bar{w}|^4}dA(w).$$

It is well known that (see e.g [8])

$$\frac{(1-|w|^2)}{|1-z\bar{w}|^2}\sim\frac{1}{(1-|z|^2)}\sim\frac{1}{\sqrt{|\Delta(z,R)|}}.$$

for $w\in\Delta(z,R)$. It is follows that there exists a positive constant C such that

$$I(f)\geq C\int_{\Delta}(1-|z|^2)^{p-2}K(1-|z|^2)dA(z)\frac{1}{|\Delta(z,R)|^{\frac{p}{2}}}\int_{\Delta(z,R)}\left|f'(w)\right|^p dA(w).$$

Then,

$$I(f)\geq C\int_{\Delta}\left|f'(z)\right|^p(1-|z|^2)^{p-2}K(1-|z|^2)dA(z).$$

This complete the proof of the theorem.

Theorem 2.2 *Let $p>2$. If the function K satisfies condition (1) and suppose that $K(t)\approx t^nK(t);\ 0<t<1,\ n\geq 0$. Then there exists a constant $C>0$ such that*

$$\int_{\Delta}\left|f'(z)\right|^p\left(1-|z|^2\right)^{p-2}K(1-|z|^2)dA(z)\geq CI(f)$$

for all analytic functions f in Δ, where $I(f)$ is as given in Lemma 2.1.

Proof. By Fubini's theorem, we can rewrite (4) as

$$I(f)\sim\int_{\Delta}\left|f'(w)\right|^p(1-|w|^2)^{p-2}dA(w)\int_{\Delta}(1-|z|^2)^{p-2}\frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4}K(1-|z|^2)dA(z).$$

$$\sim\int_{\Delta}\left|f'(w)\right|^p(1-|w|^2)^{p-2}dA(w)\int_{\Delta}\frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4}K(1-|z|^2)dA(z). \tag{5}$$

The inner integral above is nothing but the Berezin transform of the function $K(1-|z|^2)$ at the point w. The desired estimate now follows from Lemma 2.1

We can now prove the main result of this section

Theorem 2.3 *Suppose K satisfies condition (1) and satisfies all conditions of Theorems 2.1 and 2.2, then an analytic function f in Δ belongs to $Q_K(p, p-2)$ if and only if*

$$\int_\Delta \int_\Delta \frac{\left|f(z) - f(w)\right|^p}{|1 - z\bar{w}|^4} \left(1 - |z|^2\right)^{p-2} K(1 - |z|^2)) dA(z) dA(w) < \infty. \tag{6}$$

Proof. $f \in Q_K(p, p-2)$ if and only if

$$\sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|) dA(z) < \infty.$$

By a change of variables, we have $f \in Q_K(p, p-2)$ if and only if

$$\sup_{a\in\Delta} \int_\Delta \left|(f \circ \varphi_a)'(z)\right|^p \left(1 - |\varphi_a(z)|^2\right)^{p-2} K\left(1 - |z|^2\right) dA(z)$$

Replacing f by $f \circ \varphi_a$ in Theorems 2.1 and 2.2, we conclude that $f \in Q_K(p, p-2)$ iff

$$\sup_{a\in\Delta} \int_\Delta \int_\Delta \frac{\left|f \circ \varphi_a(z) - f \circ \varphi_a(w)\right|^p}{|1 - z\bar{w}|^4} \left(1 - |z|^2\right)^{p-2} K\left(1 - |z|^2\right) dA(z) dA(w) < \infty.$$

Changing variables and simplifying the result, we find that the double integral above is the same as

$$\sup_{a\in\Delta} \int_\Delta \int_\Delta \frac{\left|f(z) - f(w)\right|^p}{|1 - z\bar{w}|^4} \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) < \infty.$$

Therefore, $f \in Q_K(p, p-2)$ iff the condition (6) holds.

3 Bergman metric and $Q_K(p, q)$ spaces

In this section we give two closely related characterizations of $Q_K(p, q)$ spaces, one in terms of the Berezin transform and the other in terms of certain class of analytic functions in Bergman metric.
Given a function $f \in L^p(\Delta, dA)$ it is customary to write

$$S(f)(z) = (B(|f|^p) - |Bf(z)|^p)^{\frac{1}{p}}.$$

It easy to check that

$$\begin{aligned}(S(f)(z))^p &= \int_\Delta |f \circ \varphi_z(w) - Bf(z)|^p dA(w) \\ &= \int_\Delta |f(w) - Bf(z)|^p \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w).\end{aligned}$$

If the function f is analytic, then it is easy to see that $Bf = f$, so that

$$\begin{aligned}(S(f)(z))^p &= \int_\Delta |f \circ \varphi_z(w) - f(z)|^p dA(w) \\ &= \int_\Delta |f(w) - f(z)|^p \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w).\end{aligned}$$

We can now reformulate Theorem 3.1 as follows

Theorem 3.1 *If K satisfies condition (1), then an analytic function f in Δ belongs to $Q_K(p, p-2)$ iff*

$$\sup_{a\in\Delta} \int_\Delta (S(f)(z))^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty, \tag{7}$$

where

$$d\tau(z) = \frac{dA(z)}{(1 - |z|^2)^2}$$

is the Möbius invariant measure on the unit disk.

Proof. From Theorem 3.1

$$I_a(f) = \int_\Delta \int_\Delta \frac{\left|f(z) - f(w)\right|^p}{|1 - z\bar{w}|^4} \left(1 - |z|^2\right)^q K(1 - |\varphi_a(z)|^2) dA(z) dA(w)$$

we rewrite it as an iterated integral

$$I_a(f) = \int_\Delta \left(1 - |z|^2\right)^p K(1 - |\varphi_a(z)|^2)\, d\tau(z) \int_\Delta \frac{\left|f(z) - f(w)\right|^p}{|1 - z\bar{w}|^4}\, dA(w)$$

or

$$I_a(f) = \int_\Delta \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2)\, d\tau(z) \int_\Delta \left|f(z) - f(w)\right|^p \frac{\left(1 - |z|^2\right)^2}{|1 - z\bar{w}|^4}\, dA(w)$$

According to the calculations preceding this theorem, we have

$$I_a(f) = \int_\Delta (S(f)(z))^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) d\tau_z$$

This proves the desired result.

Now, fix a positive radius R and denote by

$$A_R(f)(z) = \frac{1}{|D(z, R)|} \int_{\Delta(z,R)} f(w) dA(w)$$

the a verge of f over the Bergman metric ball $D(z, R)$. For $p \geq 1$, we define

$$S_R(f)(z) = \left[\frac{1}{|D(z, R)|^p} \int_\Delta |f(w) - A_R(f)(z)|^p dA(w)\right]^{\frac{1}{p}}.$$

It is easy to verify that

$$(S_R(f)(z))^p = A_R(|f|^p)(z) - |A_R(f)(z)|^p.$$

Now, we prove the following theorem:

Theorem 3.2 *If K satisfies condition (1), then an analytic function f in Δ belongs to $Q_K(p, p-2)$ if and only if*

$$\sup_{a\in\Delta} \int_\Delta (S_R(f)(z))^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty, \tag{8}$$

where R is any fixed positive radius.

Proof. There exists a positive constant C which is depending on R only such that

$$S_R(f)(z) \leq C\, S(f)(z), \quad z \in \Delta,$$

where f is any function in $L^p(\Delta, dA)$. Therefore, condition (6) implies condition (7).
On the other hand, since $D(0, R)$ is an Euclidean disk centered at the origin, we can find a positive constant C which is depending on R only such that

$$|f'(0)|^p \leq C \int_{D(0,R)} |f(w) - C|^p dA(w)$$

for all analytic f in Δ and all complex constants C.
Replace f by $f \circ \varphi_z$ and replace C by $A_R(f)(z)$ then

$$(1 - |z|^2)^p |f'(z)|^p \leq C \int_{D(0,R)} |f \circ \varphi_z(w) - A_R(f)(z)|^p dA(w)$$

Make an obvious change of variables on the right hand side, we obtain

$$(1-|z|^2)^p\,|f'(z)|^p \leq C\int_{D(z,R)} |f(w)-A_R(f)(z)|^p \frac{\left(1-|z|^2\right)^2}{|1-z\bar{w}|^4}dA(w).$$

Since

$$\frac{\left(1-|z|^2\right)^2}{|1-z\bar{w}|^4} \sim \frac{1}{\left(1-|z|^2\right)^2} \sim \frac{1}{|D(z,R)|}$$

for $w\in\Delta(z,R)$, we can find another positive constant C such that

$$(1-|z|^2)^p\,|f'(z)|^p \leq C\;(S_R(f)(z))^p, \quad z\in\Delta$$

It follows that for each $a\in\Delta$ that

$$\sup_{a\in\Delta}\int_\Delta |f'(z)|^p\left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z) \leq C\sup_{a\in\Delta}\int_\Delta (S_R(f)(z))^p\left(1-|z|^2\right)^{p-2} K(1-|\varphi_a(z)|^2)d\tau(z).$$

This shows that the condition (7) implies $f\in Q_K(p,p-2)$.

Recall from [5] that a positive Borel measure μ on Δ is called a K-Carleson measure if

$$\sup_I\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)d\mu(z)<\infty,$$

where the supremum is taken over all sub-arcs $I\subset\partial\Delta$. Here, for a sub-arcs I of $\partial\Delta$, $|I|$ is the length of I and $S(I)=\{r\xi:\xi\in I, 1-|I|<r<1\}$ is the corresponding Carlesson square. Also, A positive Borel measure μ on Δ is called a vanishing $K-$ Carleson measure if

$$\lim_{|z|\to1^-}\int_\Delta K\left(\frac{1-|z|}{|I|}\right)d\mu(z)=0.$$

Theorem 3.3 *Suppose K satisfies the following two conditions:*

(a) There exists a constant $C>0$ such that $K(2t)\leq CK(t)$ for all $t>0$.

(b) The auxiliary function φ_k has the property that

$$\int_0^1 \varphi_k(s)\frac{ds}{s}<\infty.$$

Let μ be a positive Borel measure on Δ. Then μ is a K-Carleson measure if and only if

$$\sup_{a\in\Delta}\int_\Delta K(1-|\varphi_a(z)|^2)d\mu(z)<\infty.$$

Proof. Since $Q_K(p,q)$ is defined by the condition

$$\sup_{a\in\Delta}\int_\Delta |f'(z)|^p\left(1-|z|^2\right)^q K(1-|\varphi_a(z)|^2)dA(z)<\infty,$$

we see that $f\in Q_K(p,q)$ if and only if the measure $\left(1-|z|^2\right)^q|f'(z)|^p dA(z)$ is a K-Carleson measure. The following Corollary gives two analogous characterizations.

Corollary 3.1 *Suppose K satisfies condition (1) and conditions (a) and (b) in Theorem 3.3. Let $R>0$ be a fixed radius. Then the following conditions are equivalent for an analytic function f in Δ.*

(a) The function f belong to $Q_K(p,p-2)$.

(b) The measure $d\mu(z)=(S(f)(z))^p(1-|z|^2)^{p-2}d\tau(z)$ is a K-Carleson measure.

(c) The measure $d\nu(z)=(S_R(f)(z))^p(1-|z|^2)^{p-2}d\tau(z)$ is a K-Carleson measure.

Proof. This is a direct consequence of Theorems 3.1,3.2, and 3.3.
The little-oh version of the above result can be stated as follows:

Theorem 3.4 *Suppose K satisfies condition (1) and $R > 0$ is a fixed, then the following conditions are equivalent for all analytic functions f in Δ.*

(1) $f \in Q_{K,0}(p, p-2)$

(2) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta \frac{|f(z) - f(w)|^p}{|1 - z\bar{w}|^4} \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) = 0$$

(3) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta (S(f)(z))^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$$

(4) $$\lim_{|z|\to 1^-} \int_\Delta \int_\Delta (S_R(f)(z))^p \left(1 - |z|^2\right)^{p-2} K(1 - |\varphi_a(z)|^2) d\tau(z) = 0.$$

References

[1] Y. Ameur, N. Makarov and H. Hedenmalm, Berezin transform in polynomial Bergman spaces, Commun. Pure Appl. Math. 63(12)(2010), 1533 - 1584.

[2] O. Blasco and S. Pérez-Esteva, Schatten-Herz operators, Berezin transform and mixed norm spaces, Integral Equations Oper. Theory 71(1)(2011), 65 - 90.

[3] M. Engliš, and R. Otáhalová, Covariant derivatives of the Berezin transform, Trans. Am. Math. Soc. 363(10)(2011), 5111 - 5129.

[4] M. Essén and H. Wulan, On analytic and meromorphic functions and spaces of Q_K type, Illinois J. Math. 46(2002), 1233 - 1258.

[5] M. Essén, H. Wulan, and J. Xiao, Function-theoretic aspects of mobius invariant Q_K spaces, J. Funct. Anal. 230 (2006), 78 115.

[6] H. Hedenmalm, B. Korenblum, and K. Zhu, Theory of Bergman Spaces, Springer, New York, 2000.

[7] B. Korenblum and K. Zhu, Complemented invariant subspaces in Bergman spaces, J. London Math. Soc. 71(2005), 467 - 480.

[8] K. Stroethoff, Besov-type characterisations for the Bloch space, Bull. Austral. Math. Soc. 39 (1989), 405 - 420.

[9] H. Wulan and K. Zhu, Q_K spaces via higher derivative, The Rocky Mountain Journal of Mathematics, (332)(1)(2008), 329 - 350.

[10] H. Wulan and K. Zhu, Derivative - free characterizations of Q_K spaces, J. Aust. Math. Soc. 82(2007), 283 - 295.

[11] R. Zhao, On a general family of function spaces, Ann. Acad. Sci. Fenn. Math. Diss., 105(1996).

[12] K. Zhu, Operator theory in function spaces, Marcel Dekker, New York, 1990.

Coefficient bounds for certain subclasses of close-to-convex functions of Janowski type

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In this work, we aim to determine the coefficient estimates for functions in certain subclasses of close-to-convex functions of Janowski type and related functions of complex order, which are here defined by means of Cauchy-Euler type non-homogeneous differential equation. Several interesting consequences of our results are also observed.

Key words: Analytic functions; Close-to-convex; Coefficient Estimates.
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1 Introduction

We denote by $\mathcal{A}$ the class of functions $f(z)$ which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_\gamma^*$, $\mathcal{C}_\gamma$, $\mathcal{K}_\gamma$ and $\mathcal{Q}_\gamma$ be the subclasses of $\mathcal{A}$ consisting of all functions which are starlike, convex, close-to-convex and quasi convex of complex order γ ($\gamma \neq 0$) respectively, for details see [3, 5, 7]. We note that for $0 < \gamma \leq 1$, these classes coincide with the well known classes of starlike, convex and close-to-convex of order $1-\gamma$. Recently Altintaş et al.[1] considered the following class of functions denoted by $\mathcal{SC}(\gamma, \lambda, A, B)$ and defined as:

$$\mathcal{SC}(\gamma,\lambda,A,B) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)f(z) + \lambda z f'(z)} - 1, \right) \prec \frac{1+Az}{1+Bz}, z \in E \right\}, \quad (2)$$

where $-1 \leq B < A \leq 1$, $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C} - \{0\}$. Note that the classes $\mathcal{SC}(1,0,A,B) = \mathcal{S}^*[A,B]$ and $\mathcal{SC}(1,1,A,B) = \mathcal{C}[A,B]$ were introduced by Janowski [4] and are called classes of Janowski starlike and Janowski convex functions respectively. Also

$$\mathcal{SC}(\gamma,0,1,-1) = \mathcal{S}_\gamma^*, \mathcal{SC}(\gamma,1,1,-1) = \mathcal{C}_\gamma.$$

Throughout the entire paper onward we assume the restrictions $-1 \leq B < A \leq 1, 0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C} - \{0\}$ unless otherwise mentioned. Now we denote $\mathcal{KQ}(\gamma, \lambda, A, B)$ be the class of functions $f(z) \in \mathcal{A}$ if there exist a function $g(z) \in \mathcal{SC}(1, \lambda, A, B)$ such that

$$1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right) \prec \frac{1+Az}{1+Bz}, z \in E.$$

As special choices we have the following relationships

$$\begin{aligned} \mathcal{KQ}(1,0,A,B) &= \mathcal{K}[A,B], \quad \mathcal{KQ}(1,1,A,B) = \mathcal{Q}[A,B], \textit{see [Noor, [6]]} \\ \mathcal{KQ}(\gamma,0,1,-1) &= \mathcal{K}_\gamma, \quad \mathcal{KQ}(\gamma,1,1,-1) = \mathcal{Q}_\gamma. \end{aligned}$$

Motivated from the recent work of Srivastava et al. [9] and Altintaş et al. [2] the main purpose of our investigation is to derive coefficient estimates of a subfamily $\mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$ of $\mathcal{A}$, which consists of functions $f(z)$ in $\mathcal{A}$ satisfying the following Cauchy Euler type non homogenous differential equation

$$z^m \frac{d^m w}{dz^m} + {}^mC_1(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots {}^mC_m w \prod_{j=0}^{m-1}(\mu+j) = h(z)\prod_{j=0}^{m-1}(\mu+j+1), \quad (3)$$

where $w = f(z)$, $h(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$, $\mu \in \mathbb{R} - (-\infty, -1]$, $m \in N^* = \{2, 3, \cdots\}$ for details we refer to [2, 8, 9, 10, 11]. The following result which is due to Altintaş et al. [2] is essential in deriving our main results.
Lemma 1. [2]. *Let* $f(z) \in \mathcal{SC}(\gamma, \lambda, A, B)$ *and be of the form* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j + \frac{2|\gamma|(A-B)}{1-B} \right]}{(n-1)![1+\lambda(n-1)]}, \; n \in N^*.$$

2 Coefficient Estimates for functions in the class $\mathcal{KQ}(\gamma, \lambda, A, B)$

We first establish the below result for the functions in the class $\mathcal{KQ}(\gamma, \lambda, A, B)$.
Theorem 1. *Let* $f(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$ *and be defined by* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n![1+\lambda(n-1)]} + \frac{2|\gamma|}{n[1+\lambda(n-1)]}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!},\ n\in N^*. \quad (4)$$

Proof. Since $f(z) \in \mathcal{KQ}(\gamma, \lambda, A, B)$, then there exists $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belonging to the class $\mathcal{SC}(1, \lambda, A, B)$ such that

$$1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) \prec \frac{1+Az}{1+Bz},\ for z \in E,$$

where $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $G(z) = z + \sum_{n=2}^{\infty} B_n z^n$, with

$$A_n = [1 + \lambda(n-1)]a_n,\ B_n = [1 + \lambda(n-1)]b_n. \quad (5)$$

Let

$$1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)} - 1\right) = q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ for z \in E, \quad (6)$$

Since $q(z) \prec \frac{1+Az}{1+Bz}$, $z \in E$, we find that by definition of subordination

$$q(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, w(0) = 0;\ |w(z)| < 1.$$

Therefore, we have

$$|w(z)| = \left|\frac{q(z) - 1}{A - Bq(z)}\right| < 1, \quad q(z) = u + iv,$$

which further implies that

$$2u(1 - AB) > 1 - A^2 + (1 - B^2)(u^2 + v^2).$$

Also, since $|q(z)|^2 \geq (Req(z))^2$, we have

$$(1 - B^2)u^2 - 2u(1 - AB) + 1 - A^2 < 0 \Longrightarrow Req(z) > \frac{1-A}{1-B}. \quad (7)$$

From (6) and (7), we find that

$$|c_n| \leq 2\left(\frac{A-B}{1-B}\right),\ n \in \mathtt{N}. \quad (8)$$

Then from (6), we obtain

$$nA_n = B_n + \gamma\left[c_{n-1} + \sum_{k=1}^{n-1} c_k B_{n-k}\right], n \geq 2$$

Now using Lemma 1 together with (5) and (8), we have

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n!} + \frac{2\,|\gamma|}{n}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!},$$

and hence from the relation between $F(z)$ and $f(z)$ as in (5), we obtain the desired result. By assigning different specific values to the involved parameters $A, B,\ \gamma,\ \lambda$ in Theorem 1, we deduce the following interesting results. **Corollary 1.** *Let $f(z) \in \mathcal{KQ}(1,0,A,B) = \mathcal{K}[A,B]$ and be defined by* (1). *Then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n!} + \frac{2}{n}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!}\ n \in N^*.$$

Corollary 2. *Let $f(z) \in \mathcal{KQ}(\gamma,\lambda,1,-1)$ and be defined by* (1). *Then*

$$|a_n| \leq \frac{1}{[1+\lambda(n-1)]}\left[1+(n-1)\,|\gamma|\right],\ n \in N^*.$$

Corollary 3 [3]. *Let $f(z) \in \mathcal{KQ}(\gamma,0,1,-1) = \mathcal{K}(\gamma)$ and be defined by* (1). *Then*

$$|a_n| \leq 1+(n-1)\,|\gamma|\,,\ n \in N^*.$$

Corollary 4 [5]. *Let $f(z) \in \mathcal{KQ}(\gamma,1,1,-1) = \mathcal{Q}(\gamma)$ and be defined by* (1). *Then for $n \in N^* = \{2,3,4,\ldots\}$.*

$$|a_n| \leq \frac{1+(n-1)\,|\gamma|}{n},\ n \in N^*.$$

For $\gamma = 1$ in Corollary 2 and Corollary 3, we obtain the well-known coefficient estimates for close-to-convex and quasi convex functions. **Corollary 5.** *Let $f(z) \in \mathcal{KQ}(1-\alpha,\lambda,1-2\beta,-1)$ and be defined by* (1). *Then for $n \in N^*$*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+2(1-\beta)\right]}{n![1+\lambda(n-1)]} + \frac{2(1-\alpha)(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+2(1-\beta)\right]}{(n-k-1)!}.$$

Corollary 6. *Let $f(z) \in \mathcal{KQ}(1-\alpha,0,1,-1) = \mathcal{K}(1-\alpha)$ and be defined by* (1). *Then*

$$|a_n| \leq n(1-\alpha)+\alpha, \quad n \in N^* = \{2,3,4,\ldots\}.$$

Corollary 7. *Let $f(z) \in \mathcal{KQ}(1-\alpha,1,1,-1) = \mathcal{Q}(1-\alpha)$ and be defined by* (1). *Then*

$$|a_n| \leq 1-\alpha+\frac{\alpha}{n},\ n \in N^* = \{2,3,4,\ldots\}.$$

3 Coefficient Estimates of the class $\mathcal{BK}(\gamma, \lambda, A, B; \mu)$

The theorem below is our main coefficient estimates for functions in the class $\mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$.

Theorem 2. *Let* $f(z) \in \mathcal{DK}(\gamma, \lambda, A, B, m; \mu)$ *and be defined by* (1). *Then for* $n \in N^* = \{2, 3, 4, \ldots\}$

$$|a_n| \leq \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)}\left[\frac{\prod_{j=0}^{n-2}\left[j+\frac{2(A-B)}{1-B}\right]}{n![1+\lambda(n-1)]}+\frac{2|\gamma|}{n[1+\lambda(n-1)]}\frac{A-B}{1-B}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}\left[j+\frac{2(A-B)}{1-B}\right]}{(n-k-1)!}\right]. \tag{9}$$

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}(\gamma, \lambda, \beta)$,so that

$$a_n = \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)} b_n, \quad n \in N^*, \ \mu \in \mathrm{R} - (-\infty, -1].$$

Hence, by using Theorem 1, we immediately obtain the required inequality (9).

Corollary 8. *Let* $f(z) \in \mathcal{DK}(\gamma, \lambda, 1-2\beta, -1, 2; \mu)$ *and be defined by* (1). *Then for* $n \in N^* = \{2, 3, 4, \ldots\}$

$$|a_n| \leq \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)}\left[\frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]}+\frac{2|\gamma|(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!}\right].$$

References

[1] O. Altıntaş, Ö. Özkan, H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Lett. 13 (1995) 63–67.

[2] O. Altıntaş, H. Irmak, S. Owa, H.M. Srivastava, Coefficient bounds for some families of starlike and convex functions of complex order, Appl. Math. Lett. 20 (2007) 1218-1222.

[3] H. S. Al-Amiri, T. S Fernando, On close-to-convex functions of complex order, Internat. J. Math. and Math. Sci. 13(1990)321-330.

[4] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Polon. Math. 28 (1973), 297–326.

[5] O. S. Kown, S. Owa, .On quasi convex functions of complex order, Soochow J. Math. 20 (1994)241-250.

[6] K. I. Noor, Radius problems for a subclass of close-to-convex univalent functions, Internat. J. Math. Math. Sci., 15(4)(1992)719-726.

[7] K. I. Noor; Quasi-convex functions of complex order, PanAmer. Math. J. 3(2)(1993)81-90.

[8] H. M. Srivastava , S. Owa, Certain classes of analytic functions with varying arguments, J. Math. Appl. 136(1988)217-228.

[9] H.M. Srivastava, O. Altıntaş, S. K. Serenbay, Coefficient bounds for certain subclasses of starlike functions of complex order, Appl. Math. Lett. 24(2011)1359-1363.

[10] H.M. Srivastava, Q.-H. Xu, G.-P. Wu, Coefficient estimates for certain subclasses of spiral-like functions of complex order, Appl. Math. Lett. 23 (2010)763–768.Tokyo, 1983.

[11] H.M. Srivastava, S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[12] P. Wiatrowski, On the coefficient of some family of holomorphic functions, Zeszyty Nauk. Uniw. Lodz Nauk. Mat.-Przyrod Ser. 39(2)(1970)75-85.

Five-order Extrapolation Algorithms for Laplace Equation with Linear Boundary Condition*

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Abstract

Laplace equation with linear boundary condition will be converted into a boundary integral equation(BIE) with logarithmic singularity following potential theory. In this paper, a Sidi quadrature formula is introduced to approximate the logarithmic singularity integral operator with $O(h^3)$ approximate accuracy order. A similar approximate equation is also constructed for the logarithmic singular operator, which is based on coarse grid with mesh width $2h$. So an extrapolation algorithm is applied to approximate the logarithmic operator and the accuracy order is improved to $O(h^5)$. Moreover, the accuracy order is based on fine grid h. The convergence and stability are proved based on Anselone's collective compact and asymptotic compact theory. Furthermore, an asymptotic expansion with odd powers of the errors is presented with convergence rate $O(h^5)$. Using h^5-Richardson extrapolation algorithms(EAs), not only the approximation accuracy order can be improved to $O(h^7)$, but also an a posteriori error estimate can be obtained for constructing a self-adaptive algorithm. numerical examples are shown to verify its efficiency.

Keywords: boundary integral equation, Richardson extrapolation algorithm, Laplace equation, a posteriori error estimate

2000 MSC: 65N25, 65N38

1 Introduction

Laplace equation with linear boundary condition is defined as follows: to find non-zero deformation $\tilde{u}$ in the domain Ω and on the boundary Γ satisfying

$$\begin{cases} \triangle\tilde{u} = 0, \text{ in } \Omega, \\ \dfrac{\partial\tilde{u}}{\partial n} = -c\tilde{u}(x) + \tilde{f}(x), \text{ on } \Gamma, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, $\partial/(\partial n)$ is an normal outward derivative on Γ, c is a positive constant, and $\tilde{f}(x)$ is a given function.

By means of potential theory, Eq.(1) will be transformed into a boundary integral equation(BIE) as follows[1,2,3]:

$$\alpha(y)\tilde{u}(y) - \int_\Gamma k^*(y,x)\tilde{u}(x)ds_x = \int_\Gamma h^*(y,x)\frac{\partial\tilde{u}(x)}{\partial n_x}ds_x, \quad y \in \Gamma, \tag{2}$$

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where $\alpha(y) = \theta(y)/(2\pi)$ is related to the interior angle $\theta(y)$ of Ω at point $y \in \Gamma$, in particular, when y is on a smooth part of the boundary Γ, $\alpha(y) = 1/2$, and $h^*(y, x)$ is the fundamental solution:

$$\begin{cases} h^*(y,x) = -\dfrac{1}{2\pi}\ln|x-y|, \\ k^*(y,x) = -\dfrac{\partial h^*(y,x)}{\partial n}, \end{cases} \tag{3}$$

where $|x - y|$ is the distance between points x and y.

The left terms in Eq.(2) are smooth integrals and the right hand side term is characterized as a logarithmic singularity. Various numerical methods have been proposed for dealing with the singularity, such as Galerkin methods in Stephan and Wendland[4], Chandler[5], Sloan and Spence[6], and Amini and Nixon[7], collocation methods in Elschner and Graham[8] and Yan[9], quadrature methods in Sidi and Israeli[10], Saranen[11], Huang and Lü[12,13] and combined Trefftz methods in Li[14].

Extrapolation algorithms (EAs) based on asymptotic expansion about errors are effective parallel algorithms, which possesses high accuracy degree, good stability and almost optimal computational complexity. Cheng et al.[15,16] harnessed extrapolation algorithms to obtain high accuracy order for Steklov eigenvalue in Laplace equations with smoothed and polygonal boundary condition. Huang and Lü established extrapolation algorithms for solving the Steklov eigenvalue problems[3],the Helmholtz equations[17] and the Laplace equations[18] with accuracy order $O(h^3)$. After the Extrapolation algorithms, the accuracy order of the approximate solution will be improved to $O(h^5)$.

A quadrature method[19,20] is presented for solving the boundary integral equation, in which the generation of the discrete matrixes does not require any calculations of singular integrals. The logarithmic integral kernel is approximated by extrapolation algorithms derived from Sidi's quadrature rule. An asymptotic expansion about the error is obtained with convergence rate $O(h^5)$.

Note that the five order approximate solution is obtained directly and is based on fine grid h. Although there are some papers[17−20] also obtain the same accuracy order, there are three main priority for our paper: firstly, those accuracy orders are based on fine grid; secondly, because the accuracy order is not derived from the extrapolation algorithms but from the directly calculation, so there are not any errors generated from the extrapolation algorithms; finally, when an linear equation with n order is solved, there are n approximate solutions u_h can be obtained on boundary Γ with accuracy order $O(h^5)$, while not $n/2$ values from extrapolation method.

This paper is organized as follows: In Section 2 a Sidi's quadrature method is recombined to approximate integral equations for solving the approximate solution; In Section 3 an asymptotically compact theory is provided for stability and convergence, and an asymptotic expansion for approximate solution is shown with convergence rate $O(h^5)$; In Section 4 the Richardson extrapolation algorithms are applied to improve the accuracy order to $O(h^7)$; In Section 5 numerical examples illustrate the calculate progress.

2 Five order approximate methods

Assume that Γ is a smooth closed curve described by a regular parameter mapping $x(s) = (x_1(s),\ x_2(s)) : [0, 2\pi] \to \Gamma$, satisfying $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$. Let $C^{2m}[0, 2\pi]$ denote the set of $2m$ times differentiable periodic functions with the periodic 2π and $x_i(s) \in C^{2m}[0, 2\pi], i = 1, 2$. Define the following integral operators on $C^{2m}[0, 2\pi]$:

$$\begin{cases} (Ku)(s) = 2\displaystyle\int_0^{2\pi} k(t,s)\frac{\partial u(t)}{\partial n}dt \\ (Hu)(s) = 2\displaystyle\int_0^{2\pi} h(t,s)u(t)dt. \end{cases}$$

where $u(t) = \tilde{u}(x_1(t), x_2(t))$, $k(t,s) = k^*(x(t), x(s))\ |x'(t)|$ and $h(t,s) = h^*(x(t), x(s))|x'(t)|$. Because

$$h(t,s) = -\frac{1}{2\pi} \ln |x(t) - x(s)||x'(t)|,$$

so $h(t,s)$ is a logarithmic weak singular kernel and $k(t,s)$ is a smooth kernel. Then Eq.(2) is equivalent to

$$(I - K)u - cHu = Hf \tag{4}$$

where I is an identity operator, and $f = \tilde{f}(x(t))$.

Let $h = 2\pi/n$ ($n \in N$ is supposed to be an even number and so $n/2 \in N$) be the mesh width and $t_j = s_j = jh$, $(j = 0, 1, \dots, n-1)$ be the nodes. In order to approximate the integral operators K and H, a Lemma is obtained:

Lemma 1:[19] *Consider the integral $\int_0^{2\pi} G(x)dx$ with integral kernel $G(x)$. Assume that the functions $g(x)$, $\tilde{g}(x)$ are $2m$ times differentiable on $[0, 2\pi]$. Also assume that the integral kernel $G(x)$ are periodic function with period 2π. Then the following conclusion can be drawn:*

(a). If $G(x) = g(x)/(x-t) + \tilde{g}(x)$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j)$, then

$$E_n[G] = h[\tilde{g}(t) + g'(t)] + O(h^{2m}) \ \ \text{as} \ \ h \longrightarrow 0,$$

where $E_n[G] = \int_0^{2\pi} G(x)dx - Q_n[G]$ in all cases;

(b). If $G(x) = g(x)(x-t)^s + \tilde{g}(x), s > -1$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j) + h\tilde{g}(t) - 2\zeta(-s)g(t)h^{s+1}$, then

$$E_n[G] = -2\sum_{\mu=1}^{m-1} \frac{\zeta(-s-2\mu)}{(2\mu)!} g^{(2\mu)}(t) h^{2\mu+s+1} + O(h^{2m}), \ \ \text{as} \ \ h \to 0;$$

where $\varsigma(t)$ is the Riemann zeta function.

(c). If $G(x) = g(x)(x-t)^s \log|x-t| + \tilde{g}(x), s > -1$, and $Q_n[G] = h\sum_{j=1, x_j \neq t}^{n} G(x_j) + h\tilde{g}(t) + 2[\zeta'(-s) - \zeta(-s)\log h]g(t)h^{s+1}$, then

$$E_n[G] = -2\sum_{\mu=1}^{m-1} [\zeta'(-s-2\mu) - \zeta(-s-2\mu)\log h] \frac{g^{(2\mu)}(t)}{(2\mu)!} h^{2\mu+s+1} + O(h^{2m}), \ \ \text{as} \ \ h \to 0;$$

Especially, when $s = 0$, then $\zeta'(0) = -(1/2)\log(2\pi)$, and we have

$$Q_n[G] = h\sum_{j=1, x_j = t}^{n} G(x_j) + h\tilde{g}(t) + \log\left(\frac{h}{2\pi}\right) g(t)h,$$

then

$$E_n[G] = 2\sum_{\mu=1}^{m-1} \zeta'(-2\mu) \frac{g^{(2\mu)}(t)}{(2\mu)!} h^{2\mu+1} + O(h^{2m}), \ \ \text{as} \ \ h \to 0.$$

Since K is a smooth integral operator with period 2π, we obtain a high accuracy approximation when set $g(x) \equiv 0$ in case (a) of Lemma 1:

$$(K_h u)(s) = h\sum_{j=0}^{n-1} k(t_j, s)u(t_j), \tag{5}$$

with the error estimate

$$(Ku)(s) - (K_h u)(s) = O(h^{2m}). \tag{6}$$

For the logarithmic weak singular operator H, the continuous approximation of its kernel $h_n(t,\tau)$ is defined as:

$$h_n(t,s)=\begin{cases} h(t,s), & |t-s|\geq h,\\ \ln\left(\frac{h}{2\pi}|x'(s)|\right), & |t-s|<h, \end{cases} \tag{7}$$

so its approximation operator can be obtained when set $\tilde{g}(x)\equiv 0$ and $s=0$ in case (c) of Lemma 1:

$$(H_h u)(s)=h\sum_{j=0}^{n-1} h_n(t_j,s)u(t_j), \tag{8}$$

which has the following error estimate:

$$(Hu)(s)-(H_h u)(s)=2h^3\frac{\varsigma'(-2)}{2!}u^{(2)}+2\sum_{\mu=2}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}u^{(2\mu)}(s)h^{2\mu+1}+O(h^{2m}). \tag{9}$$

We can find that there is an asymptotic expansion with accuracy order $O(h^3)$ for the logarithmic singular operator. In order to improve the accuracy order from $O(h^3)$ to $O(h^5)$, a coarse grid $2h=2\pi/(n/2)=4\pi/n$ is obtained. The approximate operator based on coarse grid $2h$ is shown as:

$$(H_{2h}u)(s)=2h\sum_{j=0}^{n-1} h_n(t_j,s)u(t_j)\vartheta_j,$$

where

$$\vartheta_j=\begin{cases} 0, & \text{j is an odd number},\\ 1, & \text{j is an even number}. \end{cases}$$

The error estimate is:

$$\begin{aligned}(Hu)(s)-(H_{2h}u)(s)&=2(2h)^3\frac{\varsigma'(-2)}{2!}u^{(2)}\\ &+2\sum_{\mu=2}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}u^{(2\mu)}(s)(2h)^{2\mu+1}+O((2h)^{2m}).\end{aligned} \tag{10}$$

An extrapolation algorithm is used to counteract the item $O(h^3)$ in Eqs (9) and (10):

$$(J_h u)(s)=\frac{8}{7}(H_h u)(s)-\frac{1}{7}(H_{2h}u)(s).$$

The error for the approximate operator will be improved from $O(h^3)$ to $O(h^5)$:

$$(Hu)(s)-(J_h u)(s)=\sum_{\mu=2}^{m-1}\eta_\mu h^{2\mu+1}+O(h^{2m}), \tag{11}$$

where η_μ is some coefficients combination of the item $h^{2\mu+1}$. So the accuracy order is not only improved to $O(h^5)$, but also built on the fine grid h.

Thus we obtain the numerical approximate equations of Eq.(4):

$$(I-K_h)u_h-cJ_h u_h=J_h f_h, \tag{12}$$

where K_h and J_h are discrete matrices of order n corresponding to the operators K and H, respectively.

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3 Asymptotical compact convergence

According to the logarithmic capacity theory[3], the eigenvalues of K and K_h do not include 1. Then the Eqs. (4) and (12) can be rewritten as follows: find $u \in C[0, 2\pi]$ satisfying

$$(I - L)u = \varphi, \tag{13}$$

and find u_h satisfying

$$(I - L_h)u_h = \varphi_h, \tag{14}$$

where $L = c(I-K)^{-1}H$, $L_h = c(I-K_h)^{-1}J_h$, $\varphi = (I-K)^{-1}\ Hf$ and $\varphi_h = (I-K_h)^{-1}J_h f_h$.

Theorem 1. *The approximate operator sequence $\{L_h\}$ is an asymptotical compact[21,22] sequence and convergent to L in $C[0, 2\pi]$, i,e.*

$$L_h \overset{a.c}{\to} L, \tag{15}$$

where $\overset{a.c}{\to}$ means the asymptotically compact convergence.

This proof can be obtained similarly as the proofs in the papers[15,16].

Corollary[13,15] 1. *Under the assumption of Theorem 1, we obtain*

$$\begin{cases} \|(L_h - L)L\| \to 0 \\ \|(L_h - L)L_h\| \to 0, \ \text{as}\ h \to 0. \end{cases}$$

4 Asymptotic expansions of the approximate solutions

Theorem 2. *Suppose $u(s) \in C^{(2m)}[0, 2\pi]$, then we have the following asymptotic expansion*

$$(L_h - L)u(s) = \sum_{j=2}^{m-1} \psi_j(s)h^{2j+1} + O(h^{2m}), \tag{16}$$

where $\psi_j(s) \in C^{(2m-2j)}, j = 2, \ldots, m-1$, are functions independent of h.

Proof. According to properties of the approximate operators, there is

$$(Ku)(s) - (K_h u)(s) = O(h^{2m}). \tag{17}$$

and

$$(Hu)(s) - (J_h u)(s) = \sum_{j=2}^{m-1} \eta_j(s)h^{2j+1} + O(h^{2m}). \tag{18}$$

We consider the relationship between L_h and L:

$$\begin{aligned}(L_h - L)u &= c(I - K_h)^{-1}J_h u - c(I - K)^{-1}Ju \\ &= c(I - K_h)^{-1}J_h u - (I - K)^{-1}J_h u + c(I - K)^{-1}J_h u - (I - K)^{-1}Ju \\ &= c[(I - K_h)^{-1} - (I - K)^{-1}]J_h u + c(I - K)^{-1}(J_h - H)u \\ &= c(I - K)^{-1}(K_h - K)(I - K_h)^{-1}J_h u + c(I - K)^{-1}(J_h - H)u\end{aligned}$$

Substituting the errors of Eqs. (19) and (20) into the above equation, and setting $\psi_j(s) = c(I - K)^{-1}\eta_j(s)$, we complete the proof of Theorem 2. □

Theorem 3. *Suppose $x(t), g(t) \in C^{2m}[0,2\pi]$, Then there exists functions $\bar{\omega}_l \in C^{2m-2l}[0, 2\pi], l = 1,\ldots,m$ independent of h, such that*

$$(u-u_h)|_{t=t_j} = \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_l|_{t=t_j} + O(h^{2l}) \tag{19}$$

Proof. Because $(I-K)^{-1}$ is exist, and $J_h \overset{a.c}{\to} H$, so there is an asymptotic expansion for function φ:

$$(\varphi-\varphi_h)|_{t=t_j} = h^5\omega_2|_{t=t_j} + h^7\omega_3|_{t=t_j} + \ldots + O(h^{2m}), \tag{20}$$

where $\omega_l \in C^{2m-2l}[0,2\pi], l = 2,\ldots,m-1$.

Because u and u_h satisfy Eqs. (13) and (14) respectively, we obtain

$$\begin{aligned}(I-L_h)&(u_h-u)|_{t=t_j}\\ &= \Big[(I-L_h)u_h - (I-L)u + (I-L)u - (I-L_h)u\Big]\Big|_{t=t_j}\\ &= (\varphi_h-\varphi)|_{t=t_j} + (L-L^h)u|_{t=t_j} = h^5\phi_2|_{t=t_j} + \ldots + O(h^{2m}),\end{aligned} \tag{21}$$

where $\phi_l = \omega_l + \psi_l, l = 2,\ldots,m-1$.

Define an auxiliary equation

$$(I-L)\bar{\omega}_l = \phi_l, \quad l = 2,\ldots,m-1, \tag{22}$$

and its approximate equation

$$(I-L_h)\bar{\omega}_{lh} = \phi_{lh}, \quad l = 2,\ldots,m-1. \tag{23}$$

Substituting Eq. (25) into Eq. (23), we obtain

$$(I-L_h)(u_h - u - \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_{lh})|_{t=t_j} = O(h^{2m}). \tag{24}$$

Noticing $\bar{\omega}_{lh} \in C^{2m-2l}[0,2\pi]$, we have

$$(\bar{\omega}_l - \bar{\omega}_{lh})(t_i) = O(h^{2m-2l}). \tag{25}$$

When substitute $\bar{\omega}_{lh}$ by $\bar{\omega}_l$ and consider the asymptotic compact properties[21], we obtain

$$\big(u_h - u - \sum_{l=2}^{m-1} h^{2l+1}\bar{\omega}_l\big)|_{t=t_j} = O(h^{2m}), \tag{26}$$

so the proof is completed. □

The asymptotic expansion in Eq. (21) implies that the Richardson extrapolation[23] can be applied to improve the accuracy order. A higher accuracy order $O(h^7)$ can be obtained by computing some approximation on Γ in parallel. It can be described as follows:

Taking h and $h/2$ to solve Eq. (12) in parallel, we obtain that $u_h(t_i), u_{h/2}(t_i)$ are the solutions on Γ. According to the asymptotic expansion, we obtain

$$u_h^*(t_i) = \frac{1}{31}(32u_{h/2}(t_i) - u_h(t_i)), \tag{27}$$

and the error is $|u_h^*(t_i) - u(t_i)| = O(h^7)$.

Moreover, using $|u_h^*(t_i) - u(t_i)| = O(h^7)$, we obtain a posteriori error estimate

$$\begin{aligned}
&|u(t_i) - u_{h/2}(t_i)| \\
&\quad\le |u(t_i) - \frac{1}{32}(32u_{h/2}(t_i) - u_h(t_i))| \\
&\qquad + \frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)| \\
&\quad\le \frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)| + O(h^7).
\end{aligned}$$

Note that the upper limitation $\frac{1}{31}|u_{h/2}(t_i) - u_h(t_i)|$ can be used to construct self-adaptive algorithms.

5 Numerical examples

In this section, we consider some computational aspects of the approximate equation and present two examples to illustrate the accelerated convergence of the extrapolation algorithms.

Example 1[24]: Consider the boundary value problem satisfying

$$\begin{cases} \triangle\tilde{u} = 0, \text{ in } \Omega, \\ \dfrac{\partial\tilde{u}}{\partial n} = -\tilde{u}(x) + \tilde{f}(x), \text{ on } \Gamma, \end{cases} \tag{28}$$

where $\tilde{f}(x) = 1$ and Ω is the region

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 < 1, \tag{29}$$

with $(a, b) = (1, 2)$. The boundary Γ can be described as: $x_1 = \cos t, x_2 = 2\sin t, 0 \le t \le 2\pi$. So the analyzed solution will be obtained as $u(x) \equiv 1$.

This problem is calculated in paper [24] by Nyström method. The results is listed in Table 1 and it shows that the convergent rate is three order. The denotes in Table 1 represent the following means: $t_i = 2i\pi/10$, with $i = 1, \dots, 10$; e_i is the errors at t_i; and $\texttt{rate} = \log_2 \frac{e_i(h)}{e_i(h/2)}$.

Table 1: Errors of the Nyström solutions in paper [24].

t_i	e_i with $h = \frac{2\pi}{10}$	e_i with $h = \frac{2\pi}{20}$	rate	e_i with $h = \frac{2\pi}{40}$	rate
0.628319	0.121881E-02	0.131313E-03	3.21	0.162984E-04	3.01
1.256637	-0.241312E-02	-0.350908E-03	2.78	-0.439971E-04	3.00
1.884956	-0.241325E-02	-0.350658E-03	2.78	-0.442431E-04	2.99
2.513274	0.121870E-02	0.131397E-03	3.21	0.162478E-04	3.02
3.141593	0.163276E-02	0.189617E-03	3.11	0.236295E-04	3.00
3.769911	0.121862E-02	0.132229E-03	3.20	0.171674E-04	2.95
4.398230	-0.241311E-02	-0.351662E-03	2.78	-0.435820E-04	3.01
5.026548	-0.241343E-02	-0.351146E-03	2.78	-0.437735E-04	3.00
5.654867	0.121874E-02	0.131003E-03	3.22	0.163326E-04	3.00
6.283185	0.163256E-02	0.189412E-03	3.11	0.236443E-04	3.00

We calculate the boundary numerical solutions u_h on Γ following Eq. (12). The boundary is divided into $5 * 2^n$ with $n = 0, 1, 2, \dots$ pieces For convenience, we introduce some denotes:

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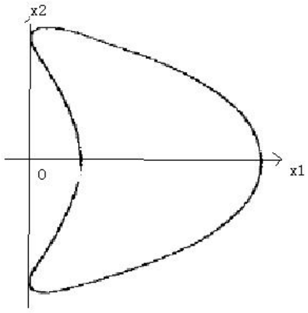

Figure 1: Boomerang-shaped domain for numerical example 2.

$e^h(P) = |u_h(P) - u(P)|$ is the error of the displacement; $r^h(P) = \log_2 e^h(P)/e^{h/2}(P)$ is the error ratio; $\bar{e}^h(P) = |u_h^*(P) - u(P)|$ is the error after Richardson extrapolation, and $p^h(P) = \frac{1}{31}|u_{h/2}(P) - u_h(P)|$ is a posteriori error estimate.

Table 2 lists the approximate values of $u_h(P)$ at points $P_1 = (a\cos 0, b\sin 0)$, $P_2 = (a\cos(\pi/5), b\sin(\pi/5))$ and $P_3 = (a\cos(2\pi/5), b\sin(2\pi/5))$.

Table 2: The errors, errors ratio of e^h, r^h and a posteriori error estimate p^h, at points $P = P_1, P_2, P_3$.

n	5	10	20	40	80
$e^h(P_1)$	2.043E-04	6.117E-06	1.848E-07	5.634E-09	1.728E-10
$r^h(P_1)$		5.062	5.049	5.036	5.027
$p^h(P_1)$		6.133E-06	1.852E-07	5.634E-09	1.727E-10
$e^h(P_2)$	7.203E-04	2.126E-05	6.419E-07	1.959E-08	6.061Ee-10
$r^h(P_2)$		5.102	5.050	5.034	5.014
$p^h(P_2)$		2.226E-05	6.428E-07	1.966E-08	6.061Ee-10
$e^h(P_3)$	4.726E-04	1.378E-05	4.096E-07	1.256E-08	3.886E-10
$r^h(P_3)$		5.100	5.073	5.028	5.014
$p^h(P_3)$		1.389E-05	4.106E-07	1.257E-08	3.886E-10

From Table 2, we can numerically see $r^h \approx 5$, that means the convergent rate is almost five order, which agrees with Theorem 3 very well.

Table 3. the errors $e^h(\theta), \tilde{e}^h(\theta)$ and errors ratio $r^h(\theta)$ when $\theta_1 = 0, \theta_2 = \pi/5$ on Γ.

n	5	10	20	40	80
$e^h(\theta_1)$	6.138E-4	1.779E-5	5.264E-7	1.585E-8	4.863E-10
$r^h(\theta_1)$		5.109	5.079	5.053	5.027
$p^h(\theta_1)$		1.791E-5	5.288E-7	1.585E-8	4.854E-10
$e^h(\theta_2)$	5.413E-4	1.574E-5	4.690E-7	1.413E-8	4.351E-10
$r^h(\theta_2)$		5.104	5.068	5.052	5.022
$p^h(\theta_2)$		1.632E-5	4.728E-7	1.403E-8	4.350E-10

Example 2[15]: Consider another boundary value problem with a non-convex boomerang-shaped cross section boundary. Similar problem is discussed for Helohmotz equation with nonlinear boundary condition in the same domain in paper [15]. The boundary Γ is illustrated in Fig.1 and described by the parametric representation:

$$x(t) = (x_1(t), x_2(t)) = (\cos t + 0.65\cos 2t + 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

We set $c = 2$ and $f = (1.5\cos t + \sin t + 1.3\sin 2t)/\sqrt{w} + 2(\cos(t) + 0.65\cos(2t) + 1.5\sin(t))$ with $w = (1.5\cos t)^2 + (\sin t + 1.3\sin 2t)^2$. Then the analytic solution is $u(t) = x_1(t) + x_2(t) = \cos t + 0.65\cos 2t + 0.65 + 1.5\sin t, t \in [0, 2\pi]$.

In Table 3 we list some errors of the $u_h(y)$ on Γ computed by formulae (14) and then the u_h at arbitrary point in Ω can be obtained following Eq.(15). We also use the denotes as used in Table 1. Evidently, from Table 3, a similar conclusion can be obtained as example 1 done.

Conclusion

Generally, there are three main advantages for the Sidi's quadrature method:

(1) Evaluating each element of discretization matrices is very simple and straightforward, which does not require any singular integrals;

(2)We can obtain a high accuracy order $O(h^5)$ and an asymptotic expansion of the errors with odd powers, which are based on fine grid h. Harnessing the Richardson extrapolation algorithms, a higher accuracy order $O(h^7)$ can be obtained.

(3)The accuracy order $O(h^5)$ of the approximate solution is obtained directly, which avoid the errors derived from the extrapolation algorithms as some articles have done.

References

[1] C.A. Brebbia, J. Telles, L. Wrobel, Boundary element techniques : theory and applications in engineering. Springer-Verlag. New York, 1984.

[2] Pan Cheng, Jin Huang and Guang Zeng, High accuracy eigensolution and its extrapolation for potential equations Appl. Math. Mech., 31(2010), 1527-1536.

[3] P.Cheng, J.Huang, G.Zeng, Splitting extrapolation algorithms for solving the boundary integral equations of Steklov problems on polygons by mechanical quadrature methods, Eng. Anal. Bound. Elem., 35(2011), 1136-1141.

[4] E.P. Stephan, W.L. Wendland, An augment Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. Appl. Anal., 18(1984),183-219.

[5] G. Chandler, Galerkin's method for boundary integral equations on polygonal domains. J. Aust. Math. Soc. Sor. B.V, 26(1984),1-13.

[6] I.H. Sloan, A. Spence, The Galerkin method for integral equations of first-kind with logarithmic kernel: theory. IMA J. Numer. Anal., 8(1988), 105-122.

[7] S. Amini, S.P. Nixon, Preconditioned multiwavelet Galerkin boundary element solution of Laplace's equation, Eng. Anal. Bound. Elem., 7(2006), 523-530.

[8] J. Elschner, I. Graham, An optimal order collocation method for first kind boundary integral equations on polygons. Numer. Math., 70(1995), 1-31.

[9] Y. Yan, The collocation method for first-kind boundary integral equations on polygonal regions. Math. Comput., 54(1990), 139-154.

[10] A. Sidi, M. Israeli, Quadrature methods for periodic singular Fredholm integral equations, J. Sci. Comput., 3(1988), 201-231.

[11] J. Saranen, The modified quadrature method for logarithmic kernel integral equations on closed curves. J. Int. Eqns. Appl., 3(1991), 575-600.

[12] P.Cheng, J. Huang, Extrapolation algorithms for solving n onlinear boundary integral equations by mechanical quadrature methods, Numerical Algorithm 58(2011), 545-554.

[13] P.Cheng, X. Luo, Z. Wang, J. Huang, Mechanical quadrature methods and extrapolation algorithms for boundary integral equations with linear boundary conditions in elasticity. J. of elasticity, 108(2012), 193-207.

[14] J. Huang, Z.C. Li , I.L. Chen, A.H.D. Cheng, Advanced quadrature methods and splitting extrapolation algorithms for first kind boundary integral equations of Laplace's equation with discontinuity solutions. Eng. Anal. Bound. Elem., 34(2010), 1003-1008.

[15] T. Lü, J. Huang, High accuracy Nyström approximations and their extrapolation for solving weakly singular integral equations of the second kind, J. Chinese Comp. Phy., 3(1997), 349-355.

[16] Z.C. Li, Combinations of method of fundamental solutions for Laplace's equation with singularities. Eng. Anal. Bound. Elem., 10(2008), 856-869.

[17] P.Cheng, J.Huang, Z.Wang, Nystrom methods and extrapolation for solving Steklov eigensolutions and its application in elasticity Numer. Meth. Part. Diff. Equ., 28(2012), 2021-2040.

[18] P.Cheng, J.Huang, G.Zeng, High Accuracy Eigensolutions in Elasticity for Boundary Integral Equations by Nystrom Method, Inte. J. Math. Comp. Sci., 7(2011)166-170.

[19] J.Huang, Z.Wang, Extrapolation algorithm for solving mixed boundary integral equations of the Helmholtz equation by mechanical quadrature methods. SIAM J. Sci. Comp., Vol. 31(6)(2009), 4115-4129.

[20] J. Huang, T. Lü, Z.C. Li, The mechanical quadrature methods and their splitting extrapolations for boundary integral equations of first kind on open arcs. Appl. Numer. Math., 59(2009),2908-2922.

[21] A.Sidi, M.Israeli, Quadrature methods for periodic singular and weakly singular Fredholm integral equations, J. Sci. Comp., 3(1988), 201-231.

[22] A.Sidi. Practical Extrapolation Methods: Theory and Applications, Cambridge University Press, 2003.

[23] P.M.Anselone, Collectively compact operator approximation theory, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

[24] P.M.Anselone, Singularity subtraction in the numerical solution of integral equations, J. Austral. Math. Soc. (Series B), 22(1981), 408-418.

[25] C.B.Lin, T.Lü, T.M.Shih, The splitting extrapolation method, World Scientific, Singapore, 1995.

[26] Y.S.Xu, Y.H.Zhao, An extrapolation method for a class of boundary integral equations, SIAM J. Math. Comp., 65(1996), 587-610.

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Sufficient conditions for univalence obtained by using first order nonlinear strong differential subordinations

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Abstract

The concept of differential subordination was introduced in [3] by S.S. Miller and P.T. Mocanu and the concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera. This last concept was applied in the special case of Briot-Bouquet strong differential subordination. In [5] the authors have developed the general theory of strong differential subordinations following the general theory introduced in [3]. In [6], the special case of first order linear strong differential subordinations was studied. Now, we study another special case, the first order nonlinear strong differential subordinations.

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1 Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$. Let A be the class of functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots,\ z \in U$.

In adition, we need the classes of convex, alpha-convex, close-to-convex and starlike (univalent) functions given respectively by $K = \{f \in A : \mathrm{Re}\, zf''(z)/f'(z) + 1 > 0\}$, $M_\alpha = \{f \in A : \frac{f(z)f'(z)}{z} \neq 0,\ \mathrm{Re}\,(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U\}$, $C = \{f \in A : \mathrm{Re}\, f'(z) > 0,\ z \in U\}$, and $S^* = \{f \in A : \mathrm{Re}\, zf'(z)/f(z) > 0\}$.

Definition 1.1 *[1], [2], [3] Let $H(z,\xi)$ be analytic in $U \times \overline{U}$ and $f(z)$ analytic and univalent in U. The function $H(z,\xi)$ is strongly subordinate to $f(z)$, written $H(z,\xi) \prec\prec f(z)$ if for each $\xi \in \overline{U}$, $H(z,\xi)$ is subordinate to $f(z)$.*

Remark 1.1 *(i) Since $f(z)$ is analytic and univalent Definition 1.1 is equivalent to $H(0,\xi) = f(0)$ and $H(U \times \overline{U}) \subset f(U)$.*

(ii) If $H(z,\xi) \equiv H(z)$ then the strong subordination becomes the usual notion of subordination.

Definition 1.2 *[4], [5, Definition 2.2.b, p. 21] We denote by Q the set of functions q that are analytic and injective in $\overline{U} \setminus E(q)$, where $E(q) = \left\{\zeta \in \partial U;\ \lim\limits_{z\to\zeta} q(z) = \infty\right\}$ and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.*

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.3 *[5, Definition 4] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n a positive integer. The class of admissible functions $\psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r,s;z,\xi) \notin \Omega \tag{A}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U \setminus E(f)$ and $m \geq n \geq 1$.

Remark 1.2 *The function* $q(z) = M\frac{Mz+a}{M+\overline{a}z}$, *with* $M > 0$ *and* $|a| < M$, *satisfies* $\Delta = q(U) = U_M = U(0, M)$, $q(0) = a$, $E(q) = \emptyset$ *and* $q \in Q$. *If* $a = 0$, *then (A) simplifies to*

$$\psi(Me^{i\theta}, Ke^{i\theta}; z, \xi) \notin \Omega \tag{A'}$$

whenever $K \geq nM$, $z \in U$, $\xi \in \overline{U}$ *and* $\theta \in \mathbb{R}$.

Remark 1.3 *The function* $q(z) = \frac{a+\overline{a}z}{1-z}$ *with* Re $a > 0$, *satisfies* $q(U) = \Delta$, $q(0) = a$, $E(q) = \{1\}$ *and* $q \in Q$. *If* $a = 1$, *then (A) simplifies to*

$$\psi(\rho i, \sigma, z, \xi) \notin \Omega, \tag{A''}$$

when $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$, $z \in U$, $\xi \in \overline{U}$ *and* $n \geq 1$.

Lemma 1.1 *[3], [4, Lemma 2.2.d, p. 24] Let* $q \in Q(a)$, *with* $q(0) = a$ *and* $p(z) = a + a_n z^n + \ldots$ *analytic in* U, *with* $p(z) \not\equiv a$, $n \geq 1$. *If* p *is not subordinate to* q, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$, *and an* $m \geq n \geq 1$ *for which* $p(U_{r_0}) \subset q(U)$
(i) $p(z_0) = q(\zeta_0)$
(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

Definition 1.4 *[6] A strong differential subordination of the form* $A(z,\xi)zp'(z) + B(z,\xi)p(z) \prec\prec h(z)$, $z \in U$, $\xi \in \overline{U}$, *where* $A(z,\xi)zp'(z) + B(z,\xi)p(z)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and* h *is an analytic and univalent function in* U, *is called first order linear strong differential subordination.*

2 Main results

Definition 2.1 *A strong differential subordination of the form*

$$A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) + D(z,\xi) \prec\prec h(z), \tag{1}$$

where $A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) + D(z,\xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and* h *is an analytic and univalent function in* U, *is called first order nonlinear strong differential subordination.*

Remark 2.1 *If* $C(z,\xi) = D(z,\xi) = 0$ *then (1) becomes a linear strong differential subordination studied in [6].*

Remark 2.2 *If* $A(z,\xi) = A(z)$, $B(z,\xi) = B(z)$, $C(z,\xi) = C(z)$, $D(z,\xi) = D(z)$ *then (1) becomes a nonlinear differential subordination studied in [7].*

Next, we find conditions for the functions p, A, B, C, D and h such that (1) holds.

Theorem 2.1 *Let* $p \in \mathcal{H}[0, n]$, $A, B, C : U \times \overline{U} \to C$ *with*

$$\text{Re } A(z,\xi) \geq 0, \quad \text{Re } [A(z,\xi) + B(z,\xi)] \geq 1 + M|C(z,\xi)| \tag{2}$$

and $A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z)$ *an analytic function in* U *for all* $\xi \in \overline{U}$. *Then*

$$A(z,\xi)zp'(z) + B(z,\xi)p(z) + C(z,\xi)p^2(z) \prec\prec Mz \tag{3}$$

implies $p(z) \prec Mz$, $M > 0$, $z \in U$.

Proof. Let $\psi(r, s; z, \xi) : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ given by Definition 1.3. For $r = p(z)$, $s = zp'(z)$, $z \in U$ we have

$$\psi(r, s; z, \xi) = A(z,\xi)s + B(z,\xi)s + C(z,\xi)r^2. \tag{4}$$

Then (3) becomes

$$\psi(r, s; z, \xi) \prec\prec Mz, \quad z \in U, \ \xi \in \overline{U}. \tag{5}$$

If we consider $h(z) = Mz$, $M > 0$ then $h(U) = U(0, M)$ and (5) is equivalent to

$$\psi(r, s; z, \xi) \in U(0, M), \quad z \in U, \ \xi \in \overline{U}. \tag{6}$$

Suppose that p is not subordinated to $h(z) = Mz$. Then, from Lemma 1.1, we have that there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ with $|\zeta_0| = 1$, such that $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$, $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$, $K \geq nM$.

By replacing r with $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$ and s with $z_0p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$ in (4) and taking into account the conditions from (2), we have

$|\psi(p(z_0), z_0p'(z_0); z_0, \xi)| = |\psi(Me^{i\theta_0}, Ke^{i\theta_0}; z_0, \xi)| = |A(z_0,\xi)Ke^{i\theta_0} + B(z_0,\xi)Me^{i\theta_0} + C(z_0,\xi)M^2e^{2i\theta_0}| = |A(z_0,\xi)K+B(z_0,\xi)M+C(z_0,\xi)M^2e^{2i\theta_0}| \ge |A(z_0,\xi)K+B(z_0,\xi)M|-M^2|C(z_0,\xi)| \ge \text{Re }[A(z_0,\xi)K+B(z_0,\xi)M]-M^2|C(z_0,\xi)| \ge K\text{Re }A(z_0,\xi)+M\text{Re }B(z_0,\xi)-M^2|C(z_0,\xi)| \ge nM\text{Re }A(z_0,\xi)+M\text{Re }B(z_0,\xi)-M^2|C(z_0,\xi)| \ge M\text{Re }[A(z_0,\xi)+B(z_0,\xi)]-M^2|C(z_0,\xi)| \ge M,$

which contradicts (6). This means the assumption made is false, hence $p(z) \prec Mz$, $M > 0$, $z \in U$. ■

Example 2.1 *Let* $A(z,\xi) = z+\xi+4$, $B(z,\xi) = 3z-2\xi+12-8i$, $C(z,\xi) = 2z-3\xi+1-\sqrt{3}i$, $M = \frac{1}{2}$. *Since* $z \in U$, $\xi \in \overline{U}$, *we have* $\text{Re }A(z,\xi) \ge 2$, $\text{Re }B(z,\xi)| \ge 7$, $|C(z,\xi)| \le 16$, $\text{Re }[A(z,\xi)+B(z,\xi)] \ge 9$.

From Theorem 2.1, we obtain: If $p \in [0,n]$, $n \in \mathbb{N}$, *and* $(z+\xi+4)zp'(z)+(3z-2\xi+12-8i)p(z)+(2z-2\xi+1-\sqrt{3}i)p^2(z)$ *is a function of* z, *analytic in* U *for all* $\xi \in \overline{U}$, *then* $(z+\xi+4)zp'(z)+(3z-2\xi+12-8i)p(z)$ $+(2z-3\xi+i-\sqrt{3}i)p^2(z) \prec\prec \frac{z}{2}$, $\quad z \in U$, $\xi \in \overline{U}$, *implies* $p(z) \prec \frac{z}{2}$, $z \in U$.

Theorem 2.2 *Let* $p \in [0,n]$, $A,B,C,D : U \times \overline{U} \to C$ *with*

$$\text{Re }A(z,\xi) \ge 0, \quad \text{Re }C(z,\xi) \ge 0, \quad \frac{n}{2}\text{Re }A(z,\xi) \ge \text{Re }D(z,\xi) \tag{7}$$

and $\text{Im }B(z,\xi) \le 2\sqrt{\left[\frac{n}{2}\text{Re }A(z,\xi)+\text{Re }C(z,\xi)\right]\ \left[\frac{n}{2}\text{Re }A(z,\xi)-\text{Re }D(z,\xi)\right]}$.

If $A(z,\xi)zp'(z)+B(z,\xi)p(z)+C(z,\xi)p^2(z)+D(z,\xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and satisfies the inequality*

$$\text{Re }[A(z,\xi)zp'(z)+B(z,\xi)p(z)+C(z,\xi)p^2(z)+D(z,\xi)] > 0 \tag{8}$$

then $\text{Re }p(z) > 0$, $z \in U$.

Proof. Let $\psi(r,s;z,\xi) : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ given by Definition 1.3. For $r = p(z)$, $s = zp'(z)$, $z \in U$ we have

$$\psi(r,s;z,\xi) = A(z,\xi)s+B(z,\xi)r+C(z,\xi)r^2+D(z,\xi), \quad z \in U,\ \xi \in \overline{U}. \tag{9}$$

Then (8) becomes

$$\text{Re }\psi(r,s;z,\xi) > 0, \quad z \in U,\ \xi \in \overline{U}. \tag{10}$$

If we consider $h(z) = \frac{1+z}{1-z}$ then $h(U) = \{w \in \mathbb{C};\ \text{Re }w > 0\}$ and (10) is equivalent to

$$\psi(r,s;z,\xi) \prec\prec \frac{1+z}{1-z}, \quad z \in U,\ \xi \in \overline{U}. \tag{11}$$

Suppose that p is not subordinated to $h(z) = \frac{1+z}{1-z}$. Then, from Lemma 1.1, we have that there exist $z_0 = r_0e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ such that $p(z_0) = h(\zeta_0) = \rho i$, $\rho \in \mathbb{R}$, $z_0p'(z_0) = m\zeta_0h'(\zeta_0) = \sigma$, $\sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1+\rho^2)$.

By replacing r with ρi and s with σ in (9) and using the conditions given by (7) we obtain

$\text{Re }\psi(p(z_0), z_0p'(z_0); z_0,\xi) = \text{Re }\psi(\rho i, \sigma; z_0, \xi) = \text{Re }[A(z_0,\xi)\sigma+B(z_0,\xi)\rho i-\rho^2C(z_0,\xi)+D(z_0,\xi)] = \sigma\text{Re }A(z_0,\xi)-\rho\text{Im }B(z_0,\xi)-\rho^2\text{Re }C(z_0,\xi)+\text{Re }D(z_0,\xi) \ge -\frac{n}{2}(1+\rho^2)\text{Re }A(z_0,\xi)-\rho\text{Im }B(z_0,\xi)-\rho^2\text{Re }C(z_0,\xi)+\text{Re }D(z_0,\xi) \ge -\rho^2\left[\frac{n}{2}\text{Re }A(z_0,\xi)+\text{Re }C(z_0,\xi)\right]-\rho\text{Im }B(z_0,\xi)-\frac{n}{2}\text{Re }A(z_0,\xi)+\text{Re }D(z_0,\xi) \le 0,$

which contradicts (10). This means the assumption made is false, hence $p(z) \prec \frac{1+z}{1-z}$, $z \in U$, which is equivalent to $\text{Re }p(z) > 0$, $z \in U$. ■

Theorem 2.3 *Let* $p \in \mathcal{H}[1,n]$, $A,B,C,D : U \times \overline{U} \to C$ *with*

$$\text{Re }A(z,\xi) \ge 0, \quad \text{Re }C(z,\xi) \ge 0, \quad \frac{n}{2}\text{Re }A(z,\xi) \ge \text{Re }D(z,\xi)+1 \tag{12}$$

and $\text{Im }B(z,\xi) \le 2\sqrt{\left[\frac{n}{2}\text{Re }A(z,\xi)+\text{Re }C(z,\xi)\right]\left[\frac{n}{2}\text{Re }A(z,\xi)-\text{Re }D(z,\xi)-1\right]}$.

If $A(z,\xi)zp'(z)+B(z,\xi)p(z)+C(z,\xi)p^2(z)+D(z,\xi)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and satisfies the nonlinear strong differential subordination*

$$A(z,\xi)zp'(z)+B(z,\xi)p(z)+C(z,\xi)p^2(z)+D(z,\xi) \prec\prec z \tag{13}$$

then $p(z) \prec \frac{1+z}{1-z}$, $z \in U$.

Proof. Let ψ given by (9). For $r = p(z)$, $s = zp'(z)$, (13) becomes

$$\psi(r, s; z, \xi) \prec\prec z, \quad z \in U,\ \xi \in \overline{U}. \tag{14}$$

If we consider $h(z) = z$, $z \in U$, then $q(U) = U$ and from (14) we have

$$\psi(r, s; z, \xi) \in U, \quad z \in U,\ \xi \in \overline{U}, \tag{15}$$

which is equivalent to

$$-1 < \text{Re } \psi(r, s; z, \xi) < 1, \quad z \in U,\ \xi \in \overline{U}. \tag{16}$$

Suppose that p is not subordinated to $q(z) = \frac{1+z}{1-z}$. Then, from Lemma 1.1 we have that there exist $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$, such that $p(z_0) = q(\zeta_0) = \rho i$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = \sigma$, $\sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$.

By replacing r with ρi and s with σ in (9) and using the conditions given by (12), we have:

Re $\psi(r, s; z_0, \xi) =$ Re $\psi(\rho i, \sigma; z_0, \xi) =$ Re $[A(z_0, \xi)\sigma + B(z_0, \xi)\rho i - C(z_0, \xi)\rho^2 + D(z_0, \xi)] = \sigma$Re $A(z_0, \xi) - \rho$Im $B(z_0, \xi) - \rho^2$Re $C(z_0, \xi) +$ Re $D(z_0, \xi) \leq -\frac{n}{2}(1+\rho^2)$Re $A(z_0, \xi) - \rho$Im $B(z_0, \xi) - \rho^2$Re $C(z_0, \xi) +$ Re $D(z_0, \xi)$ $\leq -\rho^2 \left[\frac{n}{2}\text{Re } A(z_0, \xi) + \text{Re } C(z_0, \xi)\right] - \rho\text{Im } B(z_0, \xi) - \frac{n}{2}\text{Re } A(z_0, \xi) + \text{Re } D(z_0, \xi) \leq -1$,
which contradicts (15). That means the assumption made was false, hence $p(z) \prec \frac{1+z}{1-z}$, $z \in U$. ■

References

[1] José A. Antonino and Salvador Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations 114(1994), 101-105.

[2] José A. Antonino, *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc., **43**(2006), no.2, 311-322.

[3] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), no.2, 157-172.

[4] S.S. Miller and P.T. Mocanu, *Differential subordinations. Theory and applications*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[5] Georgia Irina Oros and Gheorghe Oros, *Strong differential subordination*, Turkish Journal of Mathematics, 33 (2009), 249-257.

[6] Georgia Irina Oros, *First order strong differential superordination*, General Mathematics, vol. 15, No. 2-3 (2007), 77-87.

[7] Georgia Irina Oros and Gheorghe Oros, *On a first order nonlinear differential subordinations I*, Analele Universităţii Oradea, Fasc. Matematică, Tom. IX, 2003, 65-70.

A note on the symmetric properties for the second kind twisted q-Euler polynomials

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Abstract : In this paper, we introduce the second kind twisted q-Euler numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the the second kind twisted Euler polynomials.

Key words : the second kind Euler numbers and polynomials, the second kind twisted Euler numbers and polynomials, the second kind twisted q-Euler numbers and polynomials, alternating sums

1. Introduction

Euler numbers, Euler polynomials, q-Euler numbers, q-Euler polynomials, the second kind Euler number and the second kind Euler polynomials were studied by many authors. Euler numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics(see for details [1-9]). In this paper, we introduce the second kind twisted q-Euler numbers and polynomials. In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second twisted q-Euler polynomials and the power sums.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x) - g(y)}{x - y}$$

have a limit $l = g'(a)$ as $(x, y) \to (a, a)$. For $g \in UD(\mathbb{Z}_p)$, Kim defined the fermionic p-adic integral on $\mathbb{Z}_p$ (see [1])

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x + 1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

Let us define the second kind twisted q-Euler numbers $E_{n,q,\omega}$ and polynomials $E_{n,q,\omega}(x)$ as follows:

$$I_{-1}(\phi_\omega(y) q^y e^{(2y+1)t}) = \int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,\omega} \frac{t^n}{n!}, \tag{1.3}$$

$$I_{-1}(\phi_\omega(y) q^y e^{(2y+1+x)t}) = \int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{(2y+1+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,\omega} x) \frac{t^n}{n!}. \tag{1.4}$$

By (1.3) and (1.4), we obtain the following Witt's formula.

Theorem 1. For $\omega \in T_p$, we have

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^x (2x+1)^n d\mu_{-1}(x) = E_{n,q,\omega},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y (2y+1+x)^n d\mu_{-1}(y) = E_{n,q,\omega}(x).$$

Theorem 2. For any positive integer n, we have

$$E_{n,q,w}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q,w} x^{n-k}.$$

2. The alternating sums of powers of consecutive q-odd integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$. Let ω be the p^N-th root of unity. By using (1.4), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty} E_{n,q,\omega} \frac{t^n}{n!} = \frac{2e^t}{\omega q e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{(2n+1)t}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{(2n+2k+1)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \omega^{n-k} q^{(n-k)} e^{(2n+1)t} = \sum_{n=0}^{k-1} (-1)^{n-k} \omega^{n-k} q^{(n-k)} e^{(2n+1)t}.$$

By using (1.3)and (1.4), we obtain

$$-\frac{1}{2} \sum_{j=0}^{\infty} E_{j,q,\omega}(2k) \frac{t^j}{j!} + \frac{1}{2} (-1)^{-k} \omega^{-k} q^{-k} \sum_{j=0}^{\infty} E_{j,q,\omega} \frac{t^j}{j!} = \sum_{j=0}^{\infty} \left((-1)^{-k} \omega^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n+1)^j \right) \frac{t^j}{j!}.$$

By comparing coefficients $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n+1)^j = \frac{(-1)^{k+1} \omega^k q^k E_{j,q,\omega}(2k) + E_{j,q,\omega}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$ and ω be the p^N-th root of unity. Then we obtain

$$T_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n+1)^j = \frac{(-1)^{k+1} \omega^k q^k E_{j,q,\omega}(2k) + E_{j,q,\omega}}{2}. \tag{2.1}$$

Remark 4. For the alternating sums of powers of consecutive odd integers, we have

$$\lim_{q \to 1} T_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n (2n+1)^j = \frac{(-1)^{k+1} \omega^k E_{j,\omega}(2k) + E_{j,\omega}}{2},$$

where $E_{j,\omega}(x)$ and $E_{j,\omega}$ denote the second kind twisted Euler polynomials and the second kind twisted Euler numbers, respectively (see [5]).

3. The symmetry property of the q-deformed fermionic integral on $\mathbb{Z}_p$

In this section, we assume that $q \in \mathbb{C}_p$ and $\omega \in T_p$. In this section, we obtain recurrence identities the second twisted q-Euler polynomials and the alternating sums of powers of consecutive q-odd integers. By using (1.1), we have

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k), \text{ (see [1], [2], [3], [5])},$$

where $n \in \mathbb{N}, g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k). \tag{3.1}$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k} g(k). \tag{3.2}$$

Substituting $g(x) = \omega^x q^x e^{(2x+1)t}$ into the above, we obtain

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) = 2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j e^{(2j+1)t}. \tag{3.3}$$

After some elementary calculations, we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{\omega q e^{2t}+1}, \\ &\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) = \omega^n q^n e^{2nt}\frac{2e^t}{\omega q e^{2t}+1}. \end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t(1+\omega^n q^n e^{2nt})}{\omega q e^{2t}+1}.$$

From the above, we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x) \\ &= \frac{2\int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{nx} e^{2ntx} d\mu_{-1}(x)}. \end{aligned} \tag{3.5}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (3.3), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n}(2x+1+2n)^m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^m d\mu_{-1}(x)\right)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j (2j+1)^m\right)\frac{t^m}{m!} \end{aligned}$$

By comparing coefficients $\dfrac{t^m}{m!}$ in the above equation, we obtain

$$\begin{aligned} &\omega^n q^n \sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x+1)^m d\mu_{-1}(x) \\ &= 2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j (2j+1)^m \end{aligned}$$

By using (2.1), we have

$$\begin{aligned}&\omega^n q^n \sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p}\omega^x q^x (2x+1)^k d\mu_{-1}(x)+\int_{\mathbb{Z}_p}\omega^x q^x (2x+1)^m d\mu_{-1}(x)\\&=2T_{m,q,\omega}(n-1).\end{aligned}\tag{3.6}$$

By using (3.5) and (3.6), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we obtain

$$\frac{2\int_{\mathbb{Z}_p}\omega^x q^x e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}\omega^{nx}q^{nx}e^{2ntx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2T_{m,q,\omega}(n-1)\right)\frac{t^m}{m!}.\tag{3.7}$$

Let w_1 and w_2 be odd positive integers. By using (3.7), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\omega^{(w_1x_1+w_2x_2)}q^{(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2xt}d\mu_{-1}(x)}\\&=\frac{2e^{w_1t}e^{w_2t}e^{w_1w_2xt}(\omega^{w_1w_2}q^{w_1w_2}e^{2w_1w_2t}+1)}{(\omega^{w_1}q^{w_1}e^{2w_1t}+1)(\omega^{w_2}q^{w_2}e^{2w_2t}+1)}\end{aligned}\tag{3.8}$$

By using (3.7) and (3.8), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_1},\omega^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_2},\omega^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned}\tag{3.9}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_1},\omega^{w_1}}(w_2x)w_1^jT_{m-j,q^{w_2},\omega^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}\tag{3.10}$$

By using the symmetry in (3.9), we have

$$\begin{aligned}a&=\left(\frac{1}{2}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,q^{w_2},\omega^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,q^{w_1},\omega^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,q^{w_2},w^{w_2}}(w_1x)w_2^jT_{m-j,q^{w_1},w^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}\tag{3.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.10) and (3.11), we arrive at the following theorem:

Theorem 6. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jE_{j,q^{w_2},\omega^{w_2}}(w_1x)T_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}E_{j,q^{w_1},\omega^{w_1}}(w_2x)T_{m-j,q^{w_2},\omega^{w_2}}(w_1-1),\end{aligned}$$

where $E_{k,q,\omega}(x)$ and $T_{m,q,\omega}(k)$ denote the second kind twisted q-Euler polynomials and the alternating sums of powers of consecutive q-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j}x^{j-k}E_{k,q^{w_2},\omega^{w_2}}T_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k}x^{j-k}E_{k,q^{w_1},\omega^{w_1}}T_{m-j,q^{w_2},\omega^{w_2}}(w_1-1).\end{aligned}$$

By using (3.8), we have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\left(2\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}E_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.12}$$

By using the symmetry property in (3.12), we also have

$$\begin{aligned}a&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(\frac{2\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}\omega^{w_1w_2x}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_{-1}(x_2)\right)\left(2\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}E_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.\end{aligned}\tag{3.13}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.12) and (3.13), we have the following theorem.

Theorem 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}E_{n,q^{w_1},\omega^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}E_{n,q^{w_2},\omega^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{3.14}$$

Substituting $w_1=1$ into (3.14), we arrive at the following corollary.

Corollary 9. Let w_2 be odd positive integer. Then we obtain

$$E_{n,q,\omega}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j \omega^j q^j E_{n,q^{w_2},\omega^{w_2}} \left(\frac{x - w_2 + (2j+1)}{w_2} \right).$$

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REFERENCES

1. Kim, T.(2002). q-Volkenborn integration, Russian Journal of Mathematical physics, v.9, pp. 288-299.

2. Kim, T.(2008). Euler numbers and polynomials associated with zeta function, Abstract and Applied Analysis, Art. ID 581582.

3. Kim, T.(2007). q-Euler numbers and polynomials associated with p-adic q-integrals, Journal of Nonlinear Mathematical Physics, v.14, pp. 15-27,

4. Ryoo, C.S., Kim, T., Jang, L. C.(2007). Some relationships between the analogs of Euler numbers and polynomials, Journal of Inequalities and Applications, v.2007, ID 86052, pp. 1-22.

5. Ryoo, C.S.(2010). On the second kind twisted Euler numbers, Far East Journal of Mathematical Sciences, v.43, pp. 21-28.

6. Ryoo, C.S.(2009). Calculating zeros of the q-Euler polynomials, Proceeding of the Jangjeon Mathematical Society, v.12, pp. 253-259.

7. Ryoo, C.S.(2010). Calculating zeros of the second kind Euler polynomials, Journal of Computational Analysis and Applications, v.12, pp. 828-833.

8. Ryoo, C.S.(2012) A numerical computation of the structure of the roots of the second kind q-Euler polynomials, Journal of Computational Analysis and Applications, v.14, pp. 321-327.

9. Ryoo, C. S.(2011) A Note on the second kind q-Euler polynomials of higher order, Applied Mathematics Sciences, v.5, pp. 3421-3427.

Sufficient conditions for functions to be in a class of p-valent analytic functions

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In this article, we obtain certain simple sufficiency criteria for a subclass of p-valent analytic functions. Many known results appear as special consequences of our work. Some applications of our work to the generalized Alexander integral operator is also given.

Key words: Spiral-like functions, convolution, integral operator.
Subject classification: 30C45, 30C50.

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1 Introduction

Let $A(p,n)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_k z^k, (p\in\mathtt{N}=\{1,2,3,...\}), \tag{1}$$

which are analytic and multivalent in the open unit disk $\mathtt{U}=\{z:|z|<1\}$. For functions $f(z)$, $g(z)\in A(p,n)$ of the form (1), We define the convolution (Hadamard product) of $f(z)$ and $g(z)$ by

$$(f\star g)(z)=z^p+\sum_{k=p+n}^{\infty}a_k b_k z^k,\ (z\in\mathtt{U}). \tag{2}$$

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Also let $Q_\lambda(p,n,\alpha;g(z))$, λ is real with $|\lambda|<\frac{\pi}{2}$, $0\leq\alpha<p$, $n\in\mathtt{N}$ and $p\in\mathtt{N}$, denote the subclass of $A(p,n)$ consisting of all functions $f(z)$ which is defined with the help of convolution by

$$\Re e^{i\lambda}\frac{z\left((f\star g)'(z)\right)}{(f\star g)(z)}>\alpha\cos\lambda,(z\in\mathtt{U}). \tag{3}$$

By suitably choosing $g(z)$ in (3), we obtain the subclasses $S_\lambda^*(p,n,\alpha)$ and $C_\lambda(p,n,\alpha)$ of $A(p,n)$ which are defined, respectively, by

$$\Re e^{i\lambda}\frac{zf'(z)}{f(z)}>\alpha\cos\lambda,(z\in\mathtt{U}), \tag{4}$$

$$\Re e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\cos\lambda,(z\in\mathtt{U}). \tag{5}$$

We note that for $\lambda=0$, the classes $S_\lambda^*(p,n,\alpha)$ and $C_\lambda(p,n,\alpha)$ reduces to the classes $S_p^*(n,\alpha)$ and $C_p(n,\alpha)$ respectively studied by Goyal et al [1]. Further if we take $\alpha=0$, $p=1$ and $n=1$ in the classes (4) and (5), we obtain the class of spiral-like functions introduced by Spacek [2] and the class of Robertson functions studied by Robertson [3] respectively.

We will assume throughout our discussion, unless otherwise stated, that λ is real with $|\lambda|<\frac{\pi}{2}$, $0\leq\alpha<p$, $n\in\mathtt{N}$ and $p\in\mathtt{N}$

2 Sufficient conditions for the class $Q_\lambda(p,n,\alpha;g(z))$

To obtain our main results, we need the following Lemma due to Mocanu [4].

Lemma 2.1. If $q(z)\in A(n)$ satisfies the condition

$$\left|q'(z)-1\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}\quad(z\in\mathtt{U}), \tag{6}$$

then

$$q(z)\in S^*(n,0). \tag{7}$$

Theorem 2.1. If $f(z)\in A(p,n)$ satisfies

$$\left|\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\left\{e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-\alpha\cos\lambda-ip\sin\lambda\right\}-(p-\alpha)\cos\lambda\right|$$

$$<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\cos\lambda\ (z\in\mathtt{U}), \tag{8}$$

then $f(z)\in Q_\lambda(p,n,\alpha;g(z))$.

Proof. Let us set a function $p(z)$ by

$$p(z)=z\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}=z+\frac{e^{i\lambda}a_{p+n}b_{p+n}}{(p-\alpha)\cos\lambda}z^{n+1}+\dots \tag{9}$$

for $f(z),g(z)\in A(p,n)$. Then clearly (9) shows that $p(z)\in A(n)$.

Differentiating (9) logarithmically, we have

$$\frac{p'(z)}{p(z)}=\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}\left[\frac{(f\star g)'(z)}{(f\star g)(z)}-\frac{p}{z}\right]+\frac{1}{z} \tag{10}$$

which gives

$$\left|p'(z)-1\right| \tag{11}$$

$$=\left|\left(\frac{(f\star g)(z)}{z^p}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\frac{1}{(p-\alpha)\cos\lambda}\left\{e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-\alpha\cos\lambda-ip\sin\lambda\right\}-1\right|. \tag{12}$$

Thus using (8), we have

$$\left|p'(z)-1\right|\le\frac{n+1}{\sqrt{(n+1)^2+1}},\ (z\in\mathtt{U}). \tag{13}$$

Hence, using Lemma 2.1, we have $p(z)\in S^*(n,0)$.

From (10), we can write

$$\frac{zp'(z)}{p(z)}=\frac{1}{(p-\alpha)\cos\lambda}\left[e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}-(\alpha\cos\lambda+ip\sin\lambda)\right]. \tag{14}$$

Since $p(z)\in S^*(n,0)$, it implies that $\Re\frac{zp'(z)}{p(z)}>0$. Therefore, we get

$$\frac{1}{(p-\alpha)\cos\lambda}\left[\Re\left(e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}\right)-\alpha\cos\lambda\right]=\Re\frac{zp'(z)}{p(z)}>0 \tag{15}$$

or

$$\Re\left(e^{i\lambda}\frac{z(f\star g)'(z)}{(f\star g)(z)}\right)>\alpha\cos\lambda. \tag{16}$$

and this implies that $f(z)\in Q_\lambda(p,n,\alpha;g(z))$. By taking $g(z)$ is an identity function and Koebe p-valent functions with $\lambda=0$ in Theorem 2.1, we obtain Corollary 2.2 and Corollary 2.3 respectively proved by Goyal et.al [1].

Corollary 2.2. If $f(z)\in A(p,n)$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{p-\alpha}}\left\{z^{\frac{1-\alpha}{p-\alpha}}\frac{f'(z)}{f(z)}-\alpha z^{\frac{1-p}{p-\alpha}}\right\}-p+\alpha\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\ (z\in\mathtt{U}), \tag{17}$$

for $0\le\alpha<p$, then $f(z)\in S_p^*(n,\alpha)$.

Corollary 2.3. If $f(z)\in A(p,n)$ satisfies

$$\left|\left\{\frac{(f'(z))^{\alpha+1-p}}{pz^{p-1}}\right\}^{\frac{1}{p-\alpha}}\left\{zf''(z)+(1-\alpha)f'(z)\right\}-p+\alpha\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(p-\alpha)\ (z\in\mathtt{U}), \tag{18}$$

for $0\le\alpha<p$, then $f(z)\in C_p(n,\alpha)$.

Further If we take $n=1$ and $p=1$ in Corollary 2.2 and Corollary 2.3, we get the following result proved by Uyanik et al [5].

Corollary 2.4. If $f(z)\in A$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}-1+\alpha\right|<\frac{2}{\sqrt{5}}(1-\alpha)\ (z\in\mathtt{U}), \tag{19}$$

for $0\le\alpha<1$, then $f(z)\in S^*(\alpha)$.

Corollary 2.5. If $f(z) \in A$ satisfies

$$\left| \left(f'(z)\right)^{\frac{\alpha}{1-\alpha}} \left\{ f'(z) + \frac{1}{1-\alpha} z f''(z) \right\} - 1 \right| < \frac{2}{\sqrt{5}} \ (z \in \mathtt{U}), \tag{20}$$

for $0 \leq \alpha < 1$, then $f(z) \in C(\alpha)$.

Remark 2.1. If we put $\alpha = 0$ and $p = 1$ in Corollary 2.4 and Corollary 2.5, we get the result proved by Mocanu [6] and Nunokawa et al [7] respectively.

Theorem 2.6. If $p(z)$, given by (9), satisfies

$$\left|p''(z)\right| < \frac{n+1}{\sqrt{(n+1)^2+1}} \, (z \in E), \tag{21}$$

then $f(z) \in Q_\lambda(p, n, \alpha; g(z))$.

Proof. From (9), we have $p(z) \in A(n)$. Also

$$\begin{aligned} \left|p'(z) - 1\right| &= \left| \int_0^z p''(t)dt \right| \leq \int_0^{|z|} \left| h''(\rho e^{i\theta}) \right| d\rho & (22) \\ &\leq \frac{n+1}{\sqrt{(n+1)^2+1}} |z| \leq \frac{n+1}{\sqrt{(n+1)^2+1}}, & (23) \end{aligned}$$

where we have used (21). This proves that $p(z)$ satisfies the condition of Lemma 2.1 and therefore $p(z) \in S^*(n, 0)$, which leads $f(z) \in Q_\lambda(p, n, \alpha; g(z))$. **Theorem 2.7.** If $f(z) \in A(p, n)$ satisfies

$$\left| \left(\frac{(f \star g)(z)}{z^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} \left[\left(\frac{(f \star g)'(z)}{(f \star g)(z)} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} - \frac{p}{z} \right] \right| \leq \frac{(n+1)(p-\alpha)\cos\lambda}{2\sqrt{(n+1)^2+1}}, \tag{24}$$

then $f(z) \in Q_\lambda(p, n, \alpha; g(z))$.

Proof. Let us define a function $p(z)$ by

$$p(z) = \int_0^z \left(\frac{(f \star g)(t)}{t^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}} dt. \tag{25}$$

Then

$$zp'(z) = z \left(\frac{(f \star g)(z)}{z^p} \right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}. \tag{26}$$

Let $g(z) = zp'(z)$. Then $g(z) \in A(n)$. Consider

$$\left|g'(z) - 1\right| = \left|p'(z) + zp''(z) - 1\right| \leq \left|p'(z) - 1\right| + \left|zp''(z)\right| = \left| \int_0^z p''(t)dt \right| + \left|zp''(z)\right|$$

$$\begin{aligned} &\leq \int_0^{|z|} \left| \frac{e^{i\lambda}}{(p-\alpha)\cos\lambda} H(z) \right| dt + \left| \frac{e^{i\lambda}}{(p-\alpha)\cos\lambda} H(z) \right| |z| \\ &\leq \int_0^{|z|} \frac{(n+1)}{2\sqrt{(n+1)^2+1}} dt + \frac{(n+1)}{2\sqrt{(n+1)^2+1}} |z| < \frac{(n+1)}{\sqrt{(n+1)^2+1}}. \end{aligned} \tag{27}$$

with

$$H(z)=\left(\frac{(f\star g)(z)}{z^{p}}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}\left[\left(\frac{(f\star g)'(z)}{(f\star g)(z)}\right)^{\frac{e^{i\lambda}}{(p-\alpha)\cos\lambda}}-\frac{p}{z}\right]. \tag{28}$$

Therefore, by using Lemma 2.1, we have

$$g(z)=zp'(z)\in S^{*}(n,0). \tag{29}$$

This means that $p(z)\in C(n,0)$, which implies that $f(z)\in Q_{\lambda}(p,n,\alpha;g(z))$.

3. Generalized Alexander Integral Operator

For $f(z),g(z)\in A(p,n)$, we consider

$$G(z)=\int_{0}^{z}\left(\frac{(f*g)(t)}{t^{p}}\right)^{\gamma}dt=z+\frac{\gamma a_{p+n}b_{p+n}}{n+1}z^{n+1}+\ldots \tag{30}$$

Clearly $G(z)\in A(n)$ and when $p=1,\gamma=1$, $g(z)=\frac{z}{1-z}$, then (30) reduces to the well-known Alexander integral operator [8].

Theorem 3.1. If $\gamma\geq\frac{1}{p}$ and $f(z),g(z)\in A(p,n)$ satisfies

$$\left|\frac{\gamma\left((f*g)(z)\right)^{\frac{\gamma e^{i\lambda}}{\cos\lambda}}}{z^{\frac{p\gamma e^{i\lambda}}{\cos\lambda}+1}}\left(\frac{z\left((f*g)'(z)\right)}{(f*g)(z)}-p\right)\right|\leq\frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}}, \tag{31}$$

then $f(z)\in Q_{\lambda}(p,n,0;g(z))$.

Proof. From (30), we get

$$G'(z)=\left(\frac{(f*g)(z)}{z^{p}}\right)^{\gamma}. \tag{32}$$

Differentiating (32), logarithmically, we get

$$\frac{G''(z)}{G'(z)}=\gamma\left(\frac{(f*g)'(z)}{(f*g)(z)}-\frac{p}{z}\right). \tag{33}$$

Then by simple computation, we have,

$$\begin{aligned}\left|G''(z)\left[G'(z)\right]^{\frac{e^{i\lambda}}{\cos\lambda}-1}\right| &= \left|\gamma\left(\frac{(f*g)(z)}{z^{p}}\right)^{\frac{\gamma e^{i\lambda}}{\cos\lambda}}\left(\frac{(f*g)'(z)}{(f*g)(z)}-\frac{p}{z}\right)\right|\\ &\leq \frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}},\end{aligned}$$

where we have used (31). Therefore

$$\left|G''(z)\left[G'(z)\right]^{\frac{e^{i\lambda}}{\cos\lambda}-1}\right|\leq\frac{(n+1)\cos\lambda}{2\sqrt{(n+1)^{2}+1}} \tag{34}$$

By using Theorem 2.7 with $p=1$, $\alpha=0$ and $g(z)=\frac{z}{(1-z)^{2}}$, we have $G(z)\in C_{\lambda}(1,n,0)$.

From (33), we can write

$$\Re\left[e^{i\lambda}\left(1+\frac{zG''(z)}{G'(z)}\right)\right]=\gamma\Re e^{i\lambda}\left(\frac{z\left((f*g)'(z)\right)}{(f*g)(z)}\right)-p\gamma\cos\lambda+\cos\lambda, \tag{35}$$

or

$$Ree^{i\lambda}\left(\frac{z\left(\left(f*g\right)'\left(z\right)\right)}{\left(f*g\right)\left(z\right)}\right)>\left(p-\frac{1}{\gamma}\right)\cos\lambda\ \left(sinceG\left(z\right)\in C_{\lambda}\left(1,n,0\right)\right) \tag{36}$$

which shows that $f(z)\in Q_{\lambda}\left(p,n,0;g\left(z\right)\right)$, where $\gamma\geq\frac{1}{p}$.

References

[1] S. P. Goyal, S. K. Bansal, P. Goswami, Extension of sufficient conditions for starlikeness and convexity of order α for multivalent function, Appl. Math. Lett., 25(11)(2012), 1993-1998.

[2] L. Spacek, Prispěvek k teorii funkei prostych, Čapopis Pest. Mat. Fys., 62(1933), 12-19.

[3] M. S. Robertson, Univalent functions $f(z)$ for wich $zf'(z)$ is spiral-like, Mich. Math. J., 16(1969), 97-101.

[4] P. T. Mocanu, Some simple criteria for starlikeness and convexity, Libertas Math., 13(1993), 27-40.

[5] N. Uyanik, M. Aydogan, S. Owa, Extension of sufficient conditions for starlikeness and convexity of order α, Appl. Math. Lett., 24(9)(2011), 1393-1399.

[6] P. T. Mocanu, Some starlikeness conditions for analytic functions, Rev. Roumaine Math., Pures Appl., 33(1988), 117-124.

[7] M. Nunokawa, S. Owa, Y. Polattoglu, M. Caglar, E. Y. Duman, Some sufficient conditions for starlikeness and convexity, Turk. J. Math., 34(2010), 333-337.

[8] J. W. Alexander, Function which map the interior of the unit circle upon simple regions, Ann. Math., 1**7**(1915), 12-22.

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ADDITIVE FUNCTIONAL INEQUALITIES IN PARANORMED SPACES

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Abstract. In this paper, we investigate the following additive functional inequalities

$$\begin{aligned} \left\| \frac{1}{s} f(x) + \frac{1}{s} f(y) + f(z) + f(w) \right\| &\leq \left\| f\left(\frac{x+y}{s} + z + w \right) \right\|, \\ \left\| \frac{1}{s} f(x) + \frac{1}{s} f(y) + \frac{1}{s} f(z) + f(w) \right\| &\leq \left\| f\left(\frac{x+y+z}{s} + w \right) \right\| \end{aligned}$$

in paranormed spaces for a fixed integer s greater than 1. Furthermore, we prove the Hyers-Ulam stability of the above additive functional inequalities in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [26] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [5, 14, 16, 17, 25]). This notion was defined in normed spaces by Kolk [15].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [28] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

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In 1990, Th.M. Rassias [22] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] following the same approach as in Th.M. Rassias [21], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Th.M. Rassias and Šemrl [23] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [2], D.H. Hyers, G. Isac and Th.M. Rassias [11]).

In 1982, J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias' theorem [21] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings (see [12, 13, 18]).

In [8], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [24]. Fechner [4] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park, Cho and Han [19] proved the Hyers-Ulam stability of the following functional inequalities

$$\begin{aligned}
\|f(x) + f(y) + f(z)\| &\leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \\
\|f(x) + f(y) + f(z)\| &\leq \|f(x+y+z)\|, \\
\|f(x) + f(y) + 2f(z)\| &\leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|.
\end{aligned}$$

We proved the Hyers-Ulam stability of the following functional inequalities

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\|, \tag{1.2}$$

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + \frac{1}{s}f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y+z}{s} + w\right)\right\| \tag{1.3}$$

for a fixed integer s greater than 1.

In Section 2, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces.

In Section 3, we prove the Hyers-Ulam stability of the functional inequality (1.3) in paranormed spaces.

Throughout this paper, assume that $(X, P(\cdot))$ is a total paranormed space and that $(Y, \|\cdot\|)$ is a Banach space.

2. Hyers-Ulam stability of the functional inequality (1.2)

In this section, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces.

Proposition 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + f(z) + f(w)\right\| \leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\| \tag{2.1}$$

for all $x, y, z, w \in X$. Then f is additive.

Proof. Letting $x = y = z = w = 0$ in (2.1), we get

$$\left(\frac{2}{s} + 2\right) \|f(0)\| = \left\|\frac{2}{s}f(0) + 2f(0)\right\| \leq \|f(0)\|$$

and so

$$f(0) = 0.$$

Letting $x = y = 0$ and $w = -z$ in (2.1), we get

$$f(-z) = -f(z)$$

for all $z \in X$. Letting $x = -sz$ and $y = w = 0$ in (2.1), we get

$$f(sz) = f(sz) \qquad \& \qquad f\left(\frac{z}{s}\right) = \frac{1}{s}f(z)$$

for all $z \in X$. Letting $z = -\frac{x+y}{s}$ and $w = 0$ in (2.1), we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is additive. □

Note that $P(sx) \leq sP(x)$ for all $x \in X$.

Theorem 2.2. *Let r be a positive real number with $r < 1$, and $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} \left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + f(z) + f(w)\right\| &\leq \left\|f\left(\frac{x+y}{s} + z + w\right)\right\| \\ &+ P(x)^r + P(y)^r + P(z)^r + P(w)^r \end{aligned} \tag{2.2}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq s\left(\frac{s^r + 1}{s - s^r}\right) P(x)^r \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = w = 0$ and $z = -\frac{x}{s}$ in (2.2), we get

$$\left\|\frac{1}{s}f(x) - f\left(\frac{x}{s}\right)\right\| = \left\|\frac{1}{s}f(x) + f\left(-\frac{x}{s}\right)\right\| \leq P(x)^r + P\left(-\frac{x}{s}\right)^r$$

and so

$$\left\|\frac{1}{s}f(sx) - f(x)\right\| \leq P(sx)^r + P(-x)^r \leq (s^r + 1)P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{s^l}f(s^l x) - \frac{1}{s^m}f(s^m x)\right\| &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{s^j}f(s^j x) - \frac{1}{s^{j+1}}f(s^{j+1}x)\right\| \\ &\leq (s^r + 1)\sum_{j=l}^{m-1} \frac{s^{rj}}{s^j} P(x)^r \end{aligned} \tag{2.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.4) that the sequence $\{\frac{1}{s^n} f(s^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{s^n} f(s^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{s^n} f(s^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.3).

It follows from (2.2) that

$$\begin{aligned}
&\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| \\
&= \lim_{n\to\infty} \frac{1}{s^n} \left\|\frac{1}{s} f\left(s^n x\right) + \frac{1}{s} f\left(s^n y\right) + f\left(s^n z\right) + f\left(s^n y\right)\right\| \\
&\le \lim_{n\to\infty} \frac{1}{s^n} \left\|f\left(s^n\left(\frac{x+y}{s} + z + w\right)\right)\right\| + \lim_{n\to\infty} \frac{s^{nr}}{s^n}(P(x)^r + P(y)^r + P(z)^r + P(w)^r) \\
&= \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|
\end{aligned}$$

for all $x, y, z, w \in X$. So

$$\left\|\frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w)\right\| = \left\|h\left(\frac{x+y}{s} + z + w\right)\right\|$$

for all $x, y, z, w \in X$. By Proposition 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \frac{1}{s^n} \left\|h\left(s^n x\right) - T\left(s^n x\right)\right\| \\
&\le \frac{1}{s^n} \left(\left\|h\left(s^n x\right) - f\left(s^n x\right)\right\| + \left\|T\left(s^n x\right) - f\left(s^n x\right)\right\|\right) \\
&\le \frac{2s(s^r + 1)s^{nr}}{(s - s^r)s^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.3). □

3. Hyers-Ulam stability of the functional inequality (1.3)

In this section, we prove the Hyers-Ulam stability of the functional inequality (1.3) in paranormed spaces.

Proposition 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + \frac{1}{s}f(z) + f(w)\right\| \le \left\|f\left(\frac{x+y+z}{s} + w\right)\right\| \tag{3.1}$$

for all $x, y, z, w \in X$. Then f is additive.

Proof. Letting $x = y = z = w = 0$ in (3.1), we get

$$\left(\frac{3}{s} + 1\right)\|f(0)\| = \left\|\frac{3}{s}f(0) + f(0)\right\| \le \|f(0)\|$$

and so

$$f(0) = 0.$$

Letting $y = z = x$ and $w = -x$ in (3.1), we get

$$f(-x) = -f(x)$$

for all $x \in X$. Letting $w = -\frac{x}{s}$ and $y = z = 0$ in (3.1), we get

$$\frac{1}{s}f(x) = f\left(\frac{1}{s}x\right)$$

for all $x \in X$. Letting $z = -x - y$ and $w = 0$ in (3.1), we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is additive. □

Note that $P(sx) \le sP(x)$ for all $x \in X$.

Theorem 3.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\begin{aligned} \left\|\frac{1}{s}f(x) + \frac{1}{s}f(y) + \frac{1}{s}f(z) + f(w)\right\| &\le \left\|f\left(\frac{x+y+z}{s} + w\right)\right\| \\ &+ P(x)^r + P(y)^r + P(z)^r + P(w)^r \end{aligned} \tag{3.2}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le s\left(\frac{s^r + 1}{s - s^r}\right)P(x)^r \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = x = 0$ and $z = -\frac{x}{s}$ in (3.2), we get

$$\left\|\frac{1}{s}f(x) - f\left(\frac{x}{s}\right)\right\| = \left\|\frac{1}{s}f(x) + f\left(-\frac{x}{s}\right)\right\| \le P(x)^r + P\left(-\frac{x}{s}\right)^r$$

and so

$$\left\|\frac{1}{s}f(sx) - f(x)\right\| \le P(sx)^r + P(-x)^r \le (s^r + 1)P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{s^l}f(s^l x) - \frac{1}{s^m}f(s^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{s^j}f(s^j x) - \frac{1}{s^{j+1}}f(s^{j+1}x)\right\| \\ &\le (s^r + 1)\sum_{j=l}^{m-1}\frac{s^{rj}}{s^j}P(x)^r \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{\frac{1}{s^n}f(s^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{s^n}f(s^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{s^n}f(s^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.3).

It follows from (3.2) that

$$\begin{aligned}
&\left\| \frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w) \right\| \\
&= \lim_{n\to\infty} \frac{1}{s^n} \left\| \frac{1}{s} f\left(s^n x\right) + \frac{1}{s} f\left(s^n y\right) + f\left(s^n z\right) + f\left(s^n y\right) \right\| \\
&\le \lim_{n\to\infty} \frac{1}{s^n} \left\| f\left(s^n\left(\frac{x+y}{s} + z + w\right)\right) \right\| + \lim_{n\to\infty} \frac{s^{nr}}{s^n}(P(x)^r + P(y)^r + P(z)^r + P(w)^r) \\
&= \left\| h\left(\frac{x+y}{s} + z + w\right) \right\|
\end{aligned}$$

for all $x, y, z, w \in X$. So

$$\left\| \frac{1}{s}h(x) + \frac{1}{s}h(y) + h(z) + h(w) \right\| = \left\| h\left(\frac{x+y}{s} + z + w\right) \right\|$$

for all $x, y, z, w \in X$. By Proposition 3.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.3). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \frac{1}{s^n} \left\| h\left(s^n x\right) - T\left(s^n x\right) \right\| \\
&\le \frac{1}{s^n} \left(\left\| h\left(s^n x\right) - f\left(s^n x\right) \right\| + \left\| T\left(s^n x\right) - f\left(s^n x\right) \right\| \right) \\
&\le \frac{2s(s^r+1)s^{nr}}{(s-s^r)s^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.3). □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.

[3] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[4] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[5] J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[6] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[7] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[8] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.

[9] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[11] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[12] K. Jun and Y. Lee, *A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations*, J. Math. Anal. Appl. **297** (2004), 70–86.

[13] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.

[14] S. Karakus, *Statistical convergence on probabilistic normed spaces*, Math. Commun. **12** (2007), 11–23.

[15] E. Kolk, *The statistical convergence in Banach spaces*, Tartu Ul. Toime. **928** (1991), 41–52.

[16] M. Mursaleen, *λ-statistical convergence*, Math. Slovaca **50** (2000), 111–115.

[17] M. Mursaleen and S.A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Computat. Anal. Math. **233** (2009), 142–149.

[18] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[19] C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Inequal. Appl. **2007** (2007), Article ID 41820, 13 pages.

[20] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982) 126–130.

[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[22] Th.M. Rassias, *Problem 16; 2*, Report of the 27^{th} International Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.

[23] Th.M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[24] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.

[25] T. Šalát, *On the statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[26] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–34.

[27] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[28] A. Wilansky, *Modern Methods in Topological Vector Space*, McGraw-Hill International Book Co., New York, 1978.

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SOME IDENTITIES FOR BERNOULLI POLYNOMIALS INVOLVING CHEBYSHEV POLYNOMIALS

DAE SAN KIM, TAEKYUN KIM AND SANG-HUN LEE

Abstract. In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials associated with Chebyshev polynomials.

1. Introduction

The Bernoulli number are defined by the generating function to be

$$\frac{t}{e^t-1}=e^{Bt}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n,\quad \text{(see [3,13,14])}, \tag{1}$$

with the usual convention about replacing B^n by B_n.

As is well known, the Bernoulli polynomials are given by

$$B_n(x)=(B+x)^n=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l,\quad \text{(see [1-8])}. \tag{2}$$

From (1), we note that the recurrence relation for the Bernoulli numbers is given by

$$B_0=1,\quad (B+1)^n-B_n=\delta_{1,n},\quad \text{(see [6-8])},$$

where $\delta_{m,n}$ is the Kronecker symbol.

By (2), we get

$$\frac{dB_n(x)}{dx}=n\sum_{l=0}^{n-1}\binom{n-1}{l}B_{n-1-l}x^l=nB_{n-1}(x). \tag{3}$$

Thus, by (3), we see that

$$\int B_n(x)dx=\frac{B_{n+1}(x)}{n+1}+C,\quad \text{(see [3])}, \tag{4}$$

where C is a some constant.

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{5}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, (see [1,2,4,10,11]).

In the special case, $x=0$, $E_n(0)=E_n$ are called the n-th Euler numbers.

1

It is well known [6, 15] that Hermite polynomials are given by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{6}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$.

From (6),we have

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x), \quad H_n(x) = (-1)^n H_n(-x). \tag{7}$$

By (1) and (2), we easily get

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} E_{n-k}(x), \quad \text{(see [1-15])}, \tag{8}$$

$$E_n(x) = -2\sum_{l=0}^{n} \binom{n}{l} \frac{E_{l+1}}{l+1} E_{n-l}(x), \tag{9}$$

and

$$x^n = \frac{1}{n+1}\big(B_{n+1}(x+1) - B_{n+1}(x)\big) = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l} B_l(x). \tag{10}$$

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation

$$T_n(x) = \cos n\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{11}$$

If the range of the variable x is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$. It is known that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$, and indeed we are familiar with elementary formulas $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$, $\cdots$.

Thus, by (11), we get

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1, \cdots .$$

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \sin(n+1)\theta / \sin\theta, \quad \text{when } x = \cos\theta, \quad \text{(see [9])}. \tag{12}$$

Thus, from (12), we have

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \cdots .$$

By (11), we see that $T_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^{n-1} $(n \geq 1)$ and 1 $(n = 0)$. It is not difficult to show that $U_n(x)$ is a polynomial of degree n with integral coefficients and the leading coefficient 2^n $(n \geq 0)$. $T_n(x)$ is a solution of $(1-x^2)y'' - xy' + n^2y = 0$ and $U_n(x)$ is a solution of $(1-x^2)y'' - 3xy' + n(n+2)y = 0$. It is well known [9] that the generating functions of $T_n(x)$ and $U_n(x)$ are given by

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{13}$$

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and

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad \text{for } |x| \leq 1,\ |t| < 1. \tag{14}$$

From (11) and (12), we have

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{\pi}{2}, & \text{if } n = m > 0 \\ \pi, & \text{if } n = m = 0 \end{cases}, \tag{15}$$

and

$$\int_{-1}^{1} (1-x^2)^{1/2} U_n(x)U_m(x)dx = \frac{\pi}{2}\delta_{n,m}, \quad \text{(see [9])}. \tag{16}$$

The equations (15) and (16) are used to derive our main result in this paper.

The Rodrigues' formulae for $T_n(x)$ and $U_n(x)$ are known as follows:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n-1/2}\right), \tag{17}$$

and

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-1/2}\left(\frac{d^n}{dx^n}(1-x^2)^{n+1/2}\right). \tag{18}$$

The equations (17) and (18) are also used to derive our result related to orthogonality of Chebyshev polynomials.

From (11) and (12), we can easily derive the following equations (19) and (20):

$$T_n(x) = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \tag{19}$$

and

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}. \tag{20}$$

By the definitions of $T_n(x)$ and $U_n(x)$, we easily get

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x), \quad \frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}. \tag{21}$$

From (21), we have

$$\int U_n(x)dx = \frac{T_{n+1}(x)}{n+1}, \quad \int T_n(x)dx = \frac{nT_{n+1}(x)}{n^2-1} - \frac{xT_n(x)}{n-1}. \tag{22}$$

In this paper we derive some new and interesting identities for Bernoulli, Euler and Hermite polynomials arising from the orthogonality of the Chebyshev polynomials for the inner product space with weighted inner product.

2. Some identities for Bernoulli, Euler and Hermite polynomials involving Chebyshev polynomials

Let $\mathbf{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n$ is an inner product space with the weighted inner product

$$\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}dx, \quad \text{where } \ p(x), q(x) \in \mathbf{P}_n.$$

From (15), we note that $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ is an orthogonal basis for $\mathbf{P}_n$. Let us assume $p(x) \in \mathbf{P}_n$. Then $p(x)$ is generated by $\{T_0(x), T_1(x), \cdots, T_n(x)\}$ to be

$$p(x) = \sum_{k=0}^{n} C_k T_k(x). \tag{23}$$

By (15) and (23), we get

$$\begin{aligned} &C_k = \frac{\delta_k}{\pi}\int_{-1}^{1} \frac{T_k(x)p(x)}{\sqrt{1-x^2}}dx = \frac{\delta_k}{\pi}\frac{(-1)^k 2^k k!}{(2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)p(x)dx,\\ &\text{where } \delta_k = \begin{cases} 1, & \text{if } k=0 \\ 2, & \text{if } k>0. \end{cases} \end{aligned} \tag{24}$$

Let us take $p(x) = x^n \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\right)x^n dx \\ &= \frac{(-1)^k 2^k k!}{\pi(2k)!}\delta_k(-1)^k\frac{n!}{(n-k)!}\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx. \end{aligned} \tag{25}$$

It is easy to show that

(26)

$$\begin{aligned} &\int_{-1}^{1}(1-x^2)^{k-1/2}x^{n-k}dx = \frac{(1+(-1)^{n-k})}{2}\int_{0}^{1}(1-y)^{k-1/2}y^{\frac{n-k+1}{2}-1}dy \\ &= \frac{(1+(-1)^{n-k})}{2}\frac{\Gamma(k+1/2)\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{k+n+2}{2})} = \frac{(1+(-1)^{n-k})}{2}\frac{(n-k)!(2k)!\pi}{2^{n+k}(\frac{n+k}{2})!(\frac{n-k}{2})!k!}. \end{aligned}$$

By (25) and (26), we get

$$C_k = \begin{cases} 0, & \text{if } n-k \equiv 1 \pmod 2 \\ \frac{n!\delta_k}{2^n(\frac{n+k}{2})!(\frac{n-k}{2})!}, & \text{if } n-k \equiv 0 \pmod 2. \end{cases} \tag{27}$$

From (27), we note that

$$x^n = \sum_{k=0}^{n} C_k T_k(x) = \frac{n!}{2^{n-1}}\sum_{\substack{1 \le k \le n \\ k \equiv 1 \pmod 2}} \frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}, \tag{28}$$

where $n \equiv 1 \pmod 2$.

For $n \equiv 0 \pmod 2$, we have

$$x^n = \frac{n!}{2^n}\left\{\frac{T_0(x)}{\left((\frac{n}{2})!\right)^2} + 2\sum_{\substack{2 \le k \le n \\ k \equiv 0 \pmod 2}} \frac{T_k(x)}{(\frac{n+k}{2})!(\frac{n-k}{2})!}\right\}. \tag{29}$$

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Let us take $p(x) = B_n(x) \in \mathbf{P}_n$. Then

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^k (1-x^2)^{k-1/2} \right) B_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} (-1)^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} B_{n-k}(x) dx \\ &= \frac{2^k k! \delta_k}{\pi(2k)!} \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx. \end{aligned} \tag{30}$$

Now, we compute $\int_{-1}^{1}(1-x^2)^{k-1/2}x^l dx$.

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx &= (1+(-1)^l) \int_{0}^{1} (1-x^2)^{k-1/2} x^l dx \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2 \\ \frac{l!(2k)!\pi}{2^{2k+l}(\frac{2k+l}{2})!(\frac{l}{2})!k!}, & \text{if } l \equiv 0 \pmod 2. \end{cases} \end{aligned} \tag{31}$$

By (30) and (31), we get

$$\begin{aligned} C_k &= \frac{2^k k! \delta_k}{\pi(2k)!} \times \frac{n!}{(n-k)!} \times \frac{(2k)!\pi}{2^{2k}k!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \binom{n-k}{l} B_{n-k-l} \frac{l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \\ &= \frac{n!\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!}. \end{aligned} \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (mod\ 2)}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{\delta_k}{2^k(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \pmod 2}} \frac{\binom{n-k}{l} E_{n-k-l} l!}{2^l(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n$. From (24), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^k k! \delta_k}{\pi(2k)!} \int_{-1}^{1} \left(\frac{d^k}{dx^k} (1-x^2)^{k-1/2} \right) H_n(x) dx \\ &= \frac{(-1)^k 2^k k! \delta_k}{(2k)!\pi} \times (-1)^k 2^k \frac{n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k-1/2} H_{n-k}(x) dx \\ &= \frac{2^{2k} k! \delta_k n!}{(2k)!(n-k)!\pi} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \int_{-1}^{1} (1-x^2)^{k-1/2} x^l dx, \end{aligned} \tag{33}$$

where H_{n-k-l} is the $(n-k-l)$th Hermite number.

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By (31) and (33), we get

$$C_k = n!\delta_k \sum_{\substack{0\le l\le n-k\\ l\equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!}. \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$H_n(x) = n! \sum_{0\le k\le n} \left(\delta_k \sum_{\substack{0\le l\le n-k\\ l\equiv 0\ (mod\ 2)}} \frac{H_{n-k-l}}{(n-k-l)!(\frac{2k+l}{2})!(\frac{l}{2})!} \right) T_k(x).$$

Let $\mathbf{P}_n^* = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n^*$ is an inner product space with the weighted inner product $\langle p(x),\ q(x)\rangle = \int_{-1}^{1} \sqrt{1-x^2}p(x)q(x)dx$, where $p(x), q(x) \in \mathbf{P}_n$. Then $\{U_0(x), U_1(x), \cdots, U_n(x)\}$ is an orthogonal basis for the inner product space $\mathbf{P}_n^*$.
For $p(x) \in \mathbf{P}_n^*$, let

$$p(x) = \sum_{k=0}^{n} C_k U_k(x), \tag{35}$$

where

$$\begin{aligned} C_k &= \frac{2}{\pi}\langle p(x),\ U_k(x)\rangle = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{1/2}U_k(x)p(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2}\right)p(x)dx. \end{aligned} \tag{36}$$

Let us assume that $p(x) = x^n \in \mathbf{P}_n^*$. Then, by (36), we get

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2}\right)x^n dx \\ &= \frac{(-1)^k 2^{2k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!}\int_{-1}^{1}(1-x^2)^{k+1/2}x^{n-k}dx. \end{aligned} \tag{37}$$

It is easy to show that

(38)

$$\begin{aligned} \int_{-1}^{1}(1-x^2)^{k+1/2}x^{n-k}dx &= (1+(-1)^{n-k})\int_{0}^{1}(1-x^2)^{k+1/2}x^{n-k}dx \\ &= \begin{cases} 0, & \text{if } n-k \equiv 1 \pmod 2 \\ \frac{(n-k)!(2k+2)!\pi}{2^{n+k+2}(\frac{n+k+2}{2})!(\frac{n-k}{2})!(k+1)!}, & \text{if } n-k\equiv 0 \pmod 2). \end{cases} \end{aligned}$$

Therefore, by (37) and (38), we obtain the following proposition.

Proposition 2.3. *For $n \in \mathbb{Z}_+$, we have*

$$x^n = \frac{n!}{2^n} \sum_{\substack{0\le k\le n\\ k\equiv n\ (mod\ 2)}} \frac{k+1}{(\frac{n+k+2}{2})!(\frac{n-k}{2})!} U_k(x).$$

Let us consider $p(x) = B_n(x) \in \mathbf{P}_n^*$. From (36), we have

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) B_n(x)dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{(-1)^k n!}{(n-k)!} \int_{-1}^{1} (1-x^2)^{k+1/2} B_{n-k}(x)dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi} \times \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx. \end{aligned} \tag{39}$$

It is not difficult to show that

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx &= (1+(-1)^l) \int_0^1 (1-x^2)^{k+1/2} x^l dx \\ &= \begin{cases} 0, & \text{if } l \equiv 1 \pmod 2) \\ \frac{(2k+2)!l!\pi}{2^{2k+2+l}(\frac{2k+2+l}{2})!(k+1)!(\frac{l}{2})!}, & \text{if } l \equiv 0 \pmod 2). \end{cases} \end{aligned} \tag{40}$$

By (39) and (40), we get

$$C_k = \frac{(k+1)n!}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\text{mod } 2)}} \frac{B_{n-k-l}}{(n-k-l)! 2^l (\frac{2k+l+2}{2})! (\frac{l}{2})!}. \tag{41}$$

Therefore, by (41), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\text{mod } 2)}} \frac{B_{n-k-l}}{2^l (n-k-l)! (\frac{2k+l+2}{2})! (\frac{l}{2})!} \right) U_k(x).$$

By the same method, we can derive the following identity:

$$E_n(x) = n! \sum_{0 \le k \le n} \left(\frac{k+1}{2^k} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\text{mod } 2)}} \frac{E_{n-k-l}}{2^l (n-k-l)! (\frac{2k+l+2}{2})! (\frac{l}{2})!} \right) U_k(x).$$

Let us take $p(x) = H_n(x) \in \mathbf{P}_n^*$. Then $H_n(x) = \sum_{k=0}^{n} C_k U_k(x)$, with

$$\begin{aligned} C_k &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+1/2} \right) H_n(x)dx \\ &= \frac{2^{2k+1}(k+1)!n!}{(2k+1)!\pi(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l H_{n-k-l} \int_{-1}^{1} (1-x^2)^{k+1/2} x^l dx \\ &= n!(k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\text{mod } 2)}} \frac{H_{n-k-l}}{(n-k-l)!} \times \frac{1}{(\frac{2k+l+2}{2})!(\frac{l}{2})!}. \end{aligned} \tag{42}$$

Thus, by (42) and (43), we get

$$H_n(x) = n! \sum_{0 \le k \le n} \left((k+1) \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\text{mod } 2)}} \frac{H_{n-k-l}}{(n-k-l)! (\frac{2k+l+2}{2})! (\frac{l}{2})!} \right) U_k(x).$$

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References

[1] S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. 22 (2012), no. 3, 399-406.

[2] A. Bayad, T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys. **18** (2011), no. 2, 133-143.

[3] L. Carlitz, *Note oin the integral of the product of several Bernoulli polynomials*, J. London Math. Soc. **34**(1959), 362-363.

[4] M. Can, M. Cenkci, V. Kurt, Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions*, Adv. Stud. Contemp. Math. **18** (2009), no. 2, 135-160.

[5] L. Fox, I. B. Parker, *Chebyshev polynomials in numerical analysis*, Oxford University Press, London-New York-Toronto, Ont. 1968.

[6] D. S. Kim, T. Kim, S.-H. Rim, S, H. Lee, *Hermite polynomials and their applications associated with Bernoulli and Euler numbers*, Discrete Dynamics in Nature and Society 2012(2012), Article ID **974632**, 13 pp.

[7] D. S. Kim, T. Kim, S.-H. Lee, Y.-H. Kim, *Some identities for the product of two Bernoulli and Euler polynomials*,Advances in Difference Equations 2012(2012) Article ID **2012:95**, 14 pp.

[8] T. Kim, *Identities involving Frobenius-Euler polynomials arising from nonlinear differential equations*, J. Number Theory (Article in Press), 2012.

[9] D. S. Kim, T. Kim, *Extended Laguerre polynomials associated with Hermite, Bernoulli, and Euler numbers and polynomials*, Abstr. Appl. Anal. 2012(2012), Art. ID 957350, 15 pp.

[10] D. S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012(2012), Art. ID 463659, 12 pp.

[11] T. Kim, C. S. Ryoo, H. Yi *A note on q-Bernoulli numbers and q-Bernstein polynomials,* Ars Combin. 104 (2012), 437-447.

[12] J. C. Mason, D.C. Handscomb, *Chebyshev Polynomials*, Chapman Hall CRC , A CRC Press Company Boca Raton London New York Washington, D.C., 2003.

[13] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. **18** (2009), no. 1, 41-48.

[14] S.-H. Rim, J. Jeong, *On the modified q-Euler numbers of higher order with weight*, Adv. Stud. Contemp. Math. **22** (2012), no. 1, 93-98.

[15] C. S. Ryoo, *Some identities of the twisted q-Euler numbers and polynomials associated with q-Bernstein polynomials*, Proc. Jangjeon Math. Soc. **14** (2011), no. 2, 239-248.

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An economical aggregation algorithm for algebraic multigrid (AMG)*

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Abstract

Aggregation-based AMG method is a widely studied technique of robustness for large-scale linear systems. Some previous aggregation algorithms, belonging to a part of aggregation-based AMG method, exhibit certain excellent properties. These aggregation methods, however, have to aggregate every grid points so that these methods lead expensive computation with grid points increasing. In the paper, a property that the aggregations hold particular structure associated with certain condition is discovered to damp the computation of aggregation algorithms. Meanwhile, this property is under the condition of the system matrix derived from the 9-point Finite Difference Method (FDM) and the particular setting of grid number. Furthermore, the conclusions about multilevel, such as the setting rule of grid number and corresponding theoretical analysis, are obtained from the extension of two level issues. Computational experiments demonstrate that the CPU time of new aggregation algorithm which generates the same aggregations with previous aggregation algorithms, keeps on a low level evidently, even for the linear systems of millions grade.

Key words: Aggregation-based AMG; Aggregation algorithms; Economical computation; Poisson-like equations; Helmholtz-like equations; Millions grade problems

1 Introduction

Several methods can be utilized to solve the large-scale sparse linear systems

$$Ax = b, \tag{1}$$

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where $A \in \mathcal{R}^{N\times N}$ arises from the discretization of a scalar second-order elliptic partial differential equation (PDE). AMG method for large-scale linear systems is among the most efficient and convenient iterative methods [1, 2, 3, 4, 5, 6, 7, 8].

AMG method is composed with two parts: one is the *setup phase* and the other is the *solve phase*. *Setup phase* is associated with three parts on each level, i.e., defining coarse grids (aggregations), constructing transfer operators (i.e., prolongation and restriction operators), computing the linear systems on the coarse level, respectively. *Solve phase* involves a recursive process with solving the linear systems level by level and contains three parts mainly, i.e., the smoothing steps, the transferring of linear systems among levels and solving linear systems on the coarsest level, respectively.

AMG method is a recursive method of efficiency for large-scale linear systems with mainly recursive forms: V-cycle and W-cycle, for instance, in [9, 10, 11, 12]. It projects the large-scale problems, level by level, to the small-scale problems until the problems can be solved as accurate as possible. The most important issue is making the computed solutions approximate to the true solutions. We transfer the linear systems of different levels through the restriction and prolongation operators R and P. AMG method has the following relationship among levels

$$A_{i+1} = R_i A_i P_i, \qquad i = 1, 2, \cdots, l_{\max} - 1, \tag{2}$$

where $l_{\max}$ labels the number of levels. R_i is the restriction operator from the i-th level to the $(i+1)$-th level while P_i, $P_i = R_i^T$, is prolongation operator from the $(i+1)$-th level to the i-th level. The subscripts of A_i denote corresponding belonging levels, and the levels range from fine to coarse with i increasing.

AMG method mainly refers to classical AMG method, aggregation-based AMG method, adaptive AMG method and AMGe method [6, 7, 13, 14]. Classical AMG method is introduced by Brandt, McCormick, Ruge and Stüben [15, 16], it has been employed to solve linear systems whose coefficient matrices are M-matrices. For aggregation-based AMG method, the most critical step is the construction of the prolongation operators by the aggregation algorithms [17, 18, 19, 20, 21, 22] based on different definitions of strength of Connection. Adaptive AMG method utilizes a multigrid algorithm to enhance the efficiency of the prolongation, aiming to earn a more efficient AMG algorithm [23, 24]. The AMGe method, located in [25, 26], was first introduced as a measure to improve the robustness of AMG for the finite element problems. It is different from standard AMG method for requiring access to local element stiffness matrices (in addition to the assembled global stiffness matrices). The main differences among these AMG methods, e.g., classical AMG method, aggregation-based AMG method, adaptive AMG method and AMGe method, can be discriminated by the constructions of transfer operators and coarse grids, respectively. Particularly, transfer operators of aggregation-based AMG method can be generated by a classical aggregation algorithm, corresponding details in [17]. The motivation of our work is to acquire the aggregations, same as the aggregations of the algorithm in [17], with cheaper computation by utilizing the property discovered in this paper.

In this paper, we mainly focus on the *setup phase* and establish a novel construction, aiming to reduce the computation of constructing aggregations. During the process of generating aggregations, an excellent discovery is found that the aggregations , obtained

under the condition of the square grid number satisfying $(3i+4)\times(3i+4)$, $i \in \mathcal{N}$, are symmetric. Besides, the system matrix on the finest level should be derived from the discretization of a scalar second-order PDE with 9-point FDM. Then we make use of these properties, the symmetry of aggregations and the relationship among subscripts of the grid points, to construct a new aggregation algorithm to decrease the computation. Computational experiments present that the new aggregation algorithm gains a lower consuming-time, besides, the same aggregations compared with previous aggregation algorithm in [17]. Particularly, we have to emphasize that this paper is mainly to improve the aggregation algorithm in the *setup phase*, aiming to gain more economical computation. The *solve phase*, meanwhile, keeps unchanged while our proposed method is applied in the *setup phase*.

The paper is organized as follows. In section 2, we introduce the basic scheme of AMG method and the classical aggregation algorithm. Section 3 is about the new aggregation algorithm based on 9-point FDM. Meanwhile, some theoretical analysis and conclusions on the parameter and grid number on finest level, respectively, are presented in this section. Section 4 shows the capability of our aggregation algorithm on some numerical experiments about 2D Poisson-like equation and 2D Helmholtz-like equation. A compact conclusion will be presented in section 5.

2 Aggregation-based AMG methods

Aggregation-based AMG method clusters the fine grid of unknowns to aggregations representing the unknowns on the coarse level. Different with other methods, aggregation-based AMG method constructs the transfer operators mentioned in section 1 by aggregating the unknowns on each level. The coarsening part in classic AMG method is realized mainly by the aggregation algorithm (i.e., the *setup phase* mentioned in section 1) generating prearranged conditions for *solve phase*, e.g., aggregations, transfer operators and linear systems on coarse level. Meanwhile, it is necessary to introduce the basic AMG scheme (See the following forma). Where $A_1 = A \in \mathcal{R}^{N\times N}$, $b_1 = b \in \mathcal{R}^N$,

$y_i = \mathrm{MGM}(x_0, b_i, i)$		
If $(i = m)$	Then	$y_m = \mathrm{Solve}(A_m, b_m, e_m)$
	Else	$x_i = \mathrm{Smooth}(A_i, b_i, x_0)$
		$r_{i+1} = R_i(b_i - A_i x_i)$
		$A_{i+1} = R_i A_i P_i$
		$d_{i+1} = \mathrm{MGM}(0, r_{i+1}, i+1)$
		$\hat{x}_i = x_i + P_i d_{i+1}$
		$x_i = \mathrm{Smooth}(A_i, b_i, \hat{x}_i)$

$x_0 = \mathbf{0} \in \mathcal{R}^N$ and transfer operators $R_i = P_i^T$. The above recursive process is called V-cycle while another recursive type of AMG is called W-cycle doing twice on the fifth row. Aggregation-based AMG is divide into two parts: one is *setup phase* and the other is *solve phase* mentioned in above section. The *setup phase* may be considered as the prearranged section of the *solve phase* for solving the linear system (1), i.e., aggregations, transfer operators (i.e., R_i and P_i) and coarse linear systems A_i on each

level, respectively, so the *setup phase* part acts actually an important role in the whole process of AMG.

2.1 The classical aggregation algorithm

Before giving the new algorithm, it is necessary to introduce the classical aggregation algorithm coming from [17, 27]. The following content is about the graph $G_{A_l}(V_l, E_l)$ of the system matrices A_l on the l-th level. We have to emphasize that the goal of illustrating this classical algorithm is to present that our new algorithm generates the same aggregations with the classical algorithm.

The system matrix A, generating the graph $G_A(V, E)$, is generally gained by handling the PDE with different methods of discretization, e.g., 5-point FDM and 9-point FDM, etal. In this section, some definitions about graph theory are summarized again.

Definition 1 ([27]). *Corresponding to a sparse matrix A with symmetric sparsity pattern (i.e., $a_{i,j} \neq 0 \Leftrightarrow a_{j,i} \neq 0$), let $G_A(V, E)$ be the graph that consists of a set $V = \{v_1, v_2, v_3, \cdots, v_n\}$ of n ordered vertices (nodes, unknowns), and a set of edges E such that the edge $e_{i,j} \in E$ exists (connecting v_i and v_j) if and only if $a_{i,j} \neq 0, i \neq j$.*

For a vertex v_i, the set of neighbor vertices N_i is defined in the following form,

$$N_i = \{v_j \in V | e_{i,j} \in E\}, \tag{3}$$

$|N_i|$ denotes the number of the elements in the set N_i. The degree of a vertex v_i is $\deg(v_i) = |N_i|$.

For example, if the matrix is

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix},$$

and the $G_A(V, E)$ of the matrix A is shown in Figure 1.

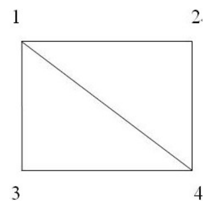


Figure 1: The matrix graph

The following contents introduce the classical aggregation algorithm and the construction of aggregations $\{A_i^l\}_{i=1}^{N_{l+1}}$ (i.e., the i-th aggregation on the l-th level), only depending on the l-th level system matrix A_l.

For a given parameter $\theta \in (0, 1]$, the strongly coupled neighborhood of the node v_i on the l-th level is defined as

$$N_i^l(\theta) = \left\{v_j \in V_l || a_{i,j}| \geq \theta \sqrt{a_{i,i} a_{j,j}}\right\}. \tag{4}$$

The classical aggregation algorithm, proposed by P. Vaněk, J. Mandel and M. Brezina [17] and utilized by Wagner [27], is presented in the following part.

Algorithm 1 Let a $N_l \times N_l$ matrix A_l with the corresponding graph $G_{A_l}(V_l, E_l)$ and $\theta \in (0,1]$ be given. The following Aggregation $(G_{A_l}(V_l, E_l))$ generates a disjoint covering $\left\{A_i^l\right\}_{i=1}^{N_{l+1}}$ of the set $V = \{v_1, v_2, v_3, \cdots, v_{N_l}\}$.

Aggregation$(G_{A_l}(V_l, E_l))$
{
initialization:
$U = \{v_i \in V_l | N_i^l(0) \neq \{v_i\}\}$;
j=0;
step 1:
$for(v_i \in U)$
{
$if(N_i^l(\theta) \subset U)$
$\{j++; A_j^l = N_i^l(\theta); U = U \backslash A_j^l;\}$
}
end

step 2:
$for(z \leq j) \quad \widetilde{A}_z^l = A_z^l; \quad end$
$for(v_i \in U)$
{
$for(z \leq j)$
{
$if(N_i^l(\theta) \cap \widetilde{A}_z^l \neq \{\}) \quad \{A_z^l = A_z^l \cup \{v_i\}; U = U \backslash \{v_i\}; break;\}$
}
end
}
end

step 3:
$for(v_i \in U)$
{
$j++; A_j^l = N_i^l(\theta) \cap U; U = U \backslash A_j^l;$
}
end
}

In the part of initialization, the set U does not contain all nodes, meanwhile, isolated nodes are not aggregated. In step 1, disjoint strongly coupled neighborhoods are selected as the initial approximation of the covering. Step 2 adds remaining nodes $v_i \in U$ to one of the sets A_z^l to which the node v_i is strongly connected if any such set exists. Finally, in step 3, the still remaining nodes $v_i \in U$ are clustered into aggregations that consist of subsets of strongly coupled neighborhoods.

The above algorithm acts crucial role for AMG method due to generating the pre-arranged information that mentioned at the beginning of this section. The above algorithm, however, runs slowly because it has to aggregate every point in the domain

and judge whether the points belong to certain aggregation. *Can we accelerate the process of above algorithm by some particular constructions?* Fortunately, next section will introduce the new discovery about the 9-point FDM based aggregation algorithm. We draw this inspiration of the discovery to develop a completely different algorithm with Algorithm 1.

3 The new aggregation algorithm

In this section, the discovery about the aggregations is illustrated clearly, meanwhile, the aggregation algorithm, according to the discovery, obtains the aggregations without through Algorithm 1 entirely but a new way of more economical computation. In classical algorithm (i.e., Algorithm 1), the final aggregations have to be gained by aggregating every point while the new way only needs to satisfy the particular condition about number of grids.

The new aggregation algorithm is based on the following definition of strongly coupled neighborhood, i.e., the eq. (4). If the problems are from the discretization of 9-point FDM, the $N_i(\theta)$, strongly coupled neighborhood of the node v_i, contains eight nodes around the v_i, e.g., $N_{10}(\theta) = \{v_2, v_3, v_4, v_9, v_{11}, v_{16}, v_{17}, v_{18}\}$ (Figure 2). Besides, the parameter θ must ensure existent according to the results in subsection 3.2.

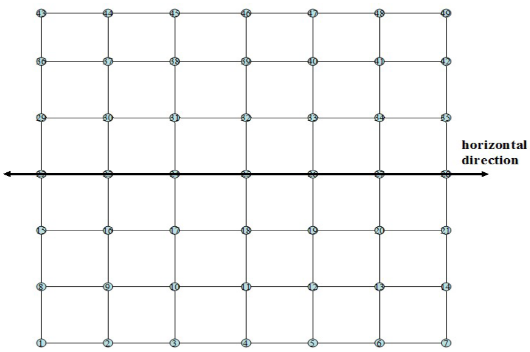
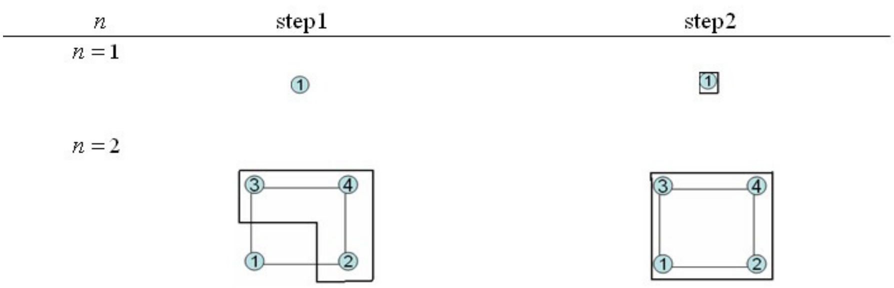


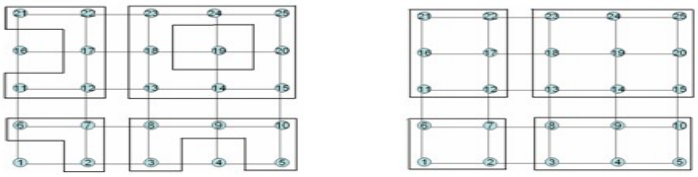
Figure 2: The instruction figure



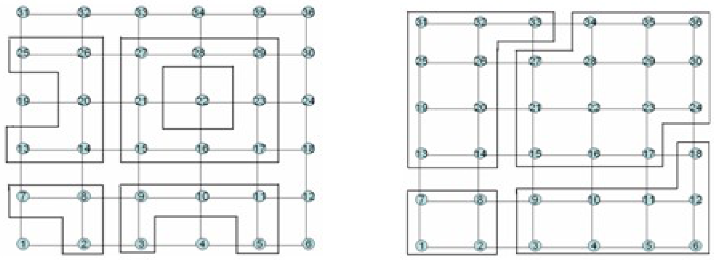
$n=3$

- $n=4$ $(i.e, n=4+3\times 0)$

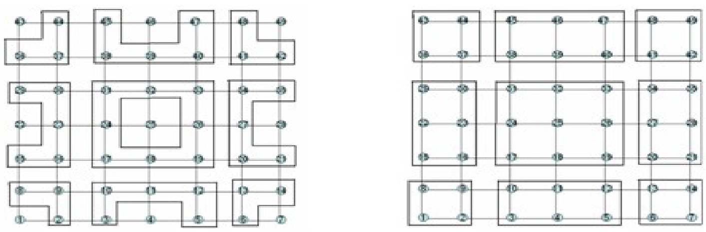
$n=5$



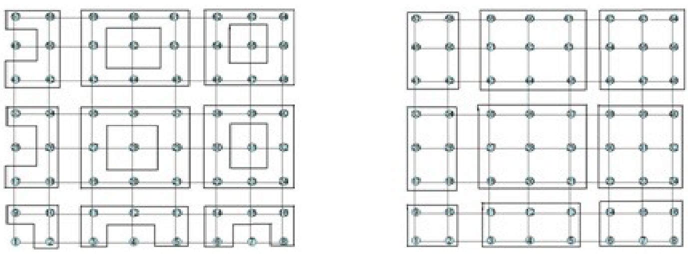
$n=6$



- $n=7$ $(i.e, n=4+3\times 1)$

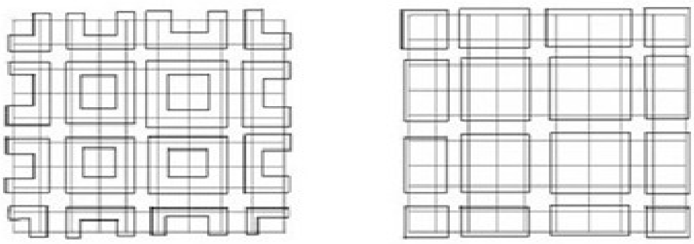


$n=8$



7

• $n = 10, (i.e, n = 4 + 3 \times 2)$



$n = \cdots$ $\cdots$ $\cdots$

Figure 3: The process of constructing aggregations according to Algorithm 1 with 9-point FDM

3.1 The discovery for constructing aggregations based on 9-point FD-M

To demonstrate our new aggregation algorithm clearly, we give the Figure 3 with small grid number n. From Figure 3, we learn that

1. When $n = 2 + 3x, (x = 0, 1, 2, \cdots)$, the aggregations are symmetrical about back diagonal direction, and the number of aggregations is $(1 + x)^2$.

2. When $n = 4 + 3x, (x = 0, 1, 2, \cdots)$, the aggregations are symmetrical about horizontal direction and vertical direction (See Figure 2), and the number of final aggregations is $(2 + x)^2$.

Particularly, when $n = 4 + 3x, (x = 0, 1, 2, \cdots)$ (See • of Figure 3), there is the property of symmetry so that the aggregations can be gained by fixed scheme easily when the grid number is set as $n = 4+3x, (x = 0, 1, 2, \cdots)$ (See details in the following algorithm). We note the subscripts from left to right and then from down to up (See Figure 2) to illustrate our algorithm clearly.

Based on the discovery, the finally aggregation algorithm is given as follows.

Algorithm 2. Consider matrix $A \in R^{N_l \times N_l}(N_l = n_l^2, n_l = 4 + 3x, x = 0, 1, 2, \cdots)$ and corresponding graph $G_{A_l}(V_l, E_l)$ and $\theta \in (0, 1]$ being given. Then Aggregation $(G_{A_l}(V_l, E_l))$ generates a disjoint covering $\{A_i^l\}_{i=1}^{N_{l+1}}$ of the set $V = \{v_1, v_2, v_3, \cdots, v_{N_l}\}$.

Aggregation($G_{A_l}(V_l, E_l)$)
{

/* firstly, we have the relation: $n_l = 4 + 3x, (x = 0, 1, 2, \cdots)$,$A_{k,j}^l = A_{(k-1)(x+2)+j}^l$ (See the following paragraph)*/

/***step 1:** get four angle's aggregations (See Figure 4 (a)) */

Get $A_{1,1}^l \quad A_{1,(x+2)}^l \quad A_{(x+2),1}^l \quad A_{(x+2),(x+2)}^l$

/* **step 2:** get the aggregations of upper boundary and lower boundary (See Figure 4 (b) (c)) */

for j=2:(x+1)
/* get the aggregations of lower boundary */

$$A^l_{1,j} = \{V_{3(j-1)}, V_{3(j-1)+1}, V_{3(j-1)+2}, V_{3(j-1)+n}, V_{3(j-1)+n+1}, V_{3(j-1)+n+2}\};$$

/* get the aggregations of upper boundary */

$$\begin{aligned} A^l_{(x+2),j} = \{&V_{3(j-1)+(3x+2)\cdot n}, V_{3(j-1)+(3x+2)\cdot n+1}, V_{3(j-1)+(3x+2)\cdot n+2}, \\ &V_{3(j-1)+(3x+2)\cdot n+n}, V_{3(j-1)+(3x+2)\cdot n+n+1}, V_{3(j-1)+(3x+2)\cdot n+n+2}\}; \end{aligned}$$

end
/* **step 3:** get the aggregations of left boundary and right boundary and central zone (See Figure 4 (d) (e) (f)) */

for k=2:(x+1)
/* get the aggregations of left boundary (See Figure 4 (d))*/

$$\begin{aligned} A^l_{k,1} = \{&V_{(3k-4)\cdot n+1}, V_{(3k-4)\cdot n+2}, \\ &V_{(3k-4)\cdot n+n+1}, V_{(3k-4)\cdot n+n+2}, \\ &V_{(3k-4)\cdot n+2n+1}, V_{(3k-4)\cdot n+2n+2}\}; \end{aligned}$$

/* get the aggregations of right boundary (See Figure 4 (e))*/

$$\begin{aligned} A^l_{k,(x+2)} = \{&V_{(3k-4)\cdot n+3x+3}, V_{(3k-4)\cdot n+3x+4}, \\ &V_{(3k-4)\cdot n+3x+3+n}, V_{(3k-4)\cdot n+3x+4+n}, \\ &V_{(3k-4)\cdot n+3x+3+2n}, V_{(3k-4)\cdot n+3x+4+2n}\}; \end{aligned}$$

/* get the aggregations of central zone (See Figure 4 (f))*/

for j=2:(x+1)

$$\begin{aligned} A^l_{k,j} = \{&V_{(3k-4)\cdot n+3(j-1)}, V_{(3k-4)\cdot n+3(j-1)+1}, V_{(3k-4)\cdot n+3(j-1)+2}, \\ &V_{(3k-4)\cdot n+3(j-1)+n}, V_{(3k-4)\cdot n+3(j-1)+n+1}, V_{(3k-4)\cdot n+3(j-1)+n+2}, \\ &V_{(3k-4)\cdot n+3(j-1)+2n}, V_{(3k-4)\cdot n+3(j-1)+2n+1}, V_{(3k-4)\cdot n+3(j-1)+2n+2}\}; \end{aligned}$$

end
end
}
where the aggregation algorithm is under the condition that PDEs are discretilized by 9-point FDM when $n = 4 + 3x, (x = 0, 1, 2, \cdots)$. We utilize a useful formula $A_{k,j} = A_{(k-1)(x+2)+j}$, matching the Algorithm 2 for two-dimensional $A_{k,j}$. The formula is easy to be proved. Seeing Figure 5, we learn that $A_1 = A_{1,1}$, $A_2 = A_{1,2}$, $A_3 = A_{1,3}$, etc. By $n = 4+3x, (x = 0, 1, 2, \cdots)$, the aggregations' number of every row is $x+2$ and

the total number of $k-1$ rows is $(k-1)(x+2)$. So $A_{k,j} = A_{(k-1)(x+2)+j}$ is proved easily. We can obtain the aggregations of boundary and central zone easily through step 2 and step 3 of the above Algorithm 2, respectively. Finally, this algorithm generates the same aggregations with classical algorithm (i.e., Algorithm 1).

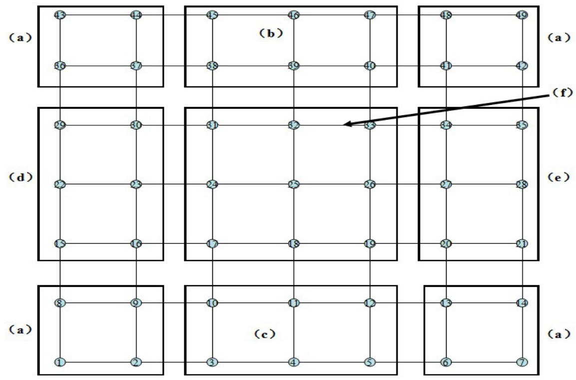


Figure 4: The instruction figure

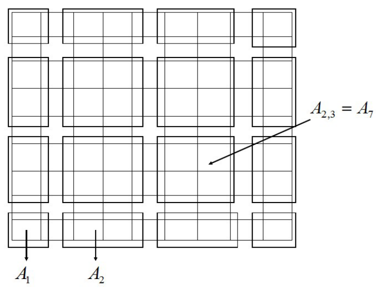


Figure 5: The instruction figure of formula $A_{k,j} = A_{(k-1)(x+2)+j}$

3.2 About the parameter θ

The parameter $\theta \in (0, 1]$ in equation (4) plays a significant role, because θ can decide the node v_j whether belongs to certain strongly coupled neighborhood $N_i(\theta)$ of node v_i. For example, if the parameter θ is smaller enough, then the corresponding strongly coupled neighborhood of node v_i will contain more nodes. Moreover, maybe the finally aggregations by the Algorithm 2 are changed obviously with slightly change of $\theta \in (0, 1]$, so it is necessary to discuss the parameter in equation (4).

Due to the discretization method (i.e., 9-point FDM) of corresponding problems, we hope the $N_i(\theta)$, the strongly coupled neighborhood of node v_i, contains the corresponding eight nodes of around v_i. We will demonstrate the existence of parameter θ firstly for this condition.

Theorem 1. *Let the strongly coupled neighborhood of node v_i be defined as equation (4). Consider the coefficient matrix $A \in R^{N\times N}$ arising from 9-point FDM, if $a_{ij} \neq$*

$0, (i \neq j)$, then, there must exist $\theta \in (0, 1]$, such that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$.

Proof. We learn that the coupled neighborhood of v_i is defined as (4)

$$N_i^l(\theta) = \left\{ v_j \in V_l | |a_{i,j}| \geq \theta \sqrt{a_{i,i} a_{j,j}} \right\}.$$

According to known conditions by 9-point FDM, all diagonal elements of matrix A are the nonzero (i.e, $a_{ii} \neq 0$), so above definition can be written as follows

$$0 < \theta \leq \frac{|a_{i,j}|}{\sqrt{a_{i,i} \cdot a_{j,j}}},$$

by $\theta \in (0, 1]$, we have

$$\theta \in (0, 1] \cap (0, \frac{|a_{i,j}|}{\sqrt{a_{i,i} \cdot a_{j,j}}}].$$

So we are sure that there must exist a θ such that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$. □

For 9-point FDM, the Theorem 1 can ensure the $N_i(\theta)$ containing the corresponding eight nodes of around v_i so that the Algorithm 2 is available. Furthermore, the simplified corollary will be presented in following part.

Corollary 1. *There exists the following definition of strongly couple neighborhood of node v_i where coefficient matrix $A \in R^{N \times N}$ arising from 9-point FDM*

$$N_i^l(\theta) = \{ v_j \in V_l | |a_{i,j} > 0 \}, \tag{5}$$

such that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$.

Proof. Due to the 9-point FDM, the coefficient matrix $A \in R^{N \times N}$ is a nine diagonal matrix that every row of A has only nine nonzero elements including $a_{i,i} \neq 0$. According to the new definition (5) and 9-point FDM, it is easy to know that $N_i(\theta)$ contains the corresponding eight nodes of around v_i, $i = 1, 2, 3, \cdots, N$. □

3.3 Extending to multilevel

This section mainly makes a discussion about extending the proposed Algorithm 2 to multilevel. According to section 3.1, if the grid number on the fine level is $(3x + 4) \times (3x+4)$ $(x = 0, 1, 2, \cdots)$, then the grid number on the next coarse level is $(x+2) \times (x+2)$ under the condition that one aggregation on the fine level generates only one grid node on the coarse level. In order to extend the relationship from two level to multilevel, two significant conclusions are given in the following analysis.

Theorem 2. *Let the number of levels of multigrid be L and assuming the gird number is $n_L = 3i + 4$ (i.e., the square grid is $(3i + 4) \times (3i+4)$) on the coarsest level. If the girds number n on the finest level satisfies the following equation*

$$n = (\sum_{j=1}^{L-1} 3^j \cdot 2) + 3^L \cdot i + 4, \tag{6}$$

then the Algorithm 2 can be extended to multilevel.

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Proof. Since the number of levels is L and the gird number on the coarsest level L is $n_L = 3i + 4$. According to the conclusions of section 3.1, the grid number on the $(L-1)$-th level should be

$$n_{L-1} = 3(n_L - 2) + 4 = 3(3 \cdot i + 4 - 2) + 4 = 3^2 \cdot i + 3 \cdot 2 + 4,$$

and the grid number on the $(L-2)$-th level should be

$$\begin{aligned} n_{L-2} &= 3(n_{L-1} - 2) + 4 = 3(3^2 \cdot i + 3 \cdot 2 + 4 - 2) + 4 \\ &= 3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 = (\sum_{j=1}^{2} 3^j \cdot 2) + 3^3 \cdot i + 4, \end{aligned}$$

and similar to above, the grids number on the $(L-3)$-th level should be

$$\begin{aligned} n_{L-3} &= 3(n_{L-2} - 2) + 4 = 3(3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 - 2) + 4 \\ &= 3^3 \cdot i + 3^2 \cdot 2 + 3 \cdot 2 + 4 = (\sum_{j=1}^{3} 3^j \cdot 2) + 3^4 \cdot i + 4, \end{aligned}$$

it is easy to extend to the finest level by mathematical induction, the grids number on the finest level should satisfy

$$n_1 = 3(n_2 - 2) + 4 = 3((\sum_{j=1}^{L-2} 3^j \cdot 2) + 3^{L-1} \cdot i + 4 - 2) + 4 = (\sum_{j=1}^{L-1} 3^j \cdot 2) + 3^L \cdot i + 4,$$

and $n = n_1$ is the grids number satisfying Algorithm 2 on the finest level and it also extends the Algorithm 2 to multilevel. □

Theorem 2 requires the grid number on coarsest level being $n_L = 3i + 4, i = 0, 1, 2, \cdots$, moreover, we can also extend to the arbitrary grids number $n_L = i, i = 0, 1, 2, \cdots$, on coarsest level.

Corollary 2. *For arbitrary grid number $n_L = i, i = 0, 1, 2, \cdots$, on the coarsest level, if the number of levels is L, then the gird number n on the finest level satisfy*

$$n = (\sum_{j=1}^{L-2} 3^j \cdot 2) + 3^{L-1} \cdot (i - 2) + 4, \tag{7}$$

then the Algorithm 2 can be extended to multilevel.

Proof. It is easy to gain the Corollary 2 by replacing $n_L = 3i+4$ in Theorem 2 with $n_L = i$. □

4 Computational experiments

All experimental problems are discretized by 9-point FDM. Before our experiments, we will introduce the 9-point FDM briefly.

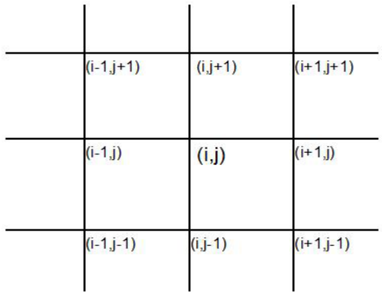

Figure 6: The instruction figure

We can get the 9-point FDM from the 5-point FDM in which one point (i, j) is only relevant to its adjacent four points (See Figure 6), i.e., $(i-1, j), (i+1, j), (i, j-1), (i, j+1)$. For example, if the elliptic PDE in a square domain is Poisson equation

$$-\Delta u = -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x, y), (x, y) \in \mathcal{R}^{[a,b]\times[a,b]}, \tag{8}$$

if $h_1 = h_2 = (b-a)/(n+1)$, then the 5-point FDM scheme can be obtained as follows

$$-\Delta_h u_{i,j} = \frac{1}{h^2}(-u_{i,j+1} - u_{i,j-1} - u_{i+1,j} - u_{i-1,j} + 4u_{i,j}) = f_{i,j}, \tag{9}$$

We define the vector $u_h = [u_{11}, u_{21}, \cdots, u_{n,1}; \cdots; u_{1,n}, u_{2,n}, \cdots, u_{n,n}]^T$ and assume zero boundary, then the finally linear system is obtained by (9)

$$\frac{1}{h^2} H u_h = g, \tag{10}$$

where

$$H = \begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B & -I \\ & & & -I & B \end{pmatrix}, B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix},$$

where the right hand vector is known beforehand and I is the identity matrix.

Then we rotate the coordinate system with $\pi/4$ so that the point (i, j) is relevant to its adjacent four points (See Figure 6), i.e., $(i-1, j+1), (i-1, j-1), (i+1, j-1), (i+1, j+1)$. By this rotation, another 5-point FDM scheme is as follows

$$-\bar{\Delta}_h u_{i,j} = \frac{1}{2h^2}(-u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} - u_{i-1,j-1} + 4u_{i,j}) = f_{i,j}, \tag{11}$$

and it also gains the similar linear system with (10) but the wider bandwidth of coefficient matrix.

Combining with the above two 5-point FDM scheme (9) and (11), the finally 9-point FDM scheme of Poisson equation [10] is determined,

$$-(\frac{2}{3}\Delta_h + \frac{1}{3}\bar{\Delta}_h)u_{ij} = f_{ij} + \frac{h^2}{12}\Delta_h f_{ij}, \tag{12}$$

where this scheme has smaller truncation error of $O(h^4)$ than 5-point FDM scheme.

Besides, some notations are necessary to be introduced. The t_i where $i = 1, 2, \cdots, L-1$, is just the CPU time of constructing aggregations by Algorithm 2 on the i-th level. $T_i, i = 1, 2, \cdots, L-1$, represents the total CPU time of generating prolongation operators by the following equation (13) and the coefficient matrices by equation (2) on the i-th level, respectively.

$$P^l_{ij} = \begin{cases} 1, & i \in A^l_j, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Moreover, the dimension N of coefficient matrix on the finest level, i.e., n^2, is computed by equation (6). Next, we will present two examples to demonstrate the efficiency of our algorithm.

4.1 Example 1: Poisson-like equation

First example is a 2D Poisson-like equation containing two scalars $\alpha, \beta \in \mathcal{R}$, it can be written in the form of

$$-(\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial y^2}) = f(x, y), (x, y) \in \Omega = [a, b] \times [a, b], \tag{14}$$

where the $f(x, y) \in \mathcal{R}$ is an arbitrary function and the boundary condition is

$$u(a, y) = u(b, y) = u(x, a) = u(x, b) = C \in \mathcal{R}. \tag{15}$$

It is easy to obtain the finally coefficient matrix $A \in \mathcal{R}^{N\times N}$ arising from the 9-point FDM of equation (14). A is nine diagonal matrix

$$A = \begin{pmatrix} B & R & & & \\ R & B & R & & \\ & \ddots & \ddots & \ddots & \\ & & R & B & R \\ & & & R & B \end{pmatrix}, B = \begin{pmatrix} e & k & & & \\ k & e & k & & \\ & \ddots & \ddots & \ddots & \\ & & k & e & k \\ & & & k & e \end{pmatrix}, R = \begin{pmatrix} p & q & & & \\ q & p & q & & \\ & \ddots & \ddots & \ddots & \\ & & q & p & q \\ & & & q & p \end{pmatrix},$$

where $e = 12(\alpha + \beta) - 4\alpha\beta, k = 2\alpha\beta - 6\alpha, p = 2\alpha\beta - 6\beta, q = -\alpha\beta$.

In Table 1, we choose the L being 3 levels and set the grid number n_L on the coarsest level being 64, 94 and 124, respectively. According to Theorem 2, the dimensions of linear systems on the finest level are 322624, 702244, 1227664, respectively. The CPU time t, constructing aggregation on each level, is very short and not exceeding 0.4s for the large-scale matrix with dimension 1227664 while the classical algorithm exceeds 1000s. The total time for the dimension with 322624, 702244, 1227664 is about 10.824s, 49.501s, 148.483s, respectively.

Table 1: CPU time for Poisson-like equation by our method with 3 levels

	t		T		Total
N	t_1	t_2	T_1	T_2	t+T
322624	0.102	0.009	10.525	0.188	10.824
702244	0.192	0.018	48.601	0.690	49.501
1227664	0.320	0.033	146.160	1.970	148.483

Table 2: CPU time for Poisson-like equation by our method with 4 levels

		t			T		Total
N	t_1	t_2	t_3	T_1	T_2	T_3	t+T
237169	0.081	0.006	0.0008	5.65	0.108	0.007	5.853
795664	0.215	0.021	0.003	62.150	0.849	0.027	63.265
1682209	0.436	0.045	0.005	274.050	3.644	0.077	278.257

In Table 2, L is 4 and the grid number n_L is 19, 34, 49, respectively. The dimensions of linear systems on the finest level are 237169, 795664, 1682209, respectively, according to Theorem 2. The consuming time t for constructing aggregations on each level is not exceeding 0.5s for the large-scale matrix with dimension 1682209 while the classical algorithm can not compute the consuming time. Total time for the dimension with 237169, 795664, 1682209 is about 5.853s, 63.265s, 278.257s, respectively. The two tables with different maximal levels illustrate that the Algorithm 2 is indeed a novel and fast method for the *setup phase* of aggregation-base AMG method.

From Table 1 and Table 2, it is easy to learn that the total time does not only contain the time t, constructing aggregations by Algorithm 2, but also the time T, constructing prolongation operators and generating the system matrices on each level. Furthermore, the mainly cost of total time is clearly T_1, because the matrices, keeping largest dimension on the finest level, are referred to vast matrix-matrix multiplication according to equation (2). The time on other levels are shorter seriously than the finest level and decreased evidently.

4.2 Example 2: Helmholtz equation

The second example containing a scalars $\omega \in \mathcal{R}$ is a 2D Helmholtz equation, the form of this equation is as follows

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \omega^2 u = f(x, y), (x, y) \in \Omega = [a, b] \times [a, b], \tag{16}$$

where the $f(x, y) \in \mathcal{R}$ can be also an arbitrary function and the boundary condition is

$$u(a, y) = u(b, y) = u(x, a) = u(x, b) = C \in \mathcal{R}, \tag{17}$$

Table 3: Time consuming for Helmholtz equation by our method with 4 levels

		t			T		Total
N	t_1	t_2	t_3	T_1	T_2	T_3	t+T
237169	0.085	0.008	0.001	5.84	0.119	0.009	6.062
795664	0.275	0.026	0.004	62.450	0.836	0.030	63.621
1682209	0.442	0.055	0.006	280.150	3.744	0.087	284.484

where $\omega \in R$ is a determined scalar, $h = (b-a)/(n+1)$. Similar to section 4.1, A is also a nine diagonal matrix

$$A = \begin{pmatrix} B & R & & & \\ R & B & R & & \\ & \ddots & \ddots & \ddots & \\ & & R & B & R \\ & & & R & B \end{pmatrix},$$

and

$$B = \begin{pmatrix} 20-2h^2\omega^2 & -4 & & & \\ -4 & 20-2h^2\omega^2 & -4 & & \\ & \ddots & \ddots & \ddots & \\ & & -4 & 20-2h^2\omega^2 & -4 \\ & & & -4 & 20-2h^2\omega^2 \end{pmatrix}, R = \begin{pmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -4 & -1 \\ & & & -1 & -4 \end{pmatrix},$$

In this example, ω, set to be 0.2, is utilized for all experiments. Actually the linear system generated by 9-point FDM has the same form with example 1, so the CPU time for the same dimension problem is almost not different. In Table 3, similar to Table 2, L is set to be 4 and n_L is also 19, 34, 49, respectively. The dimensions of linear systems on the finest level are 237169, 795664, 1682209, respectively. From Table 3, it is easy to learn that our algorithm speeds much less time while the classical one can not finish the *setup phase* within 1000s. Besides, the number of nonzero elements (NNZ) of system matrices on each level is presented in Table 4 where the NNZ1, NNZ2, NNZ3 and NNZ4 represent the number of nonzero elements on level 1, 2, 3 and 4, respectively.

Furthermore, we will try some larger scale system matrices to illustrate our Algorithm 2 all alone, i.e., the prolongation operators and system matrices on each level is out of range in the following test. L and is chosen to be 3 and n_L is set to be 229, 379, 604, 754, 904, respectively, i.e., the dimensions of the system matrices on the finest level are 4214809, 11580409, 29463184, 45941284 and 66064384, respectively. The finally shown results are in Table 5 clearly and indeed quite attractive.

5 Conclusion

This paper describes a new algorithm for constructing the aggregations in the *setup phase* of aggregation-based AMG method. The new algorithm, utilizing some particular

Table 4: NNZ on each level for Helmholtz equation by our method with 4 levels

N	NNZ1	NNZ2	NNZ3	NNZ4
237169	2128681	237169	26569	3025
795664	7150276	795664	88809	10000
1682209	15124321	1682209	187489	21025

Table 5: Time consuming of constructing aggregations by Algorithm 2 with 3 levels

N	4214809	11580409	29463184	45941284	66064384
t_1	1.028	2.815	7.002	10.844	15.553
t_2	0.091	0.260	0.713	0.112	1.620
Total	1.119	3.075	7.715	10.956	17.173

settings, e.g., the particular grid number on the finest level according to Theorem 2 and the discretization with 9-point FDM, is different with the any previous aggregation algorithms. During the process of constructing aggregations, the symmetry of the aggregations was discovered if the number of square grid satisfies the conditions of equation (5), (6) and (7). Moreover, some theoretical and practical conclusions such as Theorem 1, etal., were also illustrated in this paper. Computational experiments for Poisson-like equation and Helmholtz-like equation presented that the new aggregation algorithm captured the perfect results in the CPU time even for millions grade problems.

References

[1] A. Brandt, Algebraic multigrid theroy: the symmetric case, *Appl. Math. Comput.*, 1986, pp: 23-56

[2] K. Stüben, A review of algebraic multigrid, *J. Comput. Appl. Math.*, 1999, pp: 281-309

[3] P. Vaněk, M. Brezina and R. Tezaur, Two-grid method for linear elasticity on unstructured meshes, *SIAM J. Sci. Comput.*, 1999, pp: 900-923.

[4] U. Trottenberg, C. W. Oosterlee and A. Schüller, *Multigrid*, Academic Press, London, 2001

[5] R. D. Falgout and P. S. Vassilevski, On generalizing the AMG framwork, *SIAM J. Numerical Anal.*, 2004, pp: 1669-1693

[6] R. D. Falgout, An introduction to algebraic multigrid, *Computing in Science and Engineer*, 2006, pp: 24-33

[7] K. Stüben, *Multigrid*, chapter An introduction to algebraic multigrid, Academic Press, 2001, pp: 413-532

[8] J. K. Kraus, Algebraic multigrid based on computational molecules, 2: linear elasticity problems, *SIAM J. Sci. Comput.*, 2008, pp: 505-524

[9] W. L. Briggs, V. E. Henson and S. F. McCormick, *A Multigrid Tutorial* (Second Edition), SIAM, Philadelphia, 2000

[10] Y. Saad, *Iterative Methods for Sparse Linear Systems* (Second Edition), University of Minnesota, Minneapolis, Minnesota, 2003

[11] A. Thekale, T. Gradl, K. Klamroth and U. Rüde, Optimizing the number of multigrid cycles in the full multigrid algorithm, *Numerical Lin. Alg. Appl.*, 2010, pp: 199-210

[12] A. Napov and Y. Notay, When does two-grid optimality carry over to the V-cycle?, *Numerical Lin. Alg. Appl.*, 2010, pp: 273-290

[13] L. N. Olson, J. Schroder and R. S. Tuminaro, A new perspective on strength measures in algebraic multigrid, *Numerical Lin. Alg. Appl.*, 2010, pp: 713-733

[14] A. H. Baker, Tz. V. Kolev and U. M. Yang, Improving algebraic multigrid interpolation operators for linear elasticity problems, *Numerical Lin. Alg. Appl.*, 2010, pp: 495-517

[15] A. Brandt, S. F. Mccormick and J. W. Ruge, *Algebraic Multigrid (AMG) for Sparse Matrix Equations in Sparsity and Its Applications*, Cambridge University Press, Cambridge, 1984

[16] A. Brandt, S. F. Mccormick and J. W. Ruge, *Multigrid Methods and Applications*, vol.4 of Computation Mathematics, Springer Verlag, Berlin, 1985

[17] P. Vaněk, J. Mandel and M. Brezina, Algebraic multigrid based on smoothed aggregation for second and fourth order problems, *Computing*, 1996, pp: 179-196.

[18] M. Brezina, R. Falgout, S. MacLachlan, T. Manteuffel, S. McCormick and J. Ruge, Adaptive smoothed aggregation(αSA) multigrid, *SIAM Review*, 2005, pp: 317-346

[19] L. N. Olson and J. B. Schroder, Smoothed aggregation for Helmholtz problems, *Numerical Lin. Alg. Appl.*, 2010, pp: 361-386

[20] Y. Notay, Aggregation-based algebraic multilevel preconditioning, *SIAM J. Matrix Anal. Appl.*, 2006, pp: 998-1018

[21] A. Napov, Y. Notay, Algebraic analysis of aggregation-based multigrid, *Numerical Lin. Alg. Appl.*, 2010, pp: 539-564

[22] A. C. Muresan and Y. Notay, Analysis of aggregation-based multigrid, *SIAM J. Sci. Comput.*, 2008, pp: 1082-1103

[23] M. Brezina, R. Falgout, S. MacLachlan, T. Manteuffel and S. McCormick and J.Ruge, Adaptive algebraic multigrid, *SIAM J. Sci. Comput.*, 2006, pp: 1261-1286

[24] M. Brezina, R. Falgout, S. Maclachlan, T. Manteuffel, S. Mccormick and J. RUGE, Adaptive smoothed aggregation (αSA), *SIAM J. Sci. Comput.*, 2004, pp: 1896-1920

[25] M. Brezina, A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick and J. W. Ruge, Algebraic multigrid based on element interpolation (AMGe), *SIAM J. Sci. Comput.*, 2000, pp: 1570-1592

[26] V. E. Henson and P. S. Vassilevski, Element-free AMGe:general algorithms for computing interpolation weights in AMG, *SIAM J. Sci. Comput.*, 2001, pp: 629-650

[27] C. Wagner, *Introduction to algebraic multigrid*, Technical report, University of Heidelberg, Germany, 1999

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Relationship between lower and higher order anti-periodic boundary value problems and existence results

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Abstract

In this paper, we develop a relationship between lower and higher order classical anti-periodic boundary value problems. Some existence results for a $5th-$order anti-periodic boundary value problem of nonlinear ordinary differential equations are also presented. Our results are based on some standard tools of fixed point theory. The paper concludes with illustrative examples.

Key words and phrases: Ordinary differential equations; anti-periodic boundary conditions; existence; fixed point theorems
AMS (MOS) Subject Classifications: 334A34, 34B15, 34B27

1 Introduction

Anti-periodic boundary value problems occur in the mathematical modelling of a variety of physical processes and have recently received considerable attention. Examples include anti-periodic trigonometric polynomials in the study of interpolation problems [10], anti-periodic wavelets [8], difference equations [7, 20], ordinary, partial and abstract differential equations [1, 2, 12, 13, 14, 16, 19, 21, 22], fractional differential equations [5, 6] and impulsive differential equations [3, 4, 11, 15], etc. For some more application of anti-periodic boundary conditions in physics, see [9, 17] and the references therein.

The objective of this paper is to study a relationship between solutions of lower and higher order classical anti-periodic boundary value problems. For this purpose, we consider a $5th-$order anti-periodic boundary value problem and show that the solution for a $4th-$order anti-periodic problem follows from that of $5th-$order problem, the solution for $3rd-$order anti-periodic problem can be deduced from that of $4th-$order problem, and so on by applying a typical strategy. Expressions for Green's functions of anti-periodic problems are also presented. Besides this, we develop the existence theory for $5th-$order anti-periodic boundary value problems and illustrate it with some examples. Precisely, we consider the following problem:

$$\begin{cases} D^{(5)}x(t) = f(t, x(t)), \quad t \in [0, T], \ T > 0, \\ x(0) = -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T), \quad x'''(0) = -x'''(T), \quad x^{(iv)}(0) = -x^{(iv)}(T) \end{cases} \tag{1}$$

where f is a given continuous function.

2 Linear Problem

Lemma 2.1 *For any $y \in C[0,T]$, the unique solution for a linear $5th-$order antiperiodic boundary value problem*

$$\begin{cases} D^{(5)}x(t) = y(t), \quad t \in [0,T], \\ x(0) = -x(T),\ x'(0) = -x'(T),\ x''(0) = -x''(T),\ x'''(0) = -x'''(T),\ x^{(iv)}(0) = -x^{(iv)}(T) \end{cases} \tag{2}$$

is

$$x(t) = \int_0^T G_5(t,s)y(s)ds,$$

Where $G_5(t,s)$ is the Green's function given by

$$G_5(t,s) = \begin{cases} \frac{2(t-s)^4-(T-s)^4}{2(4!)} + \frac{(T-2t)(T-s)^3}{4(3!)} + \frac{t(T-t)(T-s)^2}{4(2!)} \\ +\frac{(6t^2T-4t^3-T^3)(T-s)}{48} + \frac{(2Tt^3-t^4-tT^3)}{48}, \quad 0 < s < t < T, \\ \\ -\frac{(T-s)^4}{2(4!)} + \frac{(T-2t)(T-s)^3}{4(3!)} + \frac{t(T-t)(T-s)^2}{4(2!)} \\ +\frac{(6t^2T-4t^3-T^3)(T-s)}{48} + \frac{(2Tt^3-t^4-tT^3)}{48}, \quad 0 < t < s < T. \end{cases} \tag{3}$$

Proof. It is well known that the integral representation for the solution of equation $D^{(5)}x(t) = y(t)$ can be written as

$$x(t) = \int_0^t \frac{(t-s)^4}{4!}y(s)ds - b_o - b_1t - b_2t^2 - b_3t^3 - b_4t^4. \tag{4}$$

where $b_o, b_1, b_2, b_3, b_4 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions of the problem (2) in (4), we find that

$$\begin{aligned} b_o &= \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}y(s)ds - \frac{T}{4}\int_0^T \frac{(T-s)^3}{3!}y(s)ds + \frac{T^3}{48}\int_0^T (T-s)y(s)ds, \\ b_1 &= \frac{1}{2}\int_0^T \frac{(T-s)^3}{3!}y(s)ds - \frac{T}{4}\int_0^T \frac{(T-s)^2}{2!}y(s)ds + \frac{T^3}{48}\int_0^T y(s)ds, \\ b_2 &= \frac{1}{4}\int_0^T \frac{(T-s)^2}{2!}y(s)ds - \frac{T}{8}\int_0^T (T-s)y(s)ds,\ b_3 = \frac{1}{12}\int_0^T (T-s)y(s)ds - \frac{T}{24}\int_0^T y(s)ds, \\ b_4 &= \frac{1}{48}\int_0^T y(s)ds. \end{aligned}$$

Substituting the values of $b_o, b_1,\ b_2,\ b_3$ and b_4 in (4), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^4}{4!}y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}y(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^3}{3!}y(s)ds \\ &+\frac{t(T-t)}{4}\int_0^T \frac{(T-s)^2}{2!}y(s)ds + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T (T-s)y(s)ds \\ &+\frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T y(s)ds. \end{aligned} \tag{5}$$

Alternatively, (5) can be written in term of Green's function as

$$x(t) = \int_0^T G_5(t,s)y(s)ds,$$

where $G_5(t,s)$ is given by (3). This completes the proof.

2.1 Relationship between higher-order and lower-order anti-periodic problems

It is found that there exists a relationship between higher-order and lower-order anti-periodic boundary value problems. For instance, by dropping the last term of (5) and replacing $\frac{(T-s)^p}{p!}$ by $\frac{(T-s)^{p-1}}{(p-1)!}$ in the resulting expression of (5), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^3}{3!} y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^3}{3!} y(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^2}{2!} y(s)ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)}{1!} y(s)ds + \frac{(6t^2T - 4t^3 - T^3)}{48}\int_0^T y(s)ds, \end{aligned} \tag{6}$$

which is the solution of $4th-$order anti-periodic boundary value problem:

$$\begin{cases} D^{(4)}x(t) = y(t), \quad t \in [0,T], \\ x(0) = -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T), \quad x'''(0) = -x'''(T), \end{cases} \tag{7}$$

In this case, Green's function $G_4(t,s)$ is

$$G_4(t,s) = \begin{cases} \frac{2(t-s)^3-(T-s)^3}{2(3!)} + \frac{(T-2t)(T-s)^2}{4(2!)} + \frac{t(T-t)(T-s)}{4} + \frac{(6t^2T-4t^3-T^3)}{48}, \ 0 < s < t < T, \\ -\frac{(T-s)^3}{2(3!)} + \frac{(T-2t)(T-s)^2}{4(2!)} + \frac{t(T-t)(T-s)}{4} + \frac{(6t^2T-4t^3-T^3)}{48}, \ \ 0 < t < s < T. \end{cases}$$

Similarly, by dropping the last term of (6) and replacing $\frac{(T-s)^p}{p!}$ by $\frac{(T-s)^{p-1}}{(p-1)!}$ in the remaining terms of (6), we obtain the solution of a third-order anti-periodic boundary value problem given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^2}{2!} y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^2}{2!} y(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)}{1!} y(s)ds + \frac{t(T-t)}{4}\int_0^T y(s)ds \\ &= \int_0^T G_3(t,s)y(s)ds, \end{aligned} \tag{8}$$

where

$$G_3(t,s) = \begin{cases} \frac{2(t-s)^2-(T-s)^2}{2(2!)} + \frac{(T-2t)(T-s)}{4} + \frac{t(T-t)}{4}, \ \ 0 < s < t < T, \\ -\frac{(T-s)^2}{2(2!)} + \frac{(T-2t)(T-s)}{4} + \frac{t(T-t)}{4}, \ \ 0 < t < s < T. \end{cases}$$

If we discard the last term of (8) and replacing $\frac{(T-s)^p}{p!}$ by $\frac{(T-s)^{p-1}}{(p-1)!}$ in the remaining terms of (8), then we get the solution of a second-order anti-periodic boundary value problem, which is given by

$$x(t) = \int_0^t \frac{(t-s)}{1!} y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)}{1!} y(s)ds + \frac{(T-2t)}{4}\int_0^T y(s)ds, \tag{9}$$

and the associated Green's function $G_2(t,s)$ is

$$G_2(t,s) = \begin{cases} \frac{2(t-s)-(T-s)}{2} + \frac{(T-2t)}{4}, \ \ 0 < s < t < T, \\ -\frac{(T-s)}{2} + \frac{(T-2t)}{4}, \ \ 0 < t < s < T. \end{cases}$$

Thus, the above strategy is quite useful to write down the solutions for lower-order anti-periodic problems once the solution of a higher-order anti-periodic problem is available.

3 Some existence results

Let $\mathrm{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$.

Define an operator $\mathcal{U} : \mathrm{C} \to \mathrm{C}$ as

$$\begin{aligned}(\mathcal{U}x)(t) &= \int_0^t \frac{(t-s)^4}{4!} f(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!} f(s,x(s))ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^3}{3!} f(s,x(s))ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^2}{2!} f(s,x(s))ds \\ &+ \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T (T-s) f(s,x(s))ds + \frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T f(s,x(s))ds, \quad t\in[0,T]. \end{aligned} \tag{10}$$

Observe that the problem (1) has solutions if and only if the operator $\mathcal{U}$ has fixed points.

To prove the existence of solutions for (1), we recall some known results.

Theorem 3.1 *([18]) let X be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$V = \{u \in X | u = \mu T u, \quad 0 < \mu < 1\}$$

is bounded. Then T has a fixed point in X

Theorem 3.2 *([18]) Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \qquad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 3.3 *([18]) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Now we are in a position to present some existence results for problem (1).

Theorem 3.4 *Assume that there exists a positive constant L_1 such that $|f(t,x)| \le L_1$ for $t \in [0,T]$, $x \in$ C. Then the problem (1) has at least one solution.*

Proof. First of all, we show that the operator $\mathcal{U}$ defined by 10 is completely continuous. Observe that continuity of the operator $\mathcal{U}$ follows from the continuity of f. Let $\Omega \subset \mathrm{C}$ be bounded. Then, $\forall x \in \Omega$, it follows by the assumption $|f(t,x)| \le L_1$ that

$$\begin{aligned}\|(\mathcal{U}x)\| \le & \sup_{t\in[0,T]}\Big\{\int_0^t \frac{(t-s)^4}{4!}|f(s,x(s))|ds + \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}|f(s,x(s))|ds \\ &+ \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^3}{3!}|f(s,x(s))|ds + \frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^2}{2!}|f(s,x(s))|ds \\ &+ \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T (T-s)|f(s,x(s))|ds + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T |f(s,x(s))|ds\Big\} \\ \le & L_1 \sup_{t\in[0,T]}\Big\{\frac{1}{4!}\int_0^t (t-s)^4 ds + \frac{1}{2(4!)}\int_0^T (T-s)^4 ds + \frac{|T-2t|}{4(3!)}\int_0^T (T-s)^3 ds \\ &+ \frac{|t(T-t)|}{4(2!)}\int_0^T (T-s)^2 ds + \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T (T-s)ds + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T ds\Big\} \\ \le & \tfrac{193T^5L_1}{3840} = L_2, \end{aligned} \tag{11}$$

which implies that $\|(\mathcal{U}x)\| \leq L_2$. Furthermore,

$$\begin{aligned} \|(\mathcal{U}x)'\| = & \sup_{t\in[0,T]} \Big\{ \int_0^t \frac{(t-s)^3}{3!}|f(s,x(s))|ds + \frac{1}{2}\int_0^T \frac{(T-s)^3}{3!}|f(s,x(s))|ds \\ & + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^2}{2!}|f(s,x(s))|ds + \frac{|t(T-t)|}{4}\int_0^T (T-s)|f(s,x(s))|ds \\ & + \frac{|6Tt^2-4t^3-T^3|}{48}\int_0^T |f(s,x(s))|ds\Big\} \\ \leq & L_1 \sup_{t\in[0,T]} \Big\{ \int_0^t \frac{(t-s)^3}{3!}ds + \frac{1}{2}\int_0^T \frac{(T-s)^3}{3!}ds + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^2}{2!}ds \\ & + \frac{|t(T-t)|}{4}\int_0^T (T-s)ds + \frac{|6Tt^2-4t^3-T^3|}{48}\int_0^T ds\Big\} \\ \leq & L_1 \tfrac{15T^4}{96} = L_3. \end{aligned} \tag{12}$$

Hence, for $t_1, t_2 \in [0,T]$, we have

$$|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \leq \int_{t_1}^{t_2} |(\mathcal{U}x)'(s)|ds \leq L_3(t_2-t_1).$$

Thus, by the foregoing arguments, one can infer that the operator $\mathcal{U}$ is equicontinuous on $[0,T]$. Hence, by the Arzela-Ascoli theorem, the operator $\mathcal{U} : \mathrm{C} \to \mathrm{C}$ is completely continuous.

Next, we consider the set

$$V = \{x \in \mathrm{C} \mid x = \mu\mathcal{U}x,\ 0<\mu<1\},$$

and show that it is bounded. Let $x \in V$, then $x = \mu\mathcal{U}x,\ 0<\mu<1$. For any $t \in [0,T]$, we have

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^4}{4!}y(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}y(s)ds + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^3}{3!}y(s)ds \\ & + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^2}{2!}y(s)ds + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T (T-s)y(s)ds + \frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T y(s)ds \\ = & \int_0^T G_5(t,s)y(s)ds \end{aligned}$$

$$\begin{aligned} |x(t)| = & \mu|(\mathcal{U}x)(t)| \leq \int_0^t \frac{(t-s)^4}{4!}|f(s,x(s))|ds + \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}|f(s,x(s))|ds \\ & + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^3}{3!}|f(s,x(s))|ds + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^2}{2!}|f(s,x(s))|ds \\ & + \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T (T-s)|f(s,x(s))|ds + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T |f(s,x(s))|ds \\ \leq & L_1\Big[\tfrac{1}{4!}\textstyle\int_0^t (t-s)^4 ds + \tfrac{1}{2(4!)}\int_0^T (T-s)^4 ds \\ & + \frac{|T-2t|}{4(3!)}\int_0^T (T-s)^3 ds + \frac{|t(T-t)|}{4(2!)}\int_0^T (T-s)^2 ds \\ & + \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T (T-s)ds + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T ds\Big] \\ \leq & \max_{t\in[0,T]}\Big\{\frac{2|t^5|+T^5}{2(5!)} + \frac{|T-2t|T^4}{4(4!)} + \frac{|t(T-t)|T^3}{4(3!)} + \frac{|6t^2T-4t^3-T^3|T^2}{48(2!)} \\ & + \frac{|2Tt^3-t^4-tT^3|T}{48}\Big\}L_1 = M_1. \end{aligned} \tag{13}$$

Thus, $\|x\| \leq M_1$ for any $t \in [0,T]$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator $\mathcal{U}$ has at least one fixed point, which means that (1) has at least one solution.

Theorem 3.5 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, and $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$. Then the problem* (1) *has at least one solution.*

Proof. By the assumption $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$, there exists a constant $r > 0$ such that $|f(t,x)| \le \delta|x|$ for $0 < |x| < r$, where $\delta > 0$ satisfies the condition

$$\max_{t\in[0,T]} \Big\{ \tfrac{2|t^5|+T^5}{2(5!)} + \tfrac{|T-2t|T^4}{4(4!)} + \tfrac{|t(T-t)|T^3}{4(3!)} + \tfrac{|6t^2T-4t^3-T^3|T^2}{48(2!)} + \tfrac{|2Tt^3-t^4-tT^3|T}{48} \Big\}\delta \le 1. \tag{14}$$

Define $\Omega_1 = \{x \in \mathrm{C} \mid \|x\| < r\}$ and take $x \in \mathrm{C}$ such that $\|x\| = r$, that is, $x \in \partial\Omega$. As before, it can be shown that $\mathcal{U}$ is completely continuous and

$$|\mathcal{U}x(t)| \le \max_{t\in[0,T]} \Big\{ \tfrac{2|t^5|+T^5}{2(5!)} + \tfrac{|T-2t|T^4}{4(4!)} + \tfrac{|t(T-t)|T^3}{4(3!))} + \tfrac{|6t^2T-4t^3-T^3|T^2}{48(2!)} + \tfrac{|2Tt^3-t^4-tT^3|T}{48} \Big\}\delta\|x\|, \tag{15}$$

which, in view of (14), yields $\|\mathcal{U}x\| \le \|x\|$, $x \in \partial\Omega$. Therefore, by Theorem 3.2, the operator $\mathcal{U}$ has at least one fixed point which corresponds to at least one solution of problem (1).

Our next existence result is based on Krasnoselskii's fixed point theorem [18].

Theorem 3.6 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumptions*

($\mathbf{A_1}$) $|f(t,x) - f(t,y)| \le L|x-y|, \forall t \in [0,T],\ x, y \in \mathbb{R}$;

($\mathbf{A_2}$) $|f(t,x)| \le \mu(t), \quad \forall (t,x) \in [0,1] \times \mathbb{R}$, *and* $\mu \in \mathbb{C}([0,T], R^+)$.

Then the problem (1) has at least one solution on $[0,T]$ if

$$L < \frac{3840}{161T^5}. \tag{16}$$

Proof. Let us define $\sup_{t\in[0,T]} |\mu(t)| = \|\mu\|$, and consider the set $B_r = \{x \in \mathrm{C} : \|x\| \le r\}$, where

$$r \ge \frac{193\|\mu\|T^5}{3840}.$$

Introduce operators χ and φ on B_r as

$$\begin{aligned}
(\chi x)(t) &= \int_0^t \frac{(t-s)^4}{4!} f(s,x(s))ds, \\
(\varphi x)(t) &= -\frac{1}{2}\int_0^T \frac{(T-s)^4}{4!} f(s,x(s))ds + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^3}{3!} f(s,x(s))ds \\
&+ \frac{1}{4}(t(T-t))\int_0^T \frac{(T-s)^2}{2!} f(s,x(s))ds + \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T (T-s)f(s,x(s))ds \\
&+ \frac{(2Tt^3-t^4-tT^3)}{48}\int_0^T f(s,x(s))ds.
\end{aligned}$$

For $x, y \in B_r$, we find that

$$\|\chi x + \varphi y\| \le \frac{193\|\mu\|T^5}{3840} \le r.$$

Thus, $\chi x + \varphi y \in B_r$. In view of (16), φ is a contraction mapping. Continuity of f implies that the operator χ is continuous. Also, χ is uniformly bounded on B_r as

$$\|\chi x\| \le \frac{\|\mu\|T^5}{5!}.$$

Now we prove the compactness of the operator χ. In view of (A_1), we define

$$\sup_{(t,x)\in[0,T]\times B_r} \|f(t,x)\| = f_m < \infty,$$

and consequently, for $t_1,\ t_2 \in [0,T],\ t_1 < t_2$, we have

$$|(\chi x)(t_2) - (\chi x)(t_1)| \leq \frac{f_m}{4!}\Big|\int_{t_1}^{t_2}(t_2-s)^4 ds - \int_0^{t_1}[(t_1-s)^4-(t_2-s)^4]ds\Big|,$$

which is independent of x and tends to zero as $t_2 \to t_1$. So χ is relatively compact on B_r. Hence, By the Arzela Ascoli theorem, χ is compact on B_r. Thus all the assumptions of Theorem 3.3 are satisfied. Therefore, the conclusion of Theorem 3.3 applies and the anti-periodic boundary value problem (1) has at least one solution on $[0,T]$. This completes the proof.

Theorem 3.7 *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the condition*

$$|f(t,x)-f(t,y)| \leq L|x-y|, \forall t\in[0,T],\ x,y\in\mathbb{R}$$

with $\rho L < 1$ where $\rho = \frac{193T^5}{3840}$. Then the anti-periodic boundary value problem (1) has a unique solution.

proof. Fixing $\sup_{t\in[0,T]}|f(t,0)| = M$ and selecting $r > \rho M/(1-\rho L)$ with $\rho = 193T^5/3840$, we show that $\mathcal{U}B_r \subset B_r$, where $B_r = \{x \in \mathrm{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
|(\mathcal{U}x)(t)| \leq\ & \int_0^t \frac{(t-s)^4}{4!}|f(s,x(s))-f(s,0)+f(s,0)|ds \\
&+\frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}|f(s,x(s))-f(s,0)+f(s,0)|ds \\
&+\frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^3}{3!}|f(s,x(s))-f(s,0)+f(s,0)|ds \\
&+\frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^2}{2!}|f(s,x(s))-f(s,0)+f(s,0)|ds \\
&+\frac{1}{48}|6t^2T-4t^3-T^3|\int_0^T (T-s)|f(s,x(s))-f(s,0)+f(s,0)|ds \\
&+\frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T |f(s,x(s))-f(s,0)+f(s,0)|ds \\
\leq\ & (Lr+M)\Big[\tfrac{1}{4!}\textstyle\int_0^t(t-s)^4ds + \tfrac{1}{2(4!)}\int_0^T(T-s)^4ds \\
&+\frac{|T-2t|}{4(3!)}\int_0^T(T-s)^3ds + \frac{|t(T-t)|}{4(2!)}\int_0^T(T-s)^2ds \\
&+\frac{1}{48}|6t^2T-4t^3-T^3|\int_0^T(T-s)ds + \frac{|2Tt^3-t^4-tT^3|}{48}\int_0^T(T-s)ds\Big] \\
\leq\ & (Lr+M)\tfrac{193T^5}{3840} \leq r,
\end{aligned} \tag{17}$$

which implies that $\|(\mathcal{U}x\| \leq r$. Thus, $\mathcal{U}x \in B_r\ \forall x \in B_r$. Hence $\mathcal{U}B_r \subset B_r$.

Now, for $x, y \in \mathrm{C}$ and for each $t \in [0,T]$, we obtain

$$\begin{aligned}
&\|(\mathcal{U}x)-(\mathcal{U}y)\| \\
\leq\ & \max_{t\in[0,T]}\Big\{\int_0^t \frac{(t-s)^4}{4!}|f(s,x(s))-f(s,y(s))|ds + \frac{1}{2}\int_0^T \frac{(T-s)^4}{4!}|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^3}{3!}|f(s,x(s))-f(s,y(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}|t(T-t)|\int_0^T \frac{(T-s)^2}{2!}|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{1}{48}|6t^2T-4t^3-T^3|\int_0^T (T-s)|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{1}{48}|2Tt^3-t^4-tT^3|\int_0^T |f(s,x(s))-f(s,y(s))|ds\Big\} \\
\le\ & L\|x-y\| \max_{t\in[0,T]}\Big\{\frac{1}{4!}\int_0^t (t-s)^4 ds+\frac{1}{2(4!)}\int_0^T (T-s)^4 ds+\frac{|T-2t|}{4(3!)}\int_0^T (T-s)^3 ds \\
&+\frac{|t(T-t)|}{4(2!)}\int_0^T (T-s)^2 ds+\frac{1}{48}|6t^2T-4t^3-T^3|\int_0^T (T-s)ds+\frac{1}{48}|2Tt^3-t^4-tT^3|\int_0^T ds\Big\} \\
\le\ & \frac{193T^5L}{3840}\|x-y\|,
\end{aligned}
$$

which depends only on the parameters T, L involved in the problem. As $\frac{193T^5L}{3840}<1$, therefore $\mathcal{U}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). This completes the proof.

Example 3.8 *Consider the following antiperiodic boundary value problem*

$$
\begin{cases}
D^{(5)}x(t)=\dfrac{e^{-\cos^2 x(t)}\Big[1+3\cos 2t+2\ln(2+3\sin^2 x(t))\Big]}{3+\sin x(t)}, \quad 0<t<1, \\
x(0)=-x(1),\quad x'(0)=-x'(1),\quad x''(0)=-x''(1), \\
x'''(0)=-x'''(1),\quad x^{(iv)}(0)=-x^{(iv)}(1),
\end{cases}
\tag{18}
$$

Clearly $|f(t,x)|\le 2+\ln 5=L_1$. So the hypothesis of Theorem 3.4 holds. Hence, by the conclusion of Theorem 3.4, there exists at least one solution for problem (18).

Example 3.9 *Consider the problem*

$$
\begin{cases}
D^{(5)}x(t)=(5+x^3(t))^{\frac{1}{2}}+2(t+1)(x-\sin x(t))-\sqrt{5}, \quad 0<t<1, \\
x(0)=-x(1),\quad x'(0)=-x'(1),\quad x''(0)=-x''(1), \\
x'''(0)=-x'''(1),,\quad x^{(iv)}(0)=-x^{(iv)}(1).
\end{cases}
\tag{19}
$$

Since

$$\lim_{x\to 0}\frac{(5+x^3)^{\frac{1}{2}}+2(t+1)(x-sinx)-\sqrt{5}}{x}=0,$$

therefore, the conclusion of Theorem 3.5 applies to problem (19).

Example 3.10 *Consider the problem*

$$
\begin{cases}
D^{(5)}x(t)=\dfrac{|x|}{1+|x|}+sin(t\sqrt{1+t^2}), \quad 0<t<1, \\
x(0)=-x(1),\quad x'(0)=-x'(1),\quad x''(0)=-x''(1), \\
x'''(0)=-x'''(1),,\quad x^{(iv)}(0)=-x^{(iv)}(1).
\end{cases}
\tag{20}
$$

Obviously $L=1$ *as*

$$|f(t,x)-f(t,y)|=\Big|\frac{|x|}{1+|x|}-\frac{|y|}{1+|y|}\Big|\le |x-y|.$$

Since $T=1$ *in this case, therefore,* $L<3840/193$. *Hence, by Theorem 3.7, problem (20) has a unique solution on* $[0,1]$.

Example 3.11 *Consider the problem*

$$\begin{cases} D^{(5)}x(t) = L(\sqrt{1+t^2}\cos t + tan^{-1}x + (t+1)^2), \quad 0 < t < 1 \\ x(0) = -x(1), \quad x'(0) = -x'(1), \quad x''(0) = -x''(1), \\ x'''(0) = -x'''(1) \quad , x^{(iv)}(0) = -x^{(iv)}(1) \end{cases} \tag{21}$$

It can easily be found that $|f(t,x) - f(t,y)| \leq L|x-y|$, where $L < 3840/161$. Thus, the conclusion of Theorem 3.6 applies to the problem (21).

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References

[1] A.R. Aftabizadeh, S. Aizicovici, N.H. Pavel, On a class of second-order anti-periodic boundary value problems. J. Math. Anal. Appl. 171 (1992), 301-320.

[2] A.R. Aftabizadeh, Y.K. Huang, N.H. Pavel, Nonlinear third-order differential equations with anti-periodic boundary conditions and some optimal control problems. J. Math. Anal. Appl. 192 (1995), 266-293.

[3] B. Ahmad, J. J. Nieto, Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions. Nonlinear Anal. 69 (2008), 3291-3298.

[4] B. Ahmad, Existence of solutions for second order nonlinear impulsive boundary value problems. Elect. J. Differential Equations, 2009(2009), No. 68, 1-7.

[5] B. Ahmad, J.J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory. Topol. Methods Nonlinear Anal. 35 (2010), 295-304.

[6] R.P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. 62 (2011), 1200-1214.

[7] A. Cabada, D.R. Vivero, Existence and uniqueness of solutions of higher-order antiperiodic dynamic equations. Adv. Difference Equ. 4 (2004), 291-310.

[8] H.L. Chen, Antiperiodic wavelets. J. Comput. Math. 14 (1996), 32-39.

[9] Y. Chen, J.J. Nieto, D. O'Regan, Antiperiodic solutions for fully nonlinear first-order differential equations. Math. Comput. Modelling 46 (2007), no. 9-10, 1183-1190.

[10] F.J. Delvos, L. Knoche, Lacunary interpolation by antiperiodic trigonometric polynomials. BIT 39 (1999), 439-450.

[11] W. Ding, Y. Xing, M. Han, Antiperiodic boundary value problems for first order impulsive functional differential equations. Appl. Math. Comput. 186 (2007), 45-53.

[12] D. Franco, J.J. Nieto, D. O'Regan, Anti-periodic boundary value problem for nonlinear first order ordinary differential equations. Math. Inequal. Appl. 6 (2003), 477-485.

[13] D. Franco, J.J. Nieto, Existence of solutions for first order ordinary differential equations with nonlinear boundary conditions. Appl. Math. Comput. 153 (2004), 793-802.

[14] B. Liu, An anti-periodic LaSalle oscillation theorem for a class of functional differential equations. J. Comput. Appl. Math. 223 (2009), 1081-1086.

[15] Z. Luo, J. Shen, J.J. Nieto, antiperiodic boundary value problem for first-order impulsive ordinary differential equations. Comput. Math. Appl. 49 (2005), 253-261.

[16] M. Nakao, Existence of an anti-periodic solution for the quasilinear wave equation with viscosity. J. Math. Anal. Appl. 204 (1996), 754764.

[17] J. Shao, Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays. Phys. Lett. A 372 (2008), 5011-5016.

[18] D.R. Smart, Fixed Point Theorems. Cambridge University Press, 1980.

[19] P. Souplet, Optimal uniqueness condition for the antiperiodic solutions of some nonlinear parabolic equations. Nonlinear Anal. 32 1998), 279286.

[20] Y. Wang, Y.M. Shi, Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions. J. Math. Anal. Appl. 309 (2005), 56-69.

[21] K. Wang, Y. Li, A note on existence of (anti-)periodic and heteroclinic solutions for a class of second-order odes. Nonlinear Anal. 70 (2009), 1711-1724.

[22] Y. Yin, Anti-periodic solutions of some semilinear parabolic boundary value problems. Dynam. Contin. Discrete Impuls. Systems 1 (1995), 283-297.

The improved (G'/G)-expansion method to the (2+1)-dimensional breaking soliton equation

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Abstract

In this article, we generate abundant traveling wave solutions of partial differential equation, namely, the (2+1)-dimensional breaking soliton equation involving parameter by applying the improved (G'/G)-expansion method. In this method, $G''+\Psi G'+\Phi G=0$ together with $F(\varphi)=\sum_{f=-U}^{U} s_f (G'/G)^f$ is implemented, where $s_f\ (f=0,\pm1,\pm2,...,\pm U)$, Ψ and Φ are constants. In addition, the obtained analytical solutions are illustrated in three different families including solitons and periodic solurions. Further, it is vital mentioning that, for a special case, some of our solutions are in good contract with those gained by other authors.

Keywords: The improved (G'/G)-expansion method, the breaking soliton equation, solitary solutions, periodic solutions, nonlinear evolution equations.

AMS Subject Classification: 35Q51, 35Q53, 37K10

1. Introduction

Nonlinear partial differential equations (PDEs) have become a useful tool for describing complex physical phenomena of mathematical physics, engineering sciences and other scientific real time application fields. Consequently, the study of analytical solutions of PDEs has now become an imperative area to researchers. In the recent past, new exact solutions may help to reveal new phenomena. A wide range of powerful methods are being introduced to obtain analytical solutions, such as, the Hirota's bilinear transformation method [1], the inverse scattering method [2], the Backlund transformation method [3], the Jacobi elliptic function expansion method [4], the tanh-coth method [5,6], the direct algebraic method [7], the F-expansion method [8,9], the Cole-Hopf transformation method [10], the Exp-function method [11-15] and others [16-20].

Recently, Wang *et al.* [21] introduced a method, called the basic (G'/G)-expansion method to construct traveling wave solutions of some nonlinear evolution equations. In this method, they employed

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$u(\xi)=\sum_{i=0}^{m} a_i (G'/G)^i$ as traveling wave solutions, where $a_m \neq 0.$ Subsequently, many researchers studied different nonlinear partial differential equations by using this method, such as, [22-27]. In recent times, this basic (G'/G)-expansion method has been extended by Zhang [28], which is called the improved (G'/G)-expansion method. In this improved method, $F(\varphi)=\sum_{f=-U}^{U} s_f (G'/G)^f$ is applied, as traveling wave solutions, where either s_{-U} or s_U may be zero, but both s_{-U} and s_U cannot be zero at a time. Consequently, many researchers implemented this powerful method for solving various differential equations to obtain abundant and more general exact traveling wave solutions, for example [29-35].

Many researchers used different methods to investigate the (2+1)-dimensional breaking soliton equation. For instance, Ping [36] constructed exact solutions of this equation by using improved Riccati equation method. In Ref [37], Wazwaz implemented modified Hirota bilinear method to obtain analytical solutions of the same equation whilst Peng [38] executed modified mapping method of the same equation for establishing exact solutions. Bekir and Uygun [39] studied this equation for obtaining traveling wave solutions via the basic (G'/G)-expansion method. In this basic (G'/G)-expansion method, $u(\xi)=\sum_{i=0}^{m} a_i (G'/G)^i$ where $a_m \neq 0,$ is considered as traveling wave solutions instead of $F(\varphi)=\sum_{f=-U}^{U} s_f (G'/G)^f$ where either s_{-U} or s_U may be zero, but both s_{-U} and s_U cannot be zero at a time.

The importance of this present work is, the (2+1)-dimensional breaking soliton equation is investigated to construct abundant traveling wave solutions including solitons, periodic and rational solutions by applying the improved (G'/G)-expansion method.

2. The improved (G'/G)-expansion method

Consider the general nonlinear partial differential equation:

$$H\left(u,u_t,u_x,u_y,u_{xt},u_{yt},u_{xy},u_{tt},u_{xx},u_{yy},\ldots\right)=0, \tag{1}$$

where $u=u(x,y,t)$ is an unknown function, H is a polynomial in $u(x,y,t)$ and the subscripts stand for the partial derivatives.

The main steps of the method [28] are:

Step 1. We suppose that combining the real variables x, y and t by a complex variable φ:

$$u(x,y,t)=F(\varphi), \qquad \varphi=x+y-Wt, \tag{2}$$

where W is the speed of the traveling wave. Now using transformation Eq. (2), Eq. (1) is transformed into an ordinary differential equation (ODE) for $u=F(\varphi)$:

$$A(F,F',F'',F''',\ldots)=0, \tag{3}$$

where A is a function of $F(\varphi)$ and the superscripts indicate the ordinary derivatives with respect to φ.

Step 2. According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [28]:

$$F(\varphi) = \sum_{f=-U}^{U} s_f \left(G'/G\right)^f \tag{4}$$

with $G = G(\varphi)$ satisfies the second order linear ODE:

$$G'' + \Psi G' + \Phi G = 0, \tag{5}$$

where $s_f \left(f = 0, \pm 1, \pm 2, ..., \pm U \right)$, Ψ and Φ are constants.

Step 4. To determine the positive integer U, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substituting Eqs. (4) and (5) into Eq. (3) with the value of U obtained in Step 4. Equating the coefficients of $\left(G'/G\right)^m$, $\left(m = 0, \pm 1, \pm 2, ...\right)$, then setting each coefficient to zero, we obtain a set of algebraic equations for $s_f \left(f = 0, \pm 1, \pm 2, ..., \pm U \right), W, \Psi$ and Φ.

Step 6. Solve the system of algebraic equations which are obtained in step 5 with the aid of algebraic software Maple and we obtain values for $s_f \left(f = 0, \pm 1, \pm 2, ..., \pm U \right)$, W, Ψ and Φ.

Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of U, we can obtain the traveling wave solutions of Eq. (1).

3. Application of the method

In this section, the (2+1)-dimensional breaking soliton equation has been investigated by applying the improved $\left(G'/G\right)$-expansion method for finding abundant new traveling wave solutions.

3.1 The (2+1)-dimensional breaking soliton equation

Let us consider the (2+1)-dimensional breaking soliton equation followed by Bekir and Uygun [39]:

$$\begin{aligned} &u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha\, u_x v = 0, \\ &u_y = v_x \end{aligned} \tag{6}$$

Making use the traveling wave transformation Eq. (2) into the Eq. (6), which yields:

$$\begin{aligned} &-WF' + \alpha F''' + 4\alpha F v' + 4\alpha F' v = 0, \\ &F' = v'. \end{aligned} \tag{7}$$

Integrating the second equation in the system and neglecting constants of integration, we find

$$F = v. \tag{8}$$

Substituting Eq. (8) into the first equation of the system, therefore, integrating with respect φ once yields:

$$-WF + 4\alpha F^2 + \alpha F'' = 0. \tag{9}$$

Taking the homogeneous balance between the nonlinear term F^2 and the highest order derivative F'' in Eq. (9), we obtain $U = 2.$

Therefore, the solution of Eq. (9) is of the form:

$$F(\varphi) = s_{-2}(G'/G)^{-2} + s_{-1}(G'/G)^{-1} + s_0 + s_1(G'/G) + s_2(G'/G)^2, \tag{10}$$

where s_{-2}, s_{-1}, s_0, s_1 and s_2 are constants to be determined.

Substituting Eq. (10) together with Eq. (5) into the Eq. (9), the left-hand side of Eq. (9) is converted into a polynomial of $(G'/G)^m, (m = 0, \pm 1, \pm 2, ...).$ According to Step 5, collecting all terms with the same power of $(G'/G).$ Then, setting each coefficient of the resulted polynomial to zero, we obtain a set of simultaneous algebraic equations (for simplicity, which are not displayed) for $s_{-2}, s_{-1}, s_0, s_1, s_2, W, \Psi$ and $\Phi.$ Solving the system of obtained algebraic equations with the help of algebraic software Maple, we obtain four different values.

Case 1:

$$s_{-2} = 0, s_{-1} = 0,\ s_0 = \frac{-3\Phi}{2},\ s_1 = \frac{-3\Psi}{2},\ s_2 = \frac{-3}{2}, W = \alpha\left(\Psi^2 - 4\Phi\right), \tag{11}$$

where Ψ and Φ are free parameters.

Case 2:

$$s_{-2} = 0,\ s_{-1} = 0,\ s_0 = \frac{-1}{4}\left(\Psi^2 + 2\Phi\right),\ s_1 = \frac{-3\Psi}{2},\ s_2 = \frac{-3}{2}, W = -\alpha\left(\Psi^2 - 4\Phi\right), \tag{12}$$

where Ψ and Φ are free parameters.

Case 3:

$$s_{-2} = \frac{-3\Phi^2}{2}, s_{-1} = \frac{-3\Psi\Phi}{2},\ s_0 = \frac{-3\Phi}{2},\ s_1 = 0,\ s_2 = 0, W = \alpha\left(\Psi^2 - 4\Phi\right), \tag{13}$$

where Ψ and Φ are free parameters.

Case 4:

$$s_{-2} = \frac{-3\Phi^2}{2}, s_{-1} = \frac{-3\Psi\Phi}{2},\ s_0 = \frac{-1}{4}\left(\Psi^2 + 2\Phi\right),\ s_1 = 0,\ s_2 = 0, W = -\alpha\left(\Psi^2 - 4\Phi\right), \tag{14}$$

where Ψ and Φ are free parameters.

Substituting the general solution Eq. (5) into Eq. (10), we obtain three different families of traveling wave solutions of Eq. (9):

Family 1: Hyperbolic function solutions:

When $\Psi^2 - 4\Phi > 0,$ we obtain

$$\begin{aligned}F(\varphi) &= s_{-2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{-2}\\
&+s_{-1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{-1}+s_0\\
&+s_{1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)\\
&+s_{2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{2},\end{aligned} \tag{15.1}$$

$$\begin{aligned}v(\varphi) &= s_{-2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{-2}\\
&+s_{-1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{-1}+s_0\\
&+s_{1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)\\
&+s_{2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\frac{H\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}{H\cosh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi+L\sinh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\,\varphi}\right)^{2},\end{aligned} \tag{15.2}$$

where H and L are arbitrary constants. If H, L, Ψ and Φ take particular values, various known results in the literature can be rediscovered.

Family 2: Trigonometric function solutions:

When $\Psi^2-4\Phi<0,$ we obtain

$$
\begin{aligned}
F(\varphi) = {} & s_{-2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{-2} \\
& + s_{-1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{-1} + s_0 \\
& s_1\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right) \\
& + s_2\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{2}
\end{aligned}
\tag{16.1}
$$

$$
\begin{aligned}
v(\varphi) = {} & s_{-2}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{-2} \\
& + s_{-1}\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{-1} + s_0 \\
& s_1\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right) \\
& + s_2\left(\frac{-\Psi}{2}+\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\frac{-H\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}{H\cos\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi+L\sin\frac{1}{2}\sqrt{4\Phi-\Psi^2}\,\varphi}\right)^{2}
\end{aligned}
\tag{16.2}
$$

where H and L are arbitrary constants. If H, L, Ψ and Φ take particular values, various known results in the literature can be rediscovered.

Family 3: Rational function solution:

When $\Psi^2 - 4\Phi = 0,$ we obtain

$$
\begin{aligned}
F(\varphi) = {} & s_{-2}\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2} + s_{-1}\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1} + s_0 + s_1\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right) \\
& + s_2\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{2},
\end{aligned}
\tag{17.1}
$$

$$\begin{aligned} v(\varphi) = {} & s_{-2}\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2} + s_{-1}\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1} + s_0 + s_1\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right) \\ & + s_2\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{2}, \end{aligned} \tag{17.2}$$

Substituting Eqs. (11), (12), (13) and (14) together with the general solution Eq. (5) into the Eq. (10), yields the hyperbolic function solution Eqs. (15.1) and (15.2), our traveling wave solutions become respectively (if $H=0$ but $L \neq 0$):

$$F_1(\varphi) = \frac{3\left(\Psi^2-4\Phi\right)}{8}\left(1-\coth\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

$$v_1(\varphi) = \frac{3\left(\Psi^2-4\Phi\right)}{8}\left(1-\coth\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

where $\varphi = x+y-\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_2(\varphi) = \frac{\left(\Psi^2-4\Phi\right)}{8}\left(1-3\coth\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

$$v_2(\varphi) = \frac{\left(\Psi^2-4\Phi\right)}{8}\left(1-3\coth\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

where $\varphi = x+y+\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_3(\varphi) = \frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2} + \Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1} + 1\right),$$

$$v_3(\varphi) = \frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2} + \Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1} + 1\right),$$

where $\varphi = x+y-\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_4(\varphi) = \frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2} + \Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}\right) - \frac{1}{4}\left(\Psi^2+2\Phi\right),$$

$$v_4(\varphi) = \frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2} + \Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\coth\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}\right) - \frac{1}{4}\left(\Psi^2+2\Phi\right),$$

where $\varphi = x+y+\alpha\left(\Psi^2-4\Phi\right)t.$

Again, substituting Eqs. (11), (12), (13) and (14) together with the general solution Eq. (5) into the Eq. (10), we obtain the hyperbolic function solution Eq. (15.1) and (15.2), exact solutions become respectively (if $L=0$ but $H \neq 0$):

$$F_5(\varphi)=\frac{3\left(\Psi^2-4\Phi\right)}{8}\left(1-\tanh\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

$$v_5(\varphi)=\frac{3\left(\Psi^2-4\Phi\right)}{8}\left(1-\tanh\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right).$$

$$F_6(\varphi)=\frac{\left(\Psi^2-4\Phi\right)}{8}\left(1-3\tanh\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right),$$

$$v_6(\varphi)=\frac{\left(\Psi^2-4\Phi\right)}{8}\left(1-3\tanh\left(\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^2\right).$$

$$F_7(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}+1\right),$$

$$v_7(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}+1\right).$$

$$F_8(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(\Psi^2+2\Phi\right),$$

$$v_8(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{\Psi^2-4\Phi}}{2}\tanh\frac{1}{2}\sqrt{\Psi^2-4\Phi}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(\Psi^2+2\Phi\right).$$

Substituting Eqs. (11), (12), (13) and (14) together with the general solution Eq. (5) into the Eq. (10), yields the trigonometric function solution Eq. (16.1) and (16.2), we obtain following solutions respectively (if $H=0$ but $L\neq 0$):

$$F_9(\varphi)=\frac{-3\left(4\Phi-\Psi^2\right)}{8}\left(1+\cot\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

$$v_9(\varphi)=\frac{-3\left(4\Phi-\Psi^2\right)}{8}\left(1+\cot\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

where $\varphi=x+y-\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_{10}(\varphi)=\frac{-\left(4\Phi-\Psi^2\right)}{8}\left(1+3\cot\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

$$v_{10}(\varphi)=\frac{-\left(4\Phi-\Psi^2\right)}{8}\left(1+3\cot\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

where $\varphi=x+y+\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_{11}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}+1\right),$$

$$v_{11}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}+1\right),$$

where $\varphi=x+y-\alpha\left(\Psi^2-4\Phi\right)t.$

$$F_{12}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(2\Phi+\Psi^2\right),$$

$$v_{12}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\cot\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(2\Phi+\Psi^2\right),$$

where $\varphi=x+y+\alpha\left(\Psi^2-4\Phi\right)t.$

Also, substituting Eqs. (11), (12), (13) and (14) together with the general solution Eq. (5) into the Eq. (10), yields the trigonometric function solution Eq. (16.1) and (16.2), our solutions become respectively (if $L=0$ but $H\neq 0$):

$$F_{13}(\varphi)=\frac{-3\left(4\Phi-\Psi^2\right)}{8}\left(1+\tan\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

$$v_{13}(\varphi)=\frac{-3\left(4\Phi-\Psi^2\right)}{8}\left(1+\tan\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right).$$

$$F_{14}(\varphi)=\frac{-\left(4\Phi-\Psi^2\right)}{8}\left(1+3\tan\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right),$$

$$v_{14}(\varphi)=\frac{-\left(4\Phi-\Psi^2\right)}{8}\left(1+3\tan\left(\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^2\right).$$

$$F_{15}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}+1\right),$$

$$v_{15}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}+1\right).$$

$$F_{16}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(2\Phi+\Psi^2\right),$$

$$v_{16}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{\sqrt{4\Phi-\Psi^2}}{2}\tan\frac{1}{2}\sqrt{4\Phi-\Psi^2}\ \varphi\right)^{-1}\right)-\frac{1}{4}\left(2\Phi+\Psi^2\right).$$

Substituting Eqs. (11), (12), (13) and (14) together with the general solution Eq. (5) into the Eq. (10), we obtain the rational function solution Eqs. (17.1) and (17.2), our wave solutions become respectively (if $\Psi^2-4\Phi=0$):

$$F_{17}(\varphi)=\frac{3}{8}\left(\left(\Psi^2-4\Phi\right)-\left(\frac{2L}{H+L\varphi}\right)^2\right),$$

$$v_{17}(\varphi)=\frac{3}{8}\left(\left(\Psi^2-4\Phi\right)-\left(\frac{2L}{H+L\varphi}\right)^2\right).$$

$$F_{18}(\varphi)=\frac{1}{8}\left(\left(\Psi^2-4\Phi\right)-3\left(\frac{2L}{H+L\varphi}\right)^2\right),$$

$$v_{18}(\varphi)=\frac{1}{8}\left(\left(\Psi^2-4\Phi\right)-3\left(\frac{2L}{H+L\varphi}\right)^2\right).$$

$$F_{19}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1}+1\right),$$

$$v_{19}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1}+1\right).$$

$$F_{20}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1}\right)-\frac{1}{4}\left(\Psi^2+2\Phi\right),$$

$$v_{20}(\varphi)=\frac{-3\Phi}{2}\left(\Phi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-2}+\Psi\left(\frac{-\Psi}{2}+\frac{L}{H+L\varphi}\right)^{-1}\right)-\frac{1}{4}\left(\Psi^2+2\Phi\right).$$

4. Results and discussion

It is important to point out that some of our traveling wave solutions are in good contract with existing results which are depicted in the table. Moreover, some of obtained solutions are shown in figure 1 to figure 8.

4.1 Table. Comparison between Bekir and Uygun [39] solutions and Newly obtained solutions

Bekir and Uygun [39] solutions	New solutions
i. If $C_1=0, C_2\neq 0, \mu=2$ and $\lambda=3$ solution Eq. (4.13) (from section 4) becomes: $u_1(\xi)=\frac{3}{8}\left(1-\coth^2\frac{1}{2}\xi\right)$ and $v_1(\xi)=\frac{3}{8}\left(1-\coth^2\frac{1}{2}\xi\right)$.	i.If $\Phi=2, \Psi=3,\ F_1(\varphi)=u_1(\xi)$ and $v_1(\varphi)=v_1(\xi)$, solutions $F_1(\varphi)$ and $v_1(\varphi)$ become: $u_1(\xi)=\frac{3}{8}\left(1-\coth^2\frac{1}{2}\xi\right)$ and $v_1(\xi)=\frac{3}{8}\left(1-\coth^2\frac{1}{2}\xi\right)$.
ii. If $C_1=0, C_2\neq 0, \mu=2$ and $\lambda=3$ solution Eq. (4.14) (from section 4) becomes:	ii. If $\Phi=2, \Psi=3,\ F_2(\varphi)=u_2(\xi)$ and $v_2(\varphi)=v_2(\xi)$, solutions $F_2(\varphi)$ and $v_2(\varphi)$

$u_2(\xi)=\frac{1}{8}\left(1-3\coth^2\frac{1}{2}\xi\right)$ and $v_2(\xi)=\frac{1}{8}\left(1-3\coth^2\frac{1}{2}\xi\right)$.	become: $u_2(\xi)=\frac{1}{8}\left(1-3\coth^2\frac{1}{2}\xi\right)$ and $v_2(\xi)=\frac{1}{8}\left(1-3\coth^2\frac{1}{2}\xi\right)$.
iii. If $C_1\neq 0, C_2=0, \mu=3$ and $\lambda=4$ solution Eq. (4.13) (from section 4) becomes: $u_1(\xi)=\frac{3}{2}\left(1-\tanh^2\xi\right)$ and $v_1(\xi)=\frac{3}{2}\left(1-\tanh^2\xi\right)$.	iii. If $\Phi=3, \Psi=4,$ $F_5(\varphi)=u_1(\xi)$ and $v_5(\varphi)=v_1(\xi)$, solutions $F_5(\varphi)$ and $v_5(\varphi)$ become: $u_1(\xi)=\frac{3}{2}\left(1-\tanh^2\xi\right)$ and $v_1(\xi)=\frac{3}{2}\left(1-\tanh^2\xi\right)$.
iv. If $C_1\neq 0, C_2=0, \mu=3$ and $\lambda=4$ solution Eq. (4.14) (from section 4) becomes: $u_2(\xi)=\frac{1}{2}\left(1-3\tanh^2\xi\right)$ and $v_2(\xi)=\frac{1}{2}\left(1-3\tanh^2\xi\right)$.	iv. If $\Phi=3, \Psi=4,$ $F_6(\varphi)=u_2(\xi)$ and $v_6(\varphi)=v_2(\xi)$, solutions $F_6(\varphi)$ and $v_6(\varphi)$ become: $u_2(\xi)=\frac{1}{2}\left(1-3\tanh^2\xi\right)$ and $v_2(\xi)=\frac{1}{2}\left(1-3\tanh^2\xi\right)$.
v. If $C_1=0, C_2\neq 0, \lambda=2$ and $\mu=3$ solution Eq. (4.15) (from section 4) becomes: $u_3(\xi)=-3\left(1+\cot^2\left(\sqrt{2}\,\xi\right)\right)$ and $v_3(\xi)=-3\left(1+\cot^2\left(\sqrt{2}\,\xi\right)\right)$.	v. If $\Phi=3, \Psi=2,$ $F_9(\varphi)=u_3(\xi)$ and $v_9(\varphi)=v_3(\xi)$, solutions $F_9(\varphi)$ and $v_9(\varphi)$ become: $u_3(\xi)=-3\left(1+\cot^2\left(\sqrt{2}\,\xi\right)\right)$ and $v_3(\xi)=-3\left(1+\cot^2\left(\sqrt{2}\,\xi\right)\right)$.
vi. If $C_1=0, C_2\neq 0, \lambda=2$ and $\mu=3$ solution Eq. (4.16) (from section 4) becomes: $u_4(\xi)=-\left(1+3\cot^2\left(\sqrt{2}\,\xi\right)\right)$ and $v_4(\xi)=-\left(1+3\cot^2\left(\sqrt{2}\,\xi\right)\right)$.	vi. If $\Phi=3, \Psi=2,$ $F_{10}(\varphi)=u_4(\xi)$ and $v_{10}(\varphi)=v_4(\xi)$, solutions $F_{10}(\varphi)$ and $v_{10}(\varphi)$ become: $u_4(\xi)=-\left(1+3\cot^2\left(\sqrt{2}\,\xi\right)\right)$ and $v_4(\xi)=-\left(1+3\cot^2\left(\sqrt{2}\,\xi\right)\right)$.
vii. If $C_1\neq 0, C_2=0, \lambda=1$ and $\mu=1$ solution Eq. (4.15) (from section 4) becomes: $u_3(\xi)=-\frac{9}{8}\left(1+\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right)$ and	vii. If $\Phi=1, \Psi=1,$ $F_{13}(\varphi)=u_3(\xi)$ and $v_{13}(\varphi)=v_3(\xi)$, solutions $F_{13}(\varphi)$ and $v_{13}(\varphi)$ become: $u_3(\xi)=-\frac{9}{8}\left(1+\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right)$ and

$v_3(\xi)=-\frac{9}{8}\left(1+\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right).$	$v_3(\xi)=-\frac{9}{8}\left(1+\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right).$
viii. If $C_1\neq 0, C_2=0, \lambda=1$ and $\mu=1$ solution Eq. (4.16) (from section 4) becomes: $u_4(\xi)=-\frac{3}{8}\left(1+3\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right)$ and $v_4(\xi)=-\frac{3}{8}\left(1+3\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right).$	viii. If $\Phi=1, \Psi=1,$ $F_{14}(\varphi)=u_4(\xi)$ and $v_{14}(\varphi)=v_4(\xi)$, solutions $F_{14}(\varphi)$ and $v_{14}(\varphi)$ become: $u_4(\xi)=-\frac{3}{8}\left(1+3\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right)$ and $v_4(\xi)=-\frac{3}{8}\left(1+3\tan^2\left(\frac{\sqrt{3}}{2}\xi\right)\right).$

Beyond the table, we obtain new exact traveling wave solutions $F_3, F_4, F_7, F_8, F_{11}, F_{12}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}$ and F_{20} which are not being established in the previous literature.

4.2 Graphical descriptions of the solutions

The graphical illustrations of some solutions are described in the following figures with the aid of commercial software Maple:

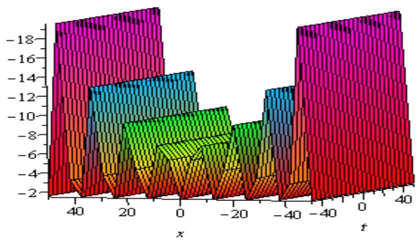

Fig. 1: Periodic solution for $\Psi=6, \Phi=10, \alpha=1.10^{-14}$

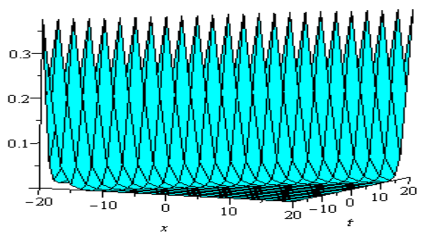

Fig. 2: Solitons solution for $\Psi=5, \Phi=6, \alpha=1$

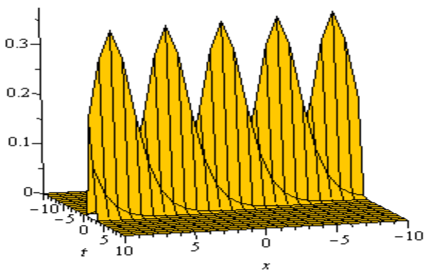

Fig. 3: Solitons solution for $\Psi=3, \Phi=2, \alpha=5$

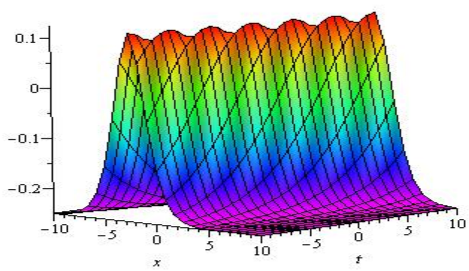

Fig. 4: Solitons solution for $\Psi=7, \Phi=12, \alpha=0.25$

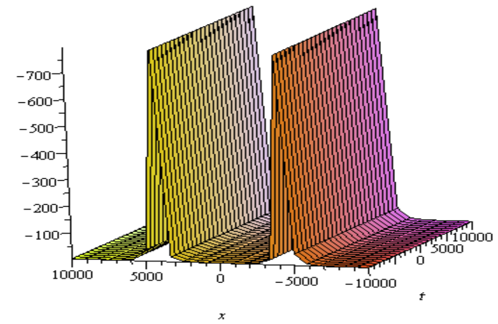

Fig. 5: Periodic solution for $\Psi = 1, \Phi = 2, \alpha = 1.10^{-9}$

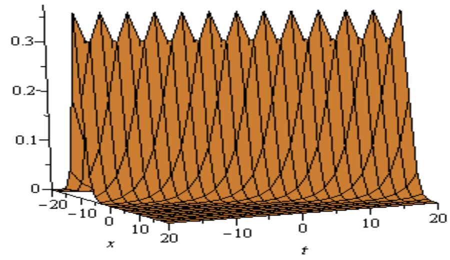

Fig. 6: Solitons solution for $\Psi = 2, \Phi = 0.75, \alpha = 0.5$

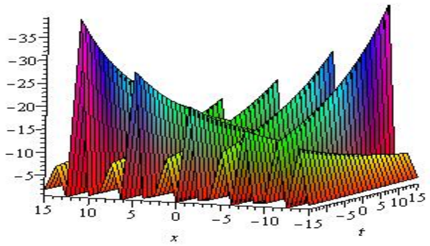

Fig. 7: Periodic solution for $\Psi = 6, \Phi = 10, \alpha = 0.001$

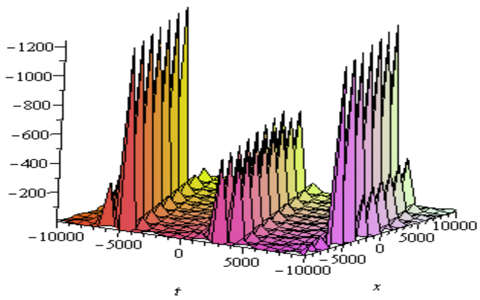

Fig. 8: Solitons solution for $\Psi = 6, \Phi = 10, \alpha = 0.75$

5. Conclusions

In this article, the improved (G'/G)-expansion method has been successfully applied for constructing abundant traveling wave solutions including solitons, periodic and rational solutions of the nonlinear evolution equation, namely, the (2+1)-dimensional breaking soliton equation. Furthermore, it is more imperative declaring that some of our solutions are being coincided with existing results, if parameter taken particular values. Moreover, the obtained solutions show that the performance of this method is reliable, effective and more general than the basic (G'/G)-expansion method because it can establish many new solutions at a time. Therefore, this straightforward and powerful method can be more successfully implemented to investigate a different class of nonlinear partial differential equations which frequently arise in engineering sciences, mathematical physics and other scientific real time application fields.

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References

[1] R.Hirota,Exact solution of the KdV equation for multiple collisions of solutions, *Phys. Rev. Lett.*, 27, 1192-1194 (1971).

[2] M.J.Ablowitz, and P.A.Clarkson,*Solitons, nonlinear evolution equations and inverse scattering transform*, Cambridge Univ. Press, Cambridge, 1991.

[3] C.Rogers,and W.F.Shadwick, *Backlund Transformations and their applications*, Academic Press, New York, 1982.

[4] S.Liu,Z.Fu,S.Liu,and Q.Zhao,Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, *Phys. Lett. A*, 289, 69–74 (2001).

[5] W.Malfliet,Solitary wave solutions of nonlinear wave equations, *Am. J. Phys*. 60, 650–654 (1992).

[6] A.M.Wazwaz,The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations, *Appl. Math. and Comput*., 188, 1467-1475 (2007).

[7] A.A.Soliman,and H.A.Abdo,New exact Solutions of nonlinear variants of the RLW, the PHI-four and Boussinesq equations based on modified extended direct algebraic method, *Int. J. Nonlinear Sci.*, 7, 274-282 (2009).

[8] M.L.Wang, and X.Z.Li,Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, *Chaos, Solitons and Fractals*, 24, 1257-1268 (2005).

[9] M.A.Abdou,The extended F-expansion method and its application for a class of nonlinear evolution equations, *Chaos, Solitons and Fractals*, 31, 95-104 (2007).

[10] A.H.Salas,and C.A.Gomez,Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation, *Math. Prob.Engr.*, Article ID 194329, 14 pages, doi: 10.1155/2010/194329 (2010).

[11] J.H.He,and X.H.Wu,Exp-function method for nonlinear wave equations, *Chaos Solitons and Fractals*, 30, 700-708 (2006).

[12] H.Naher,F.A.Abdullah,and M.A.Akbar,New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method, *J. Appl. Math.*, Article ID: 575387, 14 pages, doi: 10.1155/2012/575387 (2012).

[13] S.T.Mohyud-din,M.A.Noor,and K.I.Noor,Exp-function method for traveling wave solutions of modified Zakharov-Kuznetsov equation, *J. King Saud Univ.*, 22, 213-216 (2010).

[14] H.Naher,F.A.Abdullah,and M.A.Akbar,The exp-function method for new exact solutions of the nonlinear partial differential equations, *Int. J. Phys. Sci.*, 6, 6706-6716 (2011).

[15] A.Bekir,and A.Boz,Exact solutions for nonlinear evolution equations using Exp-function method, *Phys. Lett. A*, 372, 1619-1625 (2008).

[16] A.M.Wazwaz,A new (2+1)-dimensional Korteweg-de-Vries equation and its extension to a new (3+1)-dimensional Kadomtsev-Petviashvili equation, *Physica Scripta*, doi:10.1088/0031-8949/84/03/035010 (2011).

[17] M.Ghoreishi,A.I.B.Md.Ismail,and A.Rashid,Numerical solution of Klein-Gordon-Zakharov equations using Chebyshev Cardinal Functions, *J. Computational Anal. and Applica.*, 14, 574-582 (2012).

[18] M.Noor,K.Noor,A.Waheed,and E.A.Al-Said,An efficient method for solving system of third-order nonlinear boundary value problems, *Math. Prob. Engr.*, Article ID 250184, 14 pages, doi: 10.1155/2011/250184 (2011).

[19] M.Ghoreishi,A.I.B.Md.Ismail,and A.Rashid, The coupled viscous Burgers equations with fractional-time derivative by the homotopy perturbation method, *J. Computational Anal. and Applica.*, 13, 1054-1066 (2011).

[20] D.M.Yuan,X.L.Cheng,Method of fundamental solutions with an optimal regularization technique for the Cauchy problem of the modified Helmholtz equation, *J. Computational Anal. and Applica.*, 14, 54-66 (2012).

[21] M.Wang,X.Li,J.Zhang,The $\left(G'/G\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, 372, 417-423 (2008).

[22] R.Abazari,and R.Abazari,Hyperbolic, trigonometric and rational function solutions of Hirota-Ramani equation via $\left(G'/G\right)$-expansion method, *Math. Prob. Engr.*, Article ID: 424801, 11 pages, doi:10.1155/2011/424801 (2011).

[23] H.Naher,F.A.Abdullah,M.A.Akbar,The $\left(G'/G\right)$-expansion method for abundant traveling wave solutions of Caudrey-Dodd-Gibbon equation, *Math. Prob. Engr.*, Article ID: 218216, 11 pages, doi:10.1155/2011/218216 (2011).

[24] E.M.E.Zayed,S.Al-Joudi,Applications of an Extended $\left(G'/G\right)$-Expansion Method to Find Exact Solutions of Nonlinear PDEs in Mathematical Physics, *Math. Prob. Engr.*, Article ID 768573, 19 pages, doi:10.1155/2010/768573 (2010).

[25] A.Jabbari,H.Kheiri,A.Bekir,Exact solutions of the coupled Higgs equation and the Miccari system using He's semi-inverse method and $\left(G'/G\right)$-expansion method, *Computers and Math. with Applications*, 62, 2177-2186 (2011).

[26] T.Ozis,I.Aslan,Application of the $\left(G'/G\right)$-expansion method to Kawahara type equations using symbolic computation, *Appl. Math. Computation*, 216, 360-2365 (2010).

[27] K.A.Gepreel,Exact solutions for nonlinear PDEs with the variable coefficients in mathematical physics, *J. Info. Computing Sci.*, 6, 003-014 (2011).

[28] J.Zhang,F.Jiang,X.Zhao,An improved $\left(G'/G\right)$-expansion method for solving nonlinear evolution equations, *Int. J. Computer Math.*, 87,1716-1725 (2010).

[29] T.A.Nofel,M.Sayed,YS.Hamad,S.K.Elagan,The improved $\left(G'/G\right)$-expansion method for solving the fifth-order KdV equation, *Annals Fuzzy Math. Informatics*, 3, 9-17 (2011).

[30] Y.S.Hamad,M.Sayed,S.K.Elagan,E.R.El-Zahar,The improved $\left(G'/G\right)$-expansion method for solving (3+1)-dimensional potential-YTSF equation, *J. Modern Methods Numerical Math*., 2, 32-38 (2011).

[31] Y.M.Zhao,Y.J.Yang,W.Li,Application of the improved $\left(G'/G\right)$-expansion method for the Variant Boussinesq equations, *Appl. Math. Sci*., 5, 2855-2861 (2011).

[32] H.Naher,F.A.Abdullah,M.A.Akbar,New traveling wave solutions of the higher dimensional nonlinear evolution equation by the improved (G'/G) -expansion method, *World Appl. Sci. J.*, 16, 11-21 (2012).

[33] H.Naher,F.A.Abdullah,Some new traveling wave solutions of the nonlinear reaction diffusion equation by using the improved (G'/G) -expansion method, *Math. Prob. Engr.*, 17 pages, Article ID: 871724, doi:10.1155/2012/871724 (2012).

[34] H.Naher,F.A.Abdullah,The improved (G'/G)-expansion method for the (2+1)-dimensional modified Zakharov-Kuznetsov equation, *J. Appl. Math.*, Article ID: 438928, 20 pages, doi: 10.1155/2012/438928 (2012).

[35] H.Naher,F.A.Abdullah,A.Rashid,Some new solutions of the (3+1)-dimensional Jimbo-Miwa equation via the improved (G'/G)-expansion method, *J. Computational Anal. and Applica.*, Article ID: 146-JOCAAA (Accepted).

[36] Z.Ping,New exact solutions to breaking soliton equations and Witham-Broer-Kaup equations, *Appl. Math. Computation*, 217, 1688-1696 (2010).

[37] A.M.Wazwaz,Integrable (2+1)-dimensional and (3+1)-dimensional breaking soliton equations, *Physica Scripta*, doi: 10.1088/0031-8949/81/03/035005 (2010).

[38] Y.Peng,New exact solutions for (2+1)-dimensional breaking soliton equation. *Commun. Theor. Phys.*, 43, 205-207 (2005).

[39] A.Bekir,F.Uygun,Exact travelling wave solutions of nonlinear evolution equations by using the (G'/G)-expansion method, *Arab J. Math. Sci.*, 18, 73-85 (2012).

A Multiple Attribute Group Decision Making Method based on Generalized Interval-valued Trapezoidal Fuzzy Numbers

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Abstract. A ranking approach based on grey correlative coefficient is presented to solve the multiple attribute decision making problems in which the attribute values and the weights take the form of generalized interval-valued trapezoidal fuzzy numbers (GIVTFN). Firstly, the concept, the operational rules and the distance of GIVTFN are given, and the method of linguistic variables converted into GIVTFN is introduced. Secondly, the normalization method of the decision Matrix based on the GIVTFN is proposed, and a grey relational decision making method based on the GIVTFN is presented and decision making steps are illustrated in detail, and the alternatives is ranked based on the grey correlative coefficient. Finally, an illustrate example is given to show the effectiveness of the proposed method.

Keywords: interval-valued fuzzy number; grey correlative coefficient; multiple attribute group decision making

1 Introduction

Since the object things are fuzzy, uncertainty and Human thinking is ambiguous, the majority of multi-attribute decision-making is uncertain, which is called fuzzy multiple attribute decision-making (FMADM). Since Bellmanhe and Zadeh [2] firstly proposed the fuzzy decision making model based on the theory of fuzzy mathematics, the research on FMADM has been receiving more and more attentions, and many achievements have been made based on the various fuzzy attribute values, such as interval numbers, triangular fuzzy numbers, and trapezoidal fuzzy numbers etc. Wei and Wei [14], Men and Ji [9] proposed the grey relational analysis method with various attribute values respectively, such as interval numbers and triangular fuzzy numbers.

The concept of interval-valued fuzzy set is firstly proposed by Gorzlczany [4] and Turksen [10], and then Wang and Li [11, 12] gave the extended operations of interval-valued fuzzy numbers, and proposed the concept and properties

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of similarity coefficient of the interval-valued fuzzy numbers. Hong and Lee [5] proposed the distance of interval-valued fuzzy numbers. Ashtiani, et al [1] proposed the extended TOPSIS method based on the interval-valued triangular fuzzy numbers. Liu [6] proposed an extended TOPSIS method for multiple attribute group decision making based on generalized interval-valued trapezoidal fuzzy numbers (GIVTFN). Wei and Chen [13] proposed similarity measures between GIVTFNs for risk analysis. Liu [7] proposed some aggregation operators, such as the generalized interval-valued trapezoidal fuzzy number weighted aggregation operator (ITWA), the generalized interval-valued trapezoidal fuzzy number ordered weighted aggregation operator (ITOWA), and the generalized interval-valued trapezoidal fuzzy numbers hybrid aggregation operator (ITHA), to solve the FMAGM This paper proposed a decision making method based on the grey correlative coefficient for solving the MADM problems which the attribute weights and values are given with the form of GIVTFN.

2 The basic concept of the GIVTFN

2.1 The GIVTFN

(1) The definition of the GIVTFN [13]

Wang and Li [12] proposed the GIVTFN $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$ shown in Fig. 1. Where, $0 \leq a_1^L \leq a_2^L \leq a_3^L \leq a_4^L \leq 1$, $0 \leq a_1^U \leq a_2^U \leq a_3^U \leq a_4^U \leq 1$, $0 \leq w_{\tilde{\tilde{A}}^L} \leq w_{\tilde{\tilde{A}}^U} \leq 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$. As shown in Fig. 1, we can conclude that the GIVTFN $\tilde{\tilde{A}}$ consists of the lower value $\tilde{\tilde{A}}^L$ and the upper value $\tilde{\tilde{A}}^U$.

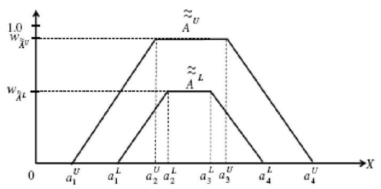

Fig. 1. generalized interval-valued trapezoidal fuzzy numbers

(2) The operational rules of the GIVTFNs [13]

Suppose that $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$, $\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{\tilde{B}}^L}), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{\tilde{B}}^U})]$ are the two GIVTFN-s, Then the operational rules are defined shown as follows:

$$\begin{aligned}\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = [&(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; min(w_{\tilde{\tilde{A}}^L}, w_{\tilde{\tilde{B}}^L})),\\ &(a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U}))]\end{aligned} \tag{1}$$

$$\begin{aligned}\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = [(a_1^L \times b_1^L, a_2^L \times b_2^L, a_3^L \times b_3^L, a_4^L \times b_4^L; min(w_{\tilde{\tilde{A}}^L}, w_{\tilde{\tilde{B}}^L})),\\ (a_1^U \times b_1^U, a_2^U \times b_2^U, a_3^U \times b_3^U, a_4^U \times b_4^U; min(w_{\tilde{\tilde{A}}^U}, w_{\tilde{\tilde{B}}^U}))]\end{aligned} \quad (2)$$

$$\lambda\tilde{\tilde{A}} = [(\lambda a_1^L, \lambda a_2^L, \lambda a_3^L, \lambda a_4^L; w_{\tilde{\tilde{A}}^L}), (\lambda a_1^U, \lambda a_2^U, \lambda a_3^U, \lambda a_4^U; w_{\tilde{\tilde{A}}^U})], \lambda > 0 \quad (3)$$

2.2 The distance between two GIVTFNs

Suppose that $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{\tilde{A}}^L}), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{\tilde{A}}^U})]$, $\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{\tilde{B}}^L}), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{\tilde{B}}^U})]$ are the two GIVTFNs, then the distance of two GIVTFNs ($\tilde{\tilde{A}} and \tilde{\tilde{B}}$) is calculated as follows:

$$d(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{(y_{\tilde{\tilde{A}}^L} - y_{\tilde{\tilde{B}}^L})^2 + (x_{\tilde{\tilde{A}}^L} - x_{\tilde{\tilde{B}}^L})^2 + (y_{\tilde{\tilde{A}}^U} - y_{\tilde{\tilde{B}}^U})^2 + (x_{\tilde{\tilde{A}}^U} - x_{\tilde{\tilde{B}}^U})^2/4} \quad (4)$$

where $(x_{\tilde{\tilde{A}}^L}, y_{\tilde{\tilde{A}}^L}), (x_{\tilde{\tilde{A}}^U}, y_{\tilde{\tilde{A}}^U}), (x_{\tilde{\tilde{B}}^L}, y_{\tilde{\tilde{B}}^L}), (x_{\tilde{\tilde{B}}^U}, y_{\tilde{\tilde{B}}^U})$ are the coordinate of COG points defined by Chen and Chen [3] for generalized trapezoidal fuzzy numbers $\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U, \tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U$ respectively.

$d(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ satisfies the following properties:

(i) if $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are the normalized GIVTFNs, then $0 \le d(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \le 1$.

(ii) $\tilde{\tilde{A}} = \tilde{\tilde{B}} \Leftrightarrow d(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 0$.

(iii) $d(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = d(\tilde{\tilde{B}}, \tilde{\tilde{A}})$.

(iv) $d(\tilde{\tilde{A}}, \tilde{\tilde{C}}) + d(\tilde{\tilde{C}}, \tilde{\tilde{B}}) \ge d(\tilde{\tilde{A}}, \tilde{\tilde{B}})$.

2.3 The method converted linguistic terms into the GIVTFNs

In the real decision making, it is difficult to adopt the form of GIVTFN to give the attribute values and weights directly by the decision makers. However, we can adopt the form of linguistic terms easily. Wei and Chen [13] proposed a method from 9-member linguistic terms to the GIVTFNs (see Table 1)

Table 1. 9-member linguistic term sets to GIVTFNs

linguistic terms (the attribute values)	linguistic terms (weights)	GIVTFNs
Absolutely-poor(AP)	Absolutely-low(AL)	[(0.00,0.00,0.00,0.00;0.8),(0.00,0.00,0.00,0.00;1.0)]
Very-poor(VP)	Very-low (VL)	[(0.00,0.00,0.02,0.07;0.8),(0.00,0.00,0.02,0.07;1.0)]
poor (P)	low (L)	[(0.04,0.10,0.18,0.23;0.8),(0.04,0.10,0.18,0.23;1.0)]
Medium-poor(MP)	Medium-low(ML)	[(0.17,0.22,0.36,0.42;0.8),(0.17,0.22,0.36,0.42;1.0)]
Medium (M)	Medium (M)	[(0.32,0.41,0.58,0.65;0.8),(0.32,0.41,0.58,0.65;1.0)]
Medium-good(MG)	Medium-high(MH)	[(0.58,0.63,0.80,0.86;0.8),(0.58,0.63,0.80,0.86;1.0)]
good(G)	high(H)	[(0.72,0.78,0.92,0.97;0.8),(0.72,0.78,0.92,0.97;1.0)]
Very-good(VG)	very-high (VH)	[(0.93,0.98,1.00,1.00;0.8),(0.93,0.98,1.00,1.00;1.0)]
Absolutely-good(AG)	Absolutely-high(AH)	[(1.00,1.00,1.00,1.00;0.8),(1.00,1.00,1.00,1.00;1.0)]

3 Group decision making method

3.1 Description the decision making problems

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ be the set of attributes, and $E = \{e_1, e_2, \ldots, e_q\}$ be the set of decision makers. Suppose that $\tilde{\tilde{a}}_{ijk} = [(a^L_{ijk1}, a^L_{ijk2}, a^L_{ijk3}, a^L_{ijk4}; w^L_{ijk}), (a^U_{ijk1}, a^U_{ijk2}, a^U_{ijk3}, a^U_{ijk4}; w^U_{ijk})]$ is the attribute value for the alternative A_i with respect to the attribute C_j given by the decision maker e_k, and $\tilde{\tilde{a}}_{ijk}$ is a GIVTFN, $\tilde{\tilde{\omega}}_{kj} = [(\omega^L_{kj1}, \omega^L_{kj2}, \omega^L_{kj3}, \omega^L_{kj4}; \eta^L_{kj}),$ $(\omega^U_{kj1}, \omega^U_{kj2}, \omega^U_{kj3}, \omega^U_{kj4}; \eta^U_{kj})]$ is the weight of attribute C_j given by the decision maker e_k, and $\tilde{\tilde{\omega}}_{kj}$ is also a GIVTFN. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ be the vector of decision makers, where λ_k is a real number, and $\Sigma^q_{k=1}\lambda_k = 1$. Then we use the attribute weights, the decision maker weights, and the attribute values to rank the alternatives.

3.2 Normalize the decision-making information

In order to eliminate the impact of different physical dimension to the decision-making result, we need normalize the decision-making information. Consider that there are generally benefit attributes (I_1) and cost attributes (I_2). The normalizing method is shown as follows:

$$\begin{aligned}\tilde{\tilde{x}}_{ijk} &= [(x^L_{ijk1}, x^L_{ijk2}, x^L_{ijk3}, x^L_{ijk4}; w^L_{ijk}), (x^U_{ijk1}, x^U_{ijk2}, x^U_{ijk3}, x^U_{ijk4}; w^U_{ijk})] \\ &= [((\frac{a^L_{ijk1}}{m_{jk}}, \frac{a^L_{ijk2}}{m_{jk}}, \frac{a^L_{ijk3}}{m_{jk}}, \frac{a^L_{ijk4}}{m_{jk}}; w^L_{ijk}), (\frac{a^U_{ijk1}}{m_{jk}}, \frac{a^U_{ijk2}}{m_{jk}}, \frac{a^U_{ijk3}}{m_{jk}}, \frac{a^U_{ijk4}}{m_{jk}}; w^U_{ijk}))]\end{aligned} \tag{5}$$

for benefit attributes, where $m_{jk} = \max_i(a^U_{ijk4})$.

$$\begin{aligned}\tilde{\tilde{x}}_{ijk} &= [(x^L_{ijk1}, x^L_{ijk2}, x^L_{ijk3}, x^L_{ijk4}; w^L_{ijk}), (x^U_{ijk1}, x^U_{ijk2}, x^U_{ijk3}, x^U_{ijk4}; w^U_{ijk})] \\ &= [((\frac{n_{jk}}{a^L_{ijk1}}, \frac{n_{jk}}{a^L_{ijk2}}, \frac{n_{jk}}{a^L_{ijk3}}, \frac{n_{jk}}{a^L_{ijk4}}; w^L_{ijk}), (\frac{n_{jk}}{a^U_{ijk1}}, \frac{n_{jk}}{a^U_{ijk2}}, \frac{n_{jk}}{a^U_{ijk3}}, \frac{n_{jk}}{a^U_{ijk4}}; w^U_{ijk}))]\end{aligned} \tag{6}$$

for cost attributes, where $n_{jk} = \min_i(a^L_{ijk1})$.

3.3 Aggregate the evaluation information of each decision maker into the collective information

According to the different attribute values and weights given by different experts, we can get the collective attribute values and weights. The steps are shown as shown as follows:

$$\begin{aligned}\tilde{\tilde{x}}_{ij} &= [(x^L_{ij1}, x^L_{ij2}, x^L_{ij3}, x^L_{ij4}; w^L_{ij}), (x^U_{ijk1}, x^U_{ijk2}, x^U_{ijk3}, x^U_{ijk4}; w^U_{ij})] = \Sigma^q_{k=1}(\lambda_k \tilde{\tilde{x}}_{ijk}) \\ &= \Sigma^q_{k=1}\{\lambda_k \times [(x^L_{ijk1}, x^L_{ijk2}, x^L_{ijk3}, x^L_{ijk4}; w^L_{ijk}), (x^U_{ijk1}, x^U_{ijk2}, x^U_{ijk3}, x^U_{ijk4}; w^U_{ijk})]\} \\ &= [(\Sigma^q_{k=1}(\lambda_k x^L_{ijk1}), \Sigma^q_{k=1}(\lambda_k x^L_{ijk2}), \Sigma^q_{k=1}(\lambda_k x^L_{ijk3}), \Sigma^q_{k=1}(\lambda_k x^L_{ijk4}); \min_k(w^L_{ijk})), \\ &\quad (\Sigma^q_{k=1}(\lambda_k x^U_{ijk1}), \Sigma^q_{k=1}(\lambda_k x^U_{ijk2}), \Sigma^q_{k=1}(\lambda_k x^U_{ijk3}), \Sigma^q_{k=1}(\lambda_k x^U_{ijk4}); \min_k(w^U_{ijk}))]\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{\tilde{\omega}}_j &= [(\omega_{j1}^L, \omega_{j2}^L, \omega_{j3}^L, \omega_{j4}^L; \eta_j^L), (\omega_{j1}^U, \omega_{j2}^U, \omega_{j3}^U, \omega_{j4}^U; \eta_j^U)] = \Sigma_{k=1}^q(\lambda_k \times \tilde{\tilde{\omega}}_{kj}) \\ &= \Sigma_{k=1}^q(\lambda_k \times [(\omega_{kj1}^L, \omega_{kj2}^L, \omega_{kj3}^L, \omega_{kj4}^L; \eta_{kj}^L), (\omega_{kj1}^U, \omega_{kj2}^U, \omega_{kj3}^U, \omega_{kj4}^U; \eta_{kj}^U)]) \\ &= [(\Sigma_{k=1}^q(\lambda_k\omega_{kj1}^L), \Sigma_{k=1}^q(\lambda_k\omega_{kj2}^L), \Sigma_{k=1}^q(\lambda_k\omega_{kj3}^L), \Sigma_{k=1}^q(\lambda_k\omega_{kj4}^L); \min_k(\eta_{kj}^L)) \\ &, (\Sigma_{k=1}^q(\lambda_k\omega_{kj1}^U), \Sigma_{k=1}^q(\lambda_k\omega_{kj2}^U), \Sigma_{k=1}^q(\lambda_k\omega_{kj3}^U), \Sigma_{k=1}^q(\lambda_k\omega_{kj4}^U); \min_k(\eta_{kj}^U))]\end{aligned} \tag{8}$$

3.4 Construct the weighted matrix

Let $\tilde{\tilde{V}} = [\tilde{\tilde{v}}_{ij}]_{m\times n}$ be the weighted matrix, then

$$\begin{aligned}\tilde{\tilde{v}}_{ij} &= [(v_{ij1}^L, v_{ij2}^L, v_{ij3}^L, v_{ij4}^L; \varpi_{ij}^L), (v_{ij1}^U, v_{ij2}^U, v_{ij3}^U, v_{ij4}^U; \varpi_{ij}^U)] = \tilde{\tilde{x}}_{ij} \otimes \tilde{\tilde{\omega}}_j \\ &= [(x_{ij1}^L\omega_{j1}^L, x_{ij2}^L\omega_{j2}^L, x_{ij3}^L\omega_{j3}^L, x_{ij4}^L\omega_{j4}^L; min(w_{ij}, \eta_j^L)), \\ &\quad (x_{ij1}^U\omega_{j1}^U, x_{ij2}^U\omega_{j2}^U, x_{ij3}^U\omega_{j3}^U, x_{ij4}^U\omega_{j4}^U; min(w_{ij}, \eta_j^U))]\end{aligned} \tag{9}$$

3.5 The decision making method based on grey relational theory

(1) Determine the positive ideal solution and the negative ideal solution of the evaluation objects. Suppose that the positive ideal solution and the negative ideal solution are $\tilde{\tilde{V}}^+ = [\tilde{\tilde{v}}_j^+]_{1\times n}, \tilde{\tilde{V}}^- = [\tilde{\tilde{v}}_j^-]_{1\times n}$, then

$$\begin{aligned}\tilde{\tilde{v}}_j^+ &= [(v_{j1}^{L+}, v_{j2}^{L+}, v_{j3}^{L+}, v_{j4}^{L+}; \varpi_j^{L+}), (v_{j1}^{U+}, v_{j2}^{U+}, v_{j3}^{U+}, v_{j4}^{U+}; \varpi_j^{U+})] \\ &= [(\max_i(v_{ij1}^L), \max_i(v_{ij2}^L), \max_i(v_{ij3}^L), \max_i(v_{ij4}^L); \max_i(\varpi_{ij}^L)), \\ &\quad (\max_i(v_{ij1}^U), \max_i(v_{ij2}^U), \max_i(v_{ij3}^U), \max_i(v_{ij4}^U); \max_i(\varpi_{ij}^U))]\end{aligned} \tag{10}$$

$$\begin{aligned}\tilde{\tilde{v}}_j^- &= [(v_{j1}^{L-}, v_{j2}^{L-}, v_{j3}^{L-}, v_{j4}^{L-}; \varpi_j^{L-}), (v_{j1}^{U-}, v_{j2}^{U-}, v_{j3}^{U-}, v_{j4}^{U-}; \varpi_j^{U-})] \\ &= [(\min_i(v_{ij1}^L), \min_i(v_{ij2}^L), \min_i(v_{ij3}^L), \min_i(v_{ij4}^L); \min_i(\varpi_{ij}^L)), \\ &\quad (\min_i(v_{ij1}^U), \min_i(v_{ij2}^U), \min_i(v_{ij3}^U), \min_i(v_{ij4}^U); \min_i(\varpi_{ij}^U))]\end{aligned} \tag{11}$$

(2) Calculate the grey correlative degree of ith alternative and the positive ideal solution with respect to the jth attribute [8].

The grey correlative coefficient of ith alternative and the positive ideal solution with respect to the jth attribute is

$$r_{ij}^+ = \frac{m + \xi M}{\Delta_{ij}^+ + \xi M}, \xi \in (0,1) \tag{12}$$

where $\Delta_{ij}^+ = d(\tilde{\tilde{v}}_j^+, \tilde{\tilde{v}}_{ij}), m = \underbrace{\min}_i \underbrace{\min}_j \Delta_{ij}^+, M = \underbrace{\max}_i \underbrace{\max}_j \Delta_{ij}^+$. ξ is a resolution coefficient, generally, $\xi = 0.5$.

The grey correlative degree of the ith alternative and the positive ideal solutions is

$$R_i^+ = \frac{1}{n}\Sigma_{j=1}^n r_{ij}^+, (i = 1, 2, \ldots, m) \tag{13}$$

(3) Calculate the grey correlative degree of ith alternative and the negative ideal solution with respect to the jth attribute [8].

The grey correlative coefficient of ith alternative and the negative ideal solution with respect to the jth attribute is

$$r_{ij}^{-} = \frac{m + \xi M}{\Delta_{ij}^{-} + \xi M}, \xi \in (0,1) \tag{14}$$

where $\Delta_{ij}^{-} = d(\tilde{\tilde{v}}_j^{-}, \tilde{\tilde{v}}_{ij})$, $m = \underbrace{\min}_{i}\underbrace{\min}_{j} \Delta_{ij}^{-}$, $M = \underbrace{\max}_{i}\underbrace{\max}_{j} \Delta_{ij}^{-}$, ξ is a resolution coefficient, generally, $\xi = 0.5$.

The grey correlative degree of the ith alternative and the negative ideal solutions is

$$R_i^{-} = \frac{1}{n}\Sigma_{j=1}^{n} r_{ij}^{-}, (i = 1, 2, \ldots, m) \tag{15}$$

(4) Calculate the grey correlative similarity coefficient of each alternative.

$$C_i = \frac{R_i^{+}}{R_i^{+} + R_i^{-}}, (i = 1, 2, \ldots, m) \tag{16}$$

The grey correlative similarity coefficient C_i satisfies the property: $0 < C_i < 1$.

(5) Rank the alternatives

Based on the grey correlative similarity coefficient, we can rank all alternatives. The bigger the grey correlative similarity coefficient is, the better prior the alternative is, or vice versa.

4 An Illustrative Example

Suppose that a Telecommunication Company intends to choose a manager for R&D department from four volunteers named $A1, A2, A3$ and $A4$ (The data came from [7]) . The decision making committee assesses the four concerned volunteers based on five attributes, including: (1) proficiency in identifying research areas (C1), (2) proficiency in administration (C2), (3) personality (C3), (4) past experience (C4) and (5) self-confidence (C5). The number of the committee members is three, labeled asDM1, DM2, DM3 respectively. Each decision maker has presented his assessment based on linguistic terms for the importance of each attribute and the evaluation information of four volunteers shown in Tables2, 3, 4 and 5 respectively. Decision steps are shown as follows:

Table 2. the attribute weights given by three DMs

	c_1	c_2	c_3	c_4	c_5
DM1	VH	H	H	VH	M
DM2	VH	H	MH	H	MH
DM3	VH	MH	MH	VH	M

Table 3. the evaluation information of four volunteers given by DM1

	c_1	c_2	c_3	c_4	c_5
a_1	VG	VG	VG	VG	VG
a_2	G	VG	VG	VG	MG
a_3	VG	MG	G	G	G
a_4	G	F	F	G	MG

Table 4. the evaluation information of four volunteers given by DM2

	c_1	c_2	c_3	c_4	c_5
a_1	G	MG	G	G	VG
a_2	G	VG	VG	VG	MG
a_3	G	G	MG	VG	G
a_4	VG	F	MG	F	G

(1) Convert the linguistic terms into the GIVTFNs, we can get the decision data expressed by interval-valued trapezoidal fuzzy numbers. (See [7]).

(2) Aggregate the individual preferences in order to obtain a collective preference value for each alternative:

$$[\tilde{\tilde{x}}_{ij}]_{4\times 5} = \left[\begin{array}{l} [(0.743, 0.797, 0.907, 0.943; 0.800), (0.743, 0.797, 0.907, 0.943; 1.000)], \\ [(0.673, 0.730, 0.880, 0.933; 0.800), (0.673, 0.730, 0.880, 0.933; 1.000)], \\ [(0.860, 0.913, 0.973, 0.990; 0.800), (0.860, 0.913, 0.973, 0.990; 1.000)], \\ [(0.743, 0.797, 0.907, 0.990; 0.800), (0.743, 0.797, 0.907, 0.943; 1.000)], \end{array}\right.$$

$$[(0.610, 0.673, 0.793, 0.837; 0.800), (0.610, 0.673, 0.793, 0.837; 1.000)],$$

$$[(0.813, 0.863, 0.933, 0.953; 0.800), (0.813, 0.863, 0.933, 0.953; 1.000)],$$

$$[(0.743, 0.797, 0.907, 0.943; 0.800), (0.743, 0.797, 0.907, 0.943; 1.000)],$$

$$[(0.523, 0.600, 0.720, 0.767; 0.800), (0.523, 0.600, 0.720, 0.767; 1.000)],$$

$$[(0.790, 0.847, 0.947, 0.980; 0.800), (0.790, 0.847, 0.947, 0.980; 1.000)],$$

$$[(0.860, 0.913, 0.973, 0.990; 0.800), (0.860, 0.913, 0.973, 0.990; 1.000)],$$

$$[(0.743, 0.797, 0.907, 0.943; 0.800), (0.743, 0.797, 0.907, 0.943; 1.000)],$$

$$[(0.493, 0.557, 0.727, 0.790; 0.800), (0.493, 0.557, 0.727, 0.790; 1.000)],$$

$$[(0.860, 0.913, 0.973, 0.990; 0.800), (0.860, 0.913, 0.973, 0.990; 1.000)],$$

$$[(0.813, 0.863, 0.933, 0.953; 0.800), (0.813, 0.863, 0.933, 0.953; 1.000)],$$

$$[(0.860, 0.913, 0.973, 0.990; 0.800), (0.860, 0.913, 0.973, 0.990; 1.000)],$$

$$[(0.657, 0.723, 0.833, 0.873; 0.800), (0.657, 0.723, 0.833, 0.873; 1.000)],$$

$$\left.\begin{array}{l} [(0.930, 0.980, 1.000, 1.000; 0.800), (0.930, 0.980, 1.000, 1.000; 1.000)] \\ [(0.627, 0.680, 0.840, 0.897; 0.800), (0.627, 0.680, 0.840, 0.897; 1.000)] \\ [(0.673, 0.730, 0.880, 0.933; 0.800), (0.673, 0.730, 0.880, 0.933; 1.000)] \\ [(0.540, 0.607, 0.767, 0.827; 0.800), (0.540, 0.607, 0.767, 0.827; 1.000)] \end{array}\right]$$

Table 5. the evaluation information of four volunteers given by DM3

	c_1	c_2	c_3	c_4	c_5
a_1	MG	F	G	VG	VG
a_2	MG	MG	G	MG	G
a_3	VG	VG	VG	VG	MG
a_4	MG	VG	MG	VG	F

$$[\tilde{\tilde{\omega}}_j]_5 = \begin{matrix}[[(0.930, 0.980, 1.000, 1.000; 0.800), (0.930, 0.980, 1.000, 1.000; 1.000)],\\ [(0.627, 0.680, 0.840, 0.897; 0.800), (0.627, 0.680, 0.840, 0.897; 1.000)],\\ [(0.627, 0.680, 0.840, 0.897; 0.800), (0.627, 0.680, 0.840, 0.897; 1.000)],\\ [(0.930, 0.980, 1.000, 1.000; 0.800), (0.930, 0.980, 1.000, 1.000; 1.000)],\\ [(0.320, 0.410, 0.580, 0.650; 0.800), (0.320, 0.410, 0.580, 0.650; 1.000)]]\end{matrix}$$

(3) Calculate the weighted decision making matrix:

$$[\tilde{\tilde{v}}_{ij}]_{4\times5} = \left[\begin{matrix}[(0.691, 0.781, 0.907, 0.943; 0.800), (0.691, 0.781, 0.907, 0.943; 1.000)],\\ [(0.626, 0.715, 0.880, 0.933; 0.800), (0.626, 0.715, 0.880, 0.933; 1.000)],\\ [(0.800, 0.895, 0.973, 0.990; 0.800), (0.800, 0.895, 0.973, 0.990; 1.000)],\\ [(0.691, 0.781, 0.907, 0.943; 0.800), (0.691, 0.781, 0.907, 0.943; 1.000)],\end{matrix}\right.$$

$$[(0.382, 0.458, 0.666, 0.750; 0.800), (0.382, 0.458, 0.666, 0.750; 1.000)],$$

$$[(0.510, 0.587, 0.784, 0.855; 0.800), (0.510, 0.587, 0.784, 0.855; 1.000)],$$

$$[(0.466, 0.542, 0.762, 0.846; 0.800), (0.466, 0.542, 0.762, 0.846; 1.000)],$$

$$[(0.328, 0.408, 0.605, 0.687; 0.800), (0.328, 0.408, 0.605, 0.687; 1.000)],$$

$$[(0.495, 0.576, 0.795, 0.879; 0.800), (0.495, 0.576, 0.795, 0.879; 1.000)],$$

$$[(0.539, 0.621, 0.818, 0.888; 0.800), (0.539, 0.621, 0.818, 0.888; 1.000)],$$

$$[(0.466, 0.542, 0.762, 0.846; 0.800), (0.466, 0.542, 0.762, 0.846; 1.000)],$$

$$[(0.309, 0.379, 0.610, 0.708; 0.800), (0.309, 0.379, 0.610, 0.708; 1.000)],$$

$$[(0.800, 0.895, 0.973, 0.990; 0.800), (0.800, 0.895, 0.973, 0.990; 1.000)],$$

$$[(0.756, 0.846, 0.933, 0.953; 0.800), (0.756, 0.846, 0.933, 0.953; 1.000)],$$

$$[(0.800, 0.895, 0.973, 0.990; 0.800), (0.800, 0.895, 0.973, 0.990; 1.000)],$$

$$[(0.611, 0.709, 0.833, 0.873; 0.800), (0.611, 0.709, 0.833, 0.873; 1.000)],$$

$$\left.\begin{matrix}[(0.298, 0.402, 0.580, 0.650; 0.800), (0.298, 0.402, 0.580, 0.650; 1.000)]\\ [(0.201, 0.279, 0.487, 0.583; 0.800), (0.201, 0.279, 0.487, 0.583; 1.000)]\\ [(0.215, 0.299, 0.510, 0.607; 0.800), (0.215, 0.299, 0.510, 0.607; 1.000)]\\ [(0.173, 0.249, 0.445, 0.537; 0.800), (0.173, 0.249, 0.445, 0.537; 1.000)]\end{matrix}\right]$$

(4) Determine the positive ideal solution and the negative ideal solution:

$$\tilde{\tilde{V}}^+ = \begin{matrix}[[(0.800, 0.895, 0.973, 0.990; 0.800), (0.800, 0.895, 0.973, 0.990; 1.000)],\\ [(0.510, 0.587, 0.784, 0.855; 0.800), (0.510, 0.587, 0.784, 0.855; 1.000)],\\ [(0.539, 0.621, 0.818, 0.888; 0.800), (0.539, 0.621, 0.818, 0.888; 1.000)],\\ [(0.800, 0.895, 0.973, 0.990; 0.800), (0.800, 0.895, 0.973, 0.990; 1.000)],\\ [(0.298, 0.402, 0.580, 0.650; 0.800), (0.298, 0.402, 0.580, 0.650; 1.000)]]\end{matrix}$$

$$\tilde{\tilde{V}}^- = \begin{matrix}[[(0.626, 0.715, 0.880, 0.933; 0.800), (0.626, 0.715, 0.880, 0.933; 1.000)],\\ [(0.328, 0.408, 0.605, 0.687; 0.800), (0.328, 0.408, 0.605, 0.687; 1.000)],\\ [(0.309, 0.379, 0.601, 0.708; 0.800), (0.309, 0.379, 0.610, 0.708; 1.000)],\\ [(0.611, 0.709, 0.833, 0.873; 0.800), (0.611, 0.709, 0.833, 0.873; 1.000)],\\ [(0.173, 0.249, 0.445, 0.537; 0.800), (0.173, 0.249, 0.445, 0.537; 1.000)]]\end{matrix}$$

(5) Calculate the grey correlative coefficient matrix:

$$R^+ = \begin{bmatrix} 0.5610 & 0.4722 & 0.7826 & 1.0000 & 1.0000\\ 0.4608 & 1.0000 & 1.0000 & 0.7178 & 0.5338\\ 1.0000 & 0.7825 & 0.6328 & 1.0000 & 0.5947\\ 0.5610 & 0.3766 & 0.3333 & 0.4045 & 0.4516\end{bmatrix}$$

$$R^- = \begin{bmatrix} 0.7207 & 0.6501 & 0.3673 & 0.4045 & 0.4516\\ 1.0000 & 0.3766 & 0.3333 & 0.4809 & 0.7451\\ 0.4608 & 0.4205 & 0.4132 & 0.4045 & 0.6521\\ 0.7207 & 1.0000 & 1.0000 & 1.0000 & 1.0000\end{bmatrix}$$

(6) Calculate the grey correlative similarity coefficient of each alternative: $C = (0.5953, 0.5584, 0.6304, 0.3106)$

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(7) Rank the alternatives:

Based on the grey correlative similarity coefficient, we can rank the alternatives: $a_3 \succ a_1 \succ a_2 \succ a_4$.

(8) Analysis:

In this example, the proposed method produces the same ranking as [1] and [7], which proves the method in this paper is effective. Comparing with [1] and [7], the advantages proposed in this paper are more general and simpler in dealing with more complex problems of fuzzy multiple attribute decision making.

5 Conclusion

Fuzzy multiple attribute decision making (FMADM) problems widely exist in the real decision-making, and the GIVTFN can be precisely express the attribute values and weights of the FMADM problems. In this paper, we proposed a decision making method based on the grey correlative coefficient to solve the MADM problems in which the attribute weights and values are given by the GIVTFN, and decision making steps were given in detail. Comparing with [1] and [7], the advantages proposed in this paper are more general and simpler in dealing with the fuzzy multiple attribute decision making. This method enriches and develops the theory and method of FMADM, and gives a new idea for solving the FMADM problems. In the future, we will research the applications of the propose method.

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References

1. B. Ashtiani, F. Haghighirad, A. Makui, G. Montazer, Extension of fuzzy TOPSIS method based on interval-valued fuzzy sets, Applied Soft Computing, 9, 457-461(2009).
2. R.E. Bellman, L.A. Zadeh, Decision-making in a fuzzy environment, Management Science, 17, 141-164 (1970).
3. S.J. Chen, S.M. Chen, A new method for handing multicriteria fuzzy decision-making problems using FN-IOWA operators, Cybernetics and Systems, 34,109-137(2003).
4. M.B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems, 21, 1-17 (1987).
5. D.H. Hong, S. Lee, Some algebraic properties and a distance measure for interval-valued fuzzy numbers, Information Sciences, 148, 1-10 (2002).

6. P.D. Liu, An Extended TOPSIS Method for Multiple Attribute Group Decision Making based on Generalized Interval-valued Trapezoidal Fuzzy Numbers, Informatica, 35, 185-196 (2011).
7. P.D. Liu, A weighted aggregation operators multi-attribute group decision-making method based on interval-valued trapezoidal fuzzy numbers, Expert Systems with Applications, 38, 1053-1060 (2011).
8. S.F. Liu, T.B. Guo, Y.G. Dang, Grey system theory and application, Beijing: Science Press, 1999.
9. F. Men, S.Q. Ji, An improved FMEA based on Fuzzy Set theory and Grey Relational Theory, Industrial engineering and management, 2, 55-59 (2008).
10. I.B. Turksen, Interval-valued strict preference with Zadeh triples, Fuzzy Sets and Systems, 78, 183-195 (1996).
11. G. Wang, X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems, 98, 331-335 (1998).
12. G. Wang, X. Li, Correlation and information energy of interval-valued fuzzy numbers, Fuzzy Sets and Systems, 103, 69-175 (2001).
13. S.H. Wei, S.M. Chen, Fuzzy risk analysis based on interval-valued fuzzy numbers, Expert Systems with Applications, 36, 2285-2299 (2009).
14. G.W. Wei, Y. Wei, Model of Grey Relational Analysis for Interval Multiple Attribute Decision Making with Preference Information on Alternatives, Chinese Journal of Management Science, 16, 158-162 (2008).

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Note on the second kind Barnes' type multiple q-Euler polynomials

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Abstract : In this paper we introduce the second kind Barnes-type multiple q-Euler numbers and polynomials, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. We give some interesting properties.

Key words : The second kind q-Euler numbers and polynomials, the second kind Barnes-type multiple q-Euler numbers and polynomials

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 2, 3].} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduced the second kind Euler numbers E_n. The second kind Euler numbers E_n are defined by the generating function(see [4]):

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$\frac{2e^t}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

In [4], we studied the second kind Euler numbers E_n and polynomials $E_n(x)$ and investigate their properties. The main aim of this paper is to study the second kind Barnes-type multiple q-Euler polynomials, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$.

2. The second kind Barnes-type multiple q-Euler polynomials

In this section, we assume that $w_1, \ldots, w_k \in \mathbb{Z}_p$ and $a_1, \ldots, a_k \in \mathbb{Z}$.
We introduce the second kind Barnes-type multiple q-Euler polynomials,

$$E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x).$$

For $k \in \mathbb{N}$, we define the second kind Barnes-type multiple q-Euler polynomials as follows:

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} q^{a_1x_1+\cdots+a_kx_k} e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \frac{2^k e^{kt}}{(q^{a_1}e^{2w_1t}+1)(q^{a_2}e^{2w_2t}+1)\cdots(q^{a_k}e^{2w_kt}+1)} e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

In the special case, $x = 0$, $E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid 0) = E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k)$ are called the second kind n-th Barnes-type multiple q-Euler numbers.

Theorem 1. For positive integers n and k, we have

$$\begin{aligned}
&E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} q^{a_1x_1+\cdots+a_kx_k}(x+2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).
\end{aligned}$$

By using the above Theorem 1, we have the following corollary.

Corollary 2. For positive integers n, we have

$$\begin{aligned}
&E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} q^{\sum_{i=1}^k a_ix_i}(2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).
\end{aligned} \tag{2.2}$$

By Theorem 1 and (2.2), we obtain

$$E_{n,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}(w_1, \ldots, w_k; a_1, \ldots, a_k), \tag{2.3}$$

where $\binom{n}{k}$ is a binomial coefficient.

In the special case, $(\underbrace{w_1,\ldots,w_k}_{k-\text{times}};\underbrace{a_1,\ldots,a_k}_{k-\text{times}})=(1,\ldots,1;1,\ldots,1)$, we have

$$E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k\mid x)=E_{n,q}^{(k)}(x),$$

where $E_{n,q}^{(k)}(x)$ denotes the second kind q-Euler polynomials of higher order(see [5]).

We define distribution relation of the second kind Barnes-type multiple q-Euler polynomials as follows: For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k\mid x)\frac{t^n}{n!}\\&=\frac{2^ke^{kmt}}{(q^{a_1m}e^{2w_1mt}+1)(q^{a_2m}e^{2w_2mt}+1)\cdots(q^{a_km}e^{2w_kmt}+1)}\\&\quad\times\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}q^{\sum_{i=1}^{k}a_il_i}e^{\left(\frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)(mt)}.\end{aligned}$$

From the above, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k\mid x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}m^n\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}q^{\sum_{i=1}^{k}a_il_i}\\&\quad\times E_{n,q^m}\left(w_1,\ldots,w_k;a_1,\ldots,a_k\mid\frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 3 (Distribution relation). For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we have

$$\begin{aligned}E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k\mid x)&=m^n\sum_{l_1,\ldots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}q^{\sum_{i=1}^{k}a_il_i}\\&\quad\times E_{n,q^m}\left(w_1,\ldots,w_k;a_1,\ldots,a_k\mid\frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right).\end{aligned}$$

From (2.1), we derive

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}q^{a_1x_1+\cdots+a_kx_k}e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\&=2^k\sum_{m_1,\ldots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}q^{\sum_{i=1}^{k}a_im_i}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.\end{aligned}\tag{2.4}$$

From (2.2) and (2.4), we note that

$$\begin{aligned} &E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x) \\ &= 2^k \sum_{m_1,\ldots m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} q^{\sum_{i=1}^k a_i m_i} (x+2w_1m_1+\cdots+2w_km_k+k)^n. \end{aligned} \tag{2.5}$$

By using binomial expansion and (2.1), we have the following addition theorem.

Theorem 4(Addition theorem). The second kind Barnes-type multiple q-Euler polynomials $E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)$ satisfies the following relation:

$$\begin{aligned} &E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x+y) \\ &= \sum_{l=0}^{n} \binom{n}{l} E_{l,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x) y^{n-l}. \end{aligned}$$

3. The second kind Barnes-type multiple q-Euler zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters $w_1,\ldots,w_k$ are positive. By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the generating function of the second kind Barnes-type multiple q-Euler polynomials, $E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)$, we define the second kind Barnes-type multiple q-Euler zeta function. This function interpolates the second kind Barnes-type multiple q-Euler polynomials at negative integers.

By (2.1), we obtain

$$\begin{aligned} F_q(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x,t) &= \frac{2^k e^{kt}}{(q^{a_1}e^{2w_1t}+1)\cdots(q^{a_k}e^{2w_kt}+1)} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Hence, by (3.1), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x)\frac{t^n}{n!} \\ &= 2^k \sum_{m_1,\ldots m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} q^{\sum_{i=1}^k a_i m_i} e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}. \end{aligned}$$

By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the above equation, we have

$$\begin{aligned} &E_{n,q}(w_1,\ldots,w_k;a_1,\ldots,a_k \mid x) \\ &= 2^k \sum_{m_1,\ldots m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} q^{\sum_{i=1}^k a_i m_i} (x+2w_1m_1+\cdots+2w_km_k+k)^n. \end{aligned} \tag{3.2}$$

By (3.2), we define the second kind Barnes-type multiple q-Euler zeta function

$$\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid s, x)$$

as follows:

Definition 1. For $s, x \in \mathbb{C}$ with $Re(x) > 0$, $a_1, \dots, a_k \in \mathbb{C}$, we define

$$\begin{aligned} &\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid s, x) \\ &= 2^k \sum_{m_1,\dots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k} q^{\sum_{i=1}^k a_i m_i}}{(x + 2w_1 m_1 + \cdots + 2w_k m_k + k)^s}. \end{aligned} \tag{3.3}$$

For $s = -l$ in (3.3) and using (3.2), we arrive at the following theorem.

Theorem 5. For positive integer l, we have

$$\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid -l, x) = E_{l,q}(w_1, \dots, w_k; a_1, \dots, a_k \mid x).$$

By (2.6), we define the second kind multiple q-Euler zeta function

$$\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid s)$$

as follows:

Definition 2. For $s \in \mathbb{C}$, we define

$$\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid s) = 2^k \sum_{m_1,\dots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k} q^{\sum_{i=1}^k a_i m_i}}{(2w_1 m_1 + \cdots + 2w_k m_k + k)^s}, \tag{3.4}$$

For $s = -l$ in (3.4) and using (2.6), we arrive at the following theorem.

Theorem 6. For positive integer l, we have

$$\zeta_q(w_1, \dots, w_k; a_1, \dots, a_k \mid -l) = E_{l,q}(w_1, \dots, w_k; a_1, \dots, a_k).$$

REFERENCES

1. T. Kim, On Euler-Barnes multiple zeta function, Russ. J. Math. Phys., 10(2003), 261-267.

2. T. Kim, Barnes-type multiple q-zeta functions and q-Euler polynomials, J. Phys. A : Math. Theor., 43(2010) 255201(11pp).

3. T. Kim, q-Volkenborn integration, Russ. J. Math. Phys., 9(2002), 288-299.

4. C. S. Ryoo, Calculating zeros of the second kind Euler polynomials, J. Comput. Anal. Appl. 12 (2010), 828-833.

5. C. S. Ryoo, A Note on the second kind q-Euler polynomials of higher order, Applied Mathematics Sciences, 5 (2011), 3421-3427.

The Approximation and Growth Problem of Dirichlet Series of Infinite Order *

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Abstract

By introducing the concept of β_U-order, we first investigate the growth of Dirichlet series of infinite order which convergence in the half plane, and a necessary and sufficient conditions on the growth of Dirichlet series with finite β_U-order has been obtained. We also investigate the error in approximating Dirichlet series of finite order β_U-order in the half plane by Dirichlet polynomials. Some relations between the error and growth of Dirichlet series of finite β_U-order have been obtained.

Key words: growth, β_U-order, approximation, Dirichlet series.

2010 Mathematics Subject Classification: 30B50, 30D15.

1 Introduction and Basic Notes

Consider Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_n s},\qquad s=\sigma+it, \tag{1}$$

where

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty\quad as\quad n\to\infty; \tag{2}$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h<+\infty, \tag{3}$$

$$\limsup_{n\to+\infty}\frac{\log^+|a_n|}{\lambda_n}=0, \tag{4}$$

then from (2), by using the similar method in [20] or [16, 17], we can get

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=E<+\infty,\qquad \limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{5}$$

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Then the abscissas of convergence and absolute convergence is 0, that is, $f(s)$ is an analytic function in the left half plane $H = \{s = \sigma + it : \sigma < 0, t \in \mathbb{R}\}$.

We denote D to be the class of all functions $f(s)$ satisfying (2)-(4) and analytic in $Res < 0$, denote $\overline{D}_\alpha$ to be the class of all functions $f(s)$ satisfying (2)-(3) and analytic in $Re \leq \alpha$ where $-\infty < \alpha < +\infty$. Thus, if $-\infty < \alpha < 0$ and $f(s) \in D$, then $f(s) \in \overline{D}_\alpha$; if $0 < \alpha < +\infty$ and $f(s) \in \overline{D}_\alpha$, then $f(s) \in D$. We denote Π_k to be the class of all exponential polynomial of degree almost k, that is,

$$\Pi_k = \left\{ \sum_{j=1}^{k} b_j e^{\lambda_j s} : (b_1, b_2, \ldots, b_k) \in \mathbb{C}^k \right\}.$$

For $f(s) \in D$,

$$M(\sigma, f) = \max_{-\infty < t < \infty} |f(\sigma + it)|, \qquad m(\sigma, f) = \max_{n \geq 1}\{|a_n| e^{\sigma \lambda_n}\}$$

are called, respectively, the maximum modulus, the maximum term of $f(s)$ for $Res = \sigma < 0$.

Definition 1.1 *Let $f(s) \in D$, the order of $f(s)$ can be defined by*

$$\rho = \limsup_{\sigma \to 0^-} \frac{\log \log^+ M(\sigma, f)}{-\log(-\sigma)},$$

where $\log^+ x = \begin{cases} \log x, & x \geq 1 \\ 0, & x < 1. \end{cases}$

For $\rho = 0, 0 < \rho < \infty, \rho = \infty$, $f(s)$ can be called zero order, finite order, infinite order Dirichlet series, respectively. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series; see [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 21] for some results.

By introducing the concept of β-order, the author [18] studied the growth of functions of infinite order which are represented by a class of Dirichlet series convergence in the half-plane, and obtained the following theorem

Theorem 1.1 *(see [18]). Let $f(s) \in D$ are of β-order $\rho_\beta (0 < \rho_\beta < \infty)$, then we have*

$$\limsup_{n \to \infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log^+ \log |a_n|} = \rho_\beta = \limsup_{\sigma \to 0^-} \frac{\beta(\log M(\sigma, f))}{-\log(-\sigma)}.$$

Remark 1.1 *The definition of β-order of Dirichlet series will be introduced in Section 2.*

Thus, a question arises naturally: *what may happen when $\rho_\beta = \infty$ in Theorem 1.1?*

The purpose of this paper is to deal with the above question by using the type function $U(x)$ in [11] to enlarge the growth of the denominator $-\log(-\sigma)$ and the main results are obtained as follows.

Theorem 1.2 *If Dirichlet series $f(s) \in D$ is of infinite β-order, then we have*

$$\limsup_{\sigma \to 0^-} \frac{\beta(\log^+ M(\sigma, F))}{\log U\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{\sigma \to 0^+} \frac{\beta(\log^+ m(\sigma, F))}{\log U\left(\frac{1}{-\sigma}\right)} = T.$$

where $0 < T < \infty$ and $U(x) = x^{\rho(x)}$ satisfies the following conditions

(i) $\rho(x)$ is monotone and $\lim_{x \to \infty} \rho(x) = \infty$;

(ii) $\lim_{x \to \infty} \frac{U(x')}{U(x)} = 1$, where $x' = x + \frac{x \log x}{\log U(x) \log^2 \log U(x)}$.

Remark 1.2 *From Lemma 2.1 and Lemma 2.2 in Section 2, we can prove the conclusion of Theorem 1.2.*

Remark 1.3 *If Dirichlet series $f(s)$ of infinite order has infinite β-order and satisfies*

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U\left(-\frac{1}{\sigma}\right)}=T,$$

then T is called the β_U-order of Dirichlet series $f(s)$.

Theorem 1.3 *Let $f(s)\in D$ are of infinite β-order, then*

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log M(\sigma,f))}{\log U(-\frac{1}{\sigma})}=T \Longleftrightarrow \limsup_{n\to\infty}\frac{\beta(\log^+|a_n|)}{\log U(\frac{\lambda_n}{\log^+|a_n|})}=T, \tag{6}$$

where $0<T<\infty$.

For $f(s)\in \overline{D}_\alpha, -\infty<\alpha<+\infty$, we denote by $E_n(f,\alpha)$ the error in approximating the function $f(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(f,\alpha)=\inf_{p\in\Pi_n}\parallel f-p\parallel_\alpha,\quad n=1,2,\dots,$$

where

$$\parallel f-p\parallel_\alpha=\max_{-\infty<t<+\infty}|f(\alpha+it)-p(\alpha+it)|.$$

In 2010, the authors [18] investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$, and obtained some equivalence relation between $E_n(f,\alpha)$ and the regular growth of $f(s)$ with finite order as follows:

Theorem 1.4 (see [18]). *Let $f(s)\in D$ be of finite order ρ, then for any real number $-\infty<\alpha<0$, we have*

$$\lim_{\sigma\to 0^-}\frac{\log^+ M(\sigma,f)}{U_1(-\frac{1}{\sigma})}=1 \Longleftrightarrow \limsup_{n\to+\infty}\frac{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}{BU_1\left(\frac{\lambda_{n+1}}{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}\right)}=1;$$

and there exists a increasing, positive integer sequence $\{n_\nu\}$ satisfying

$$\lim_{\nu\to+\infty}\frac{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_{\nu+1}}}\right]}{BU_1\left(\frac{\lambda_{n_{\nu+1}}}{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_{\nu+1}}}\right]}\right)}=1,\qquad \lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1,$$

where $B=\frac{(1+\rho)^{1+\rho}}{\rho^\rho}$ and $U_1(r)=r^{\rho(r)}$, $\rho(r)$ satisfies the following conditions:

(i) there exists a real number $r_0>0$, $\rho(r)$ is nonnegative, continuous, monotone on $[r_0,+\infty)$, and tends to ρ as $r\to+\infty$;

(ii) $\lim_{r\to+\infty}\rho'(r)r\log r=0$,;

(iii) $U_1(kr)=[k^\rho+o(1)]U_1(r)(r\to+\infty)$ for every positive integer k, and $U_1(r)$ is an increasing function on $r\geq r_0'>r_0$.

Recently, the authors [19] further investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$ when $f(s)$ has infinite order, by introducing the concept of β-order.

Theorem 1.5 (see [19]). *Let $f(s) \in D$ be of finite β-order ρ_β, then for any real number $-\infty < \alpha < 0$, we have*

$$\limsup_{n\to\infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log\log^+(E_{n-1}(f,\alpha)e^{-\alpha\lambda_n})} = \rho_\beta.$$

In this paper, we will investigate the problem on $\rho_\beta = \infty$ in Theorem 1.5 by using the type functions $U(x)$ in [11] to enlarge the growth of the denominator $-\log(-\sigma)$ and obtain the following theorem.

Theorem 1.6 *If Dirichlet series $f(s) \in D$ with infinite β-order, then for any fixed real number $-\infty < \alpha < 0$, we have*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma,f))}{\log U\left(-\frac{1}{\sigma}\right)} = T \Longleftrightarrow \limsup_{n\to\infty} \Psi_n(f,\alpha,\lambda_n) = T; \tag{7}$$

where

$$\Psi_n(f,\alpha,\lambda_n) = \frac{\beta\left(\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]\right)}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}.$$

The structure of this paper is as follows. In Section 2, we introduce the concept of β-order and give some lemmas. Section 3 is devoted to proving Theorem 1.3. Section 4 is devoted to proving Theorem 1.6.

2 Some Lemmas and The Concept of β-order

In this section, we first introduce the definition of β-order of Dirichlet series as follows.

Let $\mathfrak{F}$ be the class of all functions $\beta(x)$ satisfies the following conditions:

(i) $\beta(x)$ is defined on $[a,+\infty), a > 0$, and positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to +\infty$;

(ii) $x\beta'(x) = o(1)$ as $x \to +\infty$.

Definition 2.1 *([20]). If Dirichlet series $f(s)$ of infinite order satisfies*

$$\limsup_{\sigma\to 0^+} \frac{\beta(\log^+ M(\sigma,f))}{-\log(-\sigma)} = \rho^*,$$

where $\beta(x) \in \mathfrak{F}$, then ρ^* is called the β-order of $f(s)$.

Remark 2.1 *We can see that p-order in [4] is regard as a special case of β-order of Dirichlet series. For example, the function $h(x) = \log_p x, p \geq 2, p \in N_+$ satisfies the conditions (i) and (ii), where p is a positive integer, and $\log_1 x = \log x$ and $\log_p x = \log(\log_{p-1} x)$.*

Remark 2.2 *We can deduce that β-order is more precise than p-order to some extent. In fact, for $p(\geq 2)$ is a positive integer, we can find function $\beta(x) \in \mathfrak{F}$ and a positive real function $M(x)$ satisfying $\rho_p(M) = \infty$, $\rho_{p+1}(M) = 0$ and $\rho_\beta(M) = t(0 < t < \infty)$. For example, let $M(x) = \exp_{p+1}\{(t\log x)^{1/d}\}, \beta(x) = (\log_{p+1} x)^d$, where t is a finite positive real constant and $0 < d < 1$. Then we have*

$$\rho_p(M) = \limsup_{x\to\infty} \frac{\log_p(\log M(x))}{\log x} = \infty,\ \rho_{p+1}(M) = \limsup_{x\to\infty} \frac{\log_{p+1}(\log M(x))}{\log x} = 0,$$

and

$$\rho_\beta(M) = \limsup_{x\to\infty} \frac{\beta(\log M(x))}{\log x} = t, \quad (0 < t < \infty).$$

Remark 2.3 *When $\rho^* = \infty$ in Definition 2.1, we will say that $f(s)$ is a Dirichlet series of infinite β-order.*

Lemma 2.1 *Let $\beta(x) \in \mathfrak{F}$, $\varphi(x)$ be the function such that*

$$\limsup_{x\to\infty} \frac{\log^+ \varphi(x)}{\log x} = \varrho, \quad (0 \le \varrho < \infty),$$

and if the real function $M(x)$ satisfies $\limsup_{x\to\infty} \frac{\beta(\log M(x))}{\log x} = \nu(> 0)$. Then we have

$$\limsup_{x\to\infty} \frac{\beta(\varphi(x)\log M(x))}{\log x} = \nu.$$

Proof: The proof of this lemma can be found in [19], for the convenience of the reader, we give the process of proof of this lemma as follows.

Two following cases will be considered.

Case 1. If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, we can get that $\varphi(x) \to \infty$ as $x \to \infty$. Then, for sufficiently large x, we have $\varphi(x) > 1$. From $\beta(x) \in \mathfrak{F}$, we have $\lim_{x\to\infty} \log M(x) = \infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x) < \xi < \beta(x)\log M(x))$ satisfying

$$\frac{\beta(\varphi(x)\log M(x)) - \beta(\log M(x))}{\log(\varphi(x)\log M(x)) - \log\log M(x)} = \frac{\beta'(\xi)}{(\log \xi)'} = \xi\beta'(\xi),$$

that is,

$$\beta(\varphi(x)\log M(x)) = \beta(\log M(x)) + \log\varphi(x)\xi\beta'(\xi). \tag{8}$$

Since $x\beta'(x) = o(1)$ as $x \to +\infty$ and $\limsup_{x\to\infty} \frac{\log\varphi(x)}{\log x} = \varrho, (0 \le \varrho < \infty)$, by (8), we can get the conclusion of Lemma 2.1.

Case 2. If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that the conclusion of Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1.

The following lemma is very important in study of the growth of analytic functions represented by Dirichlet series convergent in the right half plane, which show the relation between $M(\sigma, f)$ and $m(\sigma, f)$ of such functions.

Lemma 2.2 ([20]) If Dirichlet series (1) satisfies (2) (3), then for any given $\varepsilon \in (0, 1)$ and for $\sigma(< 0)$ sufficiently reaching 0, we have

$$m(\sigma, f) \le M(\sigma, f) \le K(\varepsilon)(-\frac{1}{\sigma})m((1-\varepsilon)\sigma, f),$$

where $K(\varepsilon)$ is a positive constant depending on ε and $f(s)$.

3 The proof of Theorem 1.3

We first prove " $\Longleftarrow$ " of Theorem 1.3. Suppose that

$$\limsup_{n\to+\infty} \frac{\beta(\log^+|a_n|)}{\log U\left(\frac{\lambda_n}{\log^+|a_n|}\right)} = T, \tag{9}$$

then for any $\varepsilon(>0)$ and sufficiently large n, we have

$$\log^+ |a_n| < \gamma\left[(T+\varepsilon)\log U\left(\frac{\lambda_n}{\log^+ |a_n|}\right)\right],$$

where $\gamma(x)$ is the inverse function of $\beta(x)$. Let $V(x)$ is the inverse function of $U(x)$, then

$$\frac{\lambda_n}{\log^+ |a_n|} > V\left[\exp\left\{\frac{1}{T+\varepsilon}\beta(\log^+ |a_n|)\right\}\right],$$

and

$$\log^+ |a_n| < \frac{\lambda_n}{V\left[\exp\left\{\frac{1}{T+\varepsilon}\beta(\log^+ |a_n|)\right\}\right]}.$$

Thus, we can get

$$\log^+ |a_n| e^{\lambda_n\sigma} \le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{T+\varepsilon}\beta(\log^+ |a_n|)\right\}\right]\right)^{-1} + \sigma\right). \tag{10}$$

For any fixed sufficiently small $\sigma(<0)$, set

$$\begin{aligned} I &= \gamma\left[(T+\varepsilon)\log U\left(-\frac{1}{\sigma} - \frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right] \qquad (11)\\ &< \gamma\left[(T+\varepsilon)\log U\left(-\frac{1}{\sigma} + \frac{-\frac{1}{\sigma}\log(-\frac{1}{\sigma})}{\log U(-\frac{1}{\sigma})\log^2\log U\left(-\frac{1}{\sigma}\right)}\right)\right], \end{aligned}$$

it follows

$$-\frac{1}{\sigma} - \frac{1}{\sigma}\frac{1}{\log^2 U\left(-\frac{1}{\sigma}\right)} = V\left(\exp\left\{\frac{1}{T+\varepsilon}\beta(I)\right\}\right). \tag{12}$$

If $\log^+ |a_n| \le I$, then for sufficiently small $\sigma(<0)$, from (11) and the properties of $U(x)$, we have

$$\begin{aligned} \log^+ |a_n| e^{\lambda_n\sigma} \le \log^+ |a_n| \le I &= \gamma\left[(T+\varepsilon)\log U\left(-\frac{1}{\sigma} - \frac{1}{\sigma}\frac{1}{\log^2 U\left(\frac{1}{-\sigma}\right)}\right)\right] \\ &\le \gamma\left[(T+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \end{aligned} \tag{13}$$

If $\log^+ |a_n| > I$, then from (10) and (12), we can get

$$\begin{aligned} \log^+ |a_n| e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{T+\varepsilon}\beta(\log^+ |a_n|)\right\}\right]\right)^{-1} + \sigma\right) \\ &\le \lambda_n\left(\left(V\left[\exp\left\{\frac{1}{T+\varepsilon}\beta(I)\right\}\right]\right)^{-1} + \sigma\right) \\ &= \lambda_n\frac{\sigma}{1+\log^2 U(-\frac{1}{\sigma})} < 0. \end{aligned} \tag{14}$$

From (13) and (14), we have

$$\log m(\sigma, f) \le \gamma\left[(T+2\varepsilon)\log U(-\frac{1}{\sigma})\right]. \tag{15}$$

From (15) and Lemma 2.2, and since ε is arbitrary, it follows

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log M(\sigma,f))}{\log U(-\frac{1}{\sigma})}\le\limsup_{n\to+\infty}\frac{\beta(\log^+|a_n|)}{\log U(\frac{\lambda_n}{\log^+|a_n|})}=T.$$

Suppose that

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log M(\sigma,f))}{\log U(-\frac{1}{\sigma})}=J<\limsup_{n\to+\infty}\frac{\beta(\log^+|a_n|)}{\log U(\frac{\lambda_n}{\log^+|a_n|})}=T. \tag{16}$$

Take $\delta=\frac{T-J}{5}$, then for any positive integer n and sufficiently small $\sigma(<0)$, we have

$$\log^+|a_n|e^{\lambda_n\sigma}\le\log M(\sigma,f)<\gamma\left((J+\delta)\log U(-\frac{1}{\sigma})\right), \tag{17}$$

and from (16), there exists a subsequence $\{n(\nu)\}$ satisfying

$$\beta(\log^+|a_{n(\nu)}|)>\left((T-\delta)\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\right). \tag{18}$$

Choose the sequence $\{\sigma_\nu\}$ satisfying

$$\gamma\left((J+\delta)\log U(-\frac{1}{\sigma_\nu})\right)=\frac{\log^+|a_{n(\nu)}|}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})}. \tag{19}$$

From (17), if follows

$$\log^+|a_{n(\nu)}|e^{\lambda_{n(\nu)}\sigma_\nu}<\frac{\log^+|a_{n(\nu)}|}{1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})},$$

$$\Longrightarrow -\frac{1}{\sigma_\nu}\le\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|}\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\right),$$

$$\begin{aligned}\Longrightarrow& U(-\frac{1}{\sigma_\nu})\\ \le& U\left(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|}(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|}))\right)\\ \le&(1+\delta)U\left(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|}\right).\end{aligned} \tag{20}$$

From (19), we have

$$\begin{aligned}\log^+|a_{n(\nu)}|=&\gamma\left((J+\delta)\log U(-\frac{1}{\sigma_\nu})\right)\\ &\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem and (20), there exists a real number ξ between $\gamma((J+\delta)\log U(-\frac{1}{\sigma_\nu}))$ and $\gamma\left((J+\delta)\log U(-\frac{1}{\sigma_\nu})\right)(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+|a_{n(\nu)}|})$

$\log^2 \log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|}))$ such that

$$\begin{aligned}
\beta\left(\log^+ |a_{n(\nu)}|\right) &= \beta\left(1+\log^2 U\left(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|}\right)\gamma\left((J+\delta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\
&= \beta\left(\gamma\left((J+\delta)\log U(-\frac{1}{\sigma_\nu})\right)\right)\\
&\quad + \log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})\log^2 \log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})\right)\xi\beta'(\xi),
\end{aligned}$$

Since

$$\lim_{\nu\to\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})\log^2\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})\right)}{\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})}=0,$$

then for sufficiently large ν, from (20) we have

$$\begin{aligned}
\beta\left(\log^+ |a_{n(\nu)}|\right) &= (J+\delta)\log U(-\frac{1}{\sigma_\nu}) + K_1\xi\beta'(\xi)\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|})\\
&= (J+3\delta)\log U(\frac{\lambda_{n(\nu)}}{\log^+ |a_{n(\nu)}|}), \qquad (21)
\end{aligned}$$

where K_1 is a constant.

From (18), (21) and $\delta = \frac{T-J}{5}$, we can get a contradiction. Thus, we can get that

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log M(\sigma,f))}{\log U(-\frac{1}{\sigma})} = \limsup_{n\to+\infty}\frac{\beta(\log^+ |a_n|)}{\log U(\frac{\lambda_n}{\log^+ |a_n|})} = T.$$

Hence, the sufficiency of Theorem 1.3 is completed.

We can prove the necessity of Theorem 1.3 by using the similar argument as in the proof of the sufficiency of Theorem 1.3.

Thus, the proof of Theorem 1.3 is completed.

4 The proof of Theorem 1.6

We prove the conclusions of Theorem 1.6 by using the properties of two functions $\beta(x)$ and $U(x)$, this method is different from the previous method in [18] to some extent.

We first prove " $\Longleftarrow$ " of Theorem 1.6. Suppose that

$$\limsup_{n\to\infty}\Psi_n(f,\alpha,\lambda_n) = \limsup_{n\to\infty}\frac{\beta(\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}])}{\log U\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)} = T. \qquad (22)$$

Let

$$A_n = E_{n-1}(f,\alpha)e^{-\alpha\lambda_n},\quad n=1,2,\ldots,$$

then for any positive real number $\tau>0$, for sufficiently large n, we have

$$\log^+ A_n < \gamma\left((T+\tau)\log U\left(\frac{\lambda_n}{\log^+ A_n}\right)\right),$$

where $\gamma(x)$ is the inverse functions of $\beta(x)$. Let $V(x)$ and $U(x)$ be two reciprocally inverse functions, then we have

$$V\left(\exp\left\{\frac{1}{T+\tau}\beta(\log^+ A_n)\right\}\right) < \frac{\lambda_n}{\log^+ A_n},$$

$$\log^+ A_n \leq \lambda_n \left(V\left(\exp\left\{\frac{1}{T+\tau}\beta(\log^+ A_n)\right\}\right)\right)^{-1}.$$

Thus, we have

$$\log^+(A_n e^{\lambda_n \sigma}) \leq \lambda_n \left(\left(V\left(\exp\left\{\frac{1}{T+\tau}\beta(\log^+ A_n)\right\}\right)\right)^{-1} + \sigma\right). \tag{23}$$

For any fixed and sufficiently small $\sigma < 0$, set

$$G = \gamma\left((T+\tau)\log U\left(-\frac{1}{\sigma} - \frac{1}{\sigma \log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma} + \frac{1}{-\sigma \log^2 U\left(-\frac{1}{\sigma}\right)} = V\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right). \tag{24}$$

If $\log^+ A_n \leq G$, for sufficiently large n, let $V\left(\exp\left\{\frac{1}{T+\tau}\beta(\log^+ A_n)\right\}\right) \geq 1$, from $\sigma < 0$,(23),(24) and the definition of $U(x)$, we have

$$\begin{aligned}\log^+ A_n e^{\lambda_n \sigma} &\leq \lambda_n \left(\left(V\left(\exp\left\{\frac{1}{T+\tau}\beta(\log^+ A_n)\right\}\right)\right)^{-1} + \sigma\right) \\ &\leq G = \gamma\left((T+\tau)\log U\left(-\frac{1}{\sigma} - \frac{1}{\sigma \log^2 U\left(-\frac{1}{\sigma}\right)}\right)\right) \\ &\leq \gamma\left((T+\tau)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right).\end{aligned} \tag{25}$$

If $\log^+ A_n > G$, from (23) and (24), we have

$$\begin{aligned}\log^+ A_n e^{\lambda_n \sigma} &\leq \lambda_n \left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right)\right)^{-1} + \sigma\right) \\ &\leq \lambda_n \left(\left(\frac{1}{-\sigma} + \frac{1}{-\sigma \log^2 U\left(-\frac{1}{\sigma}\right)}\right)^{-1} + \sigma\right) < 0.\end{aligned} \tag{26}$$

For sufficiently large n, from (25) and (26), we have

$$\log^+ A_n e^{\lambda_n \sigma} \leq \gamma\left((T+\tau)\log\left[(1+o(1))U\left(-\frac{1}{\sigma}\right)\right]\right).$$

From the definition of $E_n(f,\alpha)$, there exists $p(s) \in \Pi_{n-1}$ and a real constant $K_2(>1)$ such that

$$||f-p||_\alpha \leq K_2 E_{n-1}(f,\alpha). \tag{27}$$

Since $f(s) \in \overline{D}_\alpha$ and from [20, P.16], for any real numbers $t_0, \vartheta(\neq 0)$, we have

$$\lim_{R\to+\infty} \frac{1}{R}\int_{t_0}^{R} e^{\vartheta i t} dt = 0, \quad a_n e^{\alpha\lambda_n} = \lim_{R\to\infty} \frac{1}{R}\int_{t_0}^{R} f(\alpha+it)e^{-\lambda_n i t} dt. \tag{28}$$

From (28), for any real number $x \neq 0$, we have

$$\lim_{R\to\infty} \frac{1}{R}\int_{t_0}^{R} e^{x(\alpha+it)}dt = 0. \tag{29}$$

Thus, from (28) and (29), for any $p_1(s) \in \Pi_{n-1}$, we have

$$a_n e^{\alpha\lambda_n} = \lim_{R\to\infty} \frac{1}{R}\int_{t_0}^{R} [f(\alpha+it) - p_1(\alpha+it)]e^{-\lambda_n it}dt,$$

that is,

$$|a_n|e^{\alpha\lambda_n} \leq ||f - p_1||_\alpha. \tag{30}$$

From (27) and (30), we can get

$$|a_n|e^{\alpha\lambda_n} \leq K_2 E_{n-1}(f,\alpha), \tag{31}$$

where $K_2 > 1$ is a real constant.

Since $A_n = E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}$ and τ is arbitrary, from (31), by Lemma 2.1 and Theorem 1.2, we can get

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma,f))}{\log U(-\frac{1}{\sigma})} \leq T.$$

Suppose that

$$\limsup_{\sigma\to 0^+} \frac{\beta(\log^+ M(\sigma,f))}{\log U(-\frac{1}{\sigma})} = \eta < T.$$

Thus, there exists any real number $\varepsilon(0 < \varepsilon < \frac{T-\eta}{4})$, for any sufficient small $\sigma < 0$, from Lemma 2.2, we have

$$\log M(\sigma,f) \leq \gamma\left((\eta+\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{32}$$

For any sufficiently small $\sigma < 0$ and $-\infty < \alpha < \sigma < 0$, we have

$$E_{n-1}(f,\alpha) \leq ||f - p_{n-1}||_\alpha \leq \sum_{k=n}^{\infty} |a_k|e^{\lambda_k\alpha} \leq M(\sigma,f)\sum_{k=n}^{\infty} e^{\lambda_n(\alpha-\sigma)}, \tag{33}$$

where $p_{n-1}(s) = \sum_{k=1}^{n-1} a_k e^{\lambda_k s}$. From (3), we take $0 < h' < h$ satisfying $\lambda_{n+1} - \lambda_n \geq h'$ for a sub-sequence of $\{n\}$. Thus, for sufficiently small $\sigma < 0$ such that $\sigma \geq \frac{\alpha}{2}$, from (33) we have

$$\begin{aligned}
E_{n-1}(f,\alpha) &\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\sum_{k=n}^{\infty} e^{(\lambda_k-\lambda_n)(\alpha-\sigma)} \\
&\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}e^{-\frac{\alpha}{2}h'n}\sum_{k=n}^{\infty} e^{\frac{\alpha}{2}h'k} \\
&= M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\left(1 - e^{\frac{\alpha}{2}h'}\right)^{-1}.
\end{aligned}$$

Then for sufficiently small $\sigma < 0$ and $-\infty < \alpha < \sigma < 0$, we have

$$M(\sigma,f) \geq K_3 E_{n-1}(f,\alpha)e^{-\lambda_n(\alpha-\sigma)} = K_3 A_n e^{\lambda_n\sigma}, \tag{34}$$

where $K_3 = 1 - e^{\frac{\alpha}{2}h'}$. From (32) and (34), noting the properties of the function $\gamma(x)$, we have

$$\log^+ A_n e^{\lambda_n \sigma} \leq \log M(\sigma, f) \leq \gamma\left((\eta + 2\varepsilon)\log U(-\frac{1}{\sigma})\right). \tag{35}$$

From (22), there exists a subsequence $\{\lambda_{n(p)}\}$, for sufficiently large p, we have

$$\beta(\log^+ A_{n(p)}) > (T - \varepsilon)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{36}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\gamma\left((\eta + 2\varepsilon)\log U(-\frac{1}{\sigma_p})\right) = \frac{\log^+ A_{n(p)}}{1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}. \tag{37}$$

From (35) and (37), we get

$$\begin{aligned}\log^+ A_{n(p)} + \lambda_{n(p)}\sigma_p &\leq \gamma\left((\eta + 2\varepsilon)\log U(-\frac{1}{\sigma_p})\right)\\ &= \frac{\log^+ A_{n(p)}}{1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})},\end{aligned}$$

that is,

$$-\frac{1}{\sigma_p} \leq \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right).$$

Thus, we have

$$\begin{aligned}&U(-\frac{1}{\sigma_p})\\ \leq &U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\ \leq &U^{1+o(1)}\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right).\end{aligned} \tag{38}$$

From (37) and (38), we have

$$\begin{aligned}&\log^+ A_{n(p)}\\ =&\gamma\left((\eta + 2\varepsilon)\log U(\frac{1}{\sigma_p})\right)\left(1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem and (38), there exists a real number ξ between $\gamma\left((\eta + 2\varepsilon)\log U(\frac{1}{\sigma_p})\right)$ and $\gamma\left((\eta + 2\varepsilon)\log U(\frac{1}{\sigma_p})\right)(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$

$\log^2 \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}))$ such that

$$\begin{aligned}
&\beta\left(\log^+ A_{n(p)}\right)\\
=&\beta\left(\gamma\left((\eta+2\varepsilon)\log U(\frac{1}{\sigma_p})\right)(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}))\right)\\
=&\beta\left(\gamma\left((\eta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\
&+\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\xi\beta'(\xi),
\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\log^2\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)}{\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}=0,$$

then for sufficiently large p, we have

$$\begin{aligned}
\beta\left(\log^+ A_{n(p)}\right)=&(\eta+2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\\
&+K_3\xi\beta'(\xi)\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}),
\end{aligned}\tag{39}$$

where K_3 is a constant.

From (36),(39) and $0<\varepsilon<\frac{T-\eta}{4}$, we can get a contradiction. Thus, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U(-\frac{1}{\sigma})}=T.$$

Hence, the sufficiency of Theorem 1.6 is completed.

We can prove the necessity of Theorem 1.6 by using the similar argument as in the proof of the sufficiency of Theorem 1.6.

Thus, the proof of Theorem 1.6 is completed.

References

[1] P.V. Filevich, M.N. Sheremeta, Regularly increasing entire Dirichlet series, Mathematical Notes 74 (2003), 110-122; Translated from Matematicheskie Zametki 74 (2003), 118-131.

[2] Z. S. Gao , The growth of entire functions represented by Dirichlet series, Acta Mathematica Sinica 42 (1999), 741-748(in Chinese).

[3] Z. D. Gu, D. C. Sun, The reuglar growth of Dirichlet series on the whole plane, Acta Mathematica Scientia 31 (2011), 991-997(in Chinese).

[4] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. 22 (4) (1998), 385-405.

[5] Y.Y. Kong, On some q-order and q-types of Dirichlet-Hadamard function, Acta Mathematica Sinica, Chinese Series. 52(6) (2009), 1165-1172.

[6] M.S. Liu, The regular growth of Dirichlet series of finite order in the half plane, J. Sys. Sci. & Math. Scis. 22(2) (2002), 229-238.

[7] A. Mishkelyavichyus, A Tauberian theorem for the Laplace-Stieltjes integral and the Dirichlet series (Russian) Litovsk. Mast. Sb. 29(4) (1989), 745-753.

[8] A. Nautiyal, On the coefficients of analytic Dirichlet series of fast growth, Indian J. Pure Appl. Math. 15(10),1984, 1102-1114.

[9] J.H. Ning, C.F. Yi, W.P. Huang, Regular growth of the generalized Dirichlet series, Acta Mathematica Scientia 31 (2012), 379-386(in Chinese).

[10] L.N. Shang, Z.S. Gao, Entire functions defined by Dirichlet series, J. Math. Anal. Appl. 339, (2008), 853-862.

[11] D. C. Sun, The existence theorem of Nevanlinna direction, Chin. Ann. of Math. 7A (1986), 212-221 (in Chinese).

[12] D.C. Sun, On the distribution of values of random Dirichlet series II, Chinese Ann. Math. Ser. B 11(1) (1990), 33-44.

[13] D.C. Sun, The growth of Dirichlet series, J.Analysis, 3, 1995, 73-86.

[14] D.C. Sun, T.W. Chen, Random Dirichlet series of infinite order, Acta Mathematica Sinica 44 (2001), 259-268(in Chinese).

[15] D.C. Sun, Z.S. Gao, The growth of Dirichlet series in the half plane, Acta Mathematica Scientia. 22A(4) (2002), 557-563.

[16] G. Valiron, Entire function and Borel's directions, Proc. Nat. Acad. Sci. USA. 20 (1934), 211-215.

[17] G. Valiron, Fonction entireres d'order fini et fonction méromorphes, Genere, L'Enseigenment Mathematique 1960.

[18] H.Y. Xu, C.F. Yi, The approximation problem of Dirichlet series of finite order in the half plane, Acta Mathematica Sinica 53 (3) (2010), 617-624(in Chinese).

[19] H.Y. Xu, C.F. Yi, The growth and approoximation of Dirichlet series of infinite order, Advances in Mathematics, in press(in Chinese).

[20] J.R. Yu, X.Q. Ding, F.J. Tian, On The Distribution of Values of Dirichlet Series And Random Dirichlet Series, Wuhan: Press in Wuhan University, 2004(in Chinese).

[21] J.R. Yu, Dirichlet series and the random Dirichlet series, Beijing, Science Press, 1997.

[22] J.R. Yu, Some properties of random Dirichlet series (in Chinese), Acta Math. Sinica 21 (1978), 97-118.

[23] J.R. Yu, Sur les droites de Borel de certaines fonction entières, Annales École Norm. Sup. 68(3) (1951), 65-104.

[24] J.R. Yu, D.C. Sun, On the distribution of values of random Dirichlet series(I), Lectures on Comp. Anal. Singapore, World Scientific, 1988.

WEAK AND STRONG CONVERGENCE THEOREMS OF PROXIMAL POINT ALGORITHM FOR SOLVING GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FINDING ZEROES OF MAXIMAL MONOTONE OPERATORS IN BANACH SPACES

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Abstract. Based on the results proposed by Li and Song [Modified proximal-point algorithm for maximal monotone operators in Banach spaces], J. Optim. Theory appl. 138 (2008) 45-64.], we modify and generate our new iterative scheme for solving generalized mixed equilibrium problems and finding zeroes of maximal monotone operators in a Banach space under the appropriate conditions. We also prove strong and weak convergence theorems of this proximal point algorithm and give an example with numerical test which corresponding to our main results. Furthermore, we also consider the convex minimization problem and the problem of finding a zero point of an α-inverse strongly monotone operator as its appplications.

1. Introduction

Let E be a Banach space, C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of S (see [22]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F(S)}$. A mapping S from C into itself is said to be *relatively nonexpansive* [21, 26, 37] if $\widehat{F(S)} = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [6, 7]. A mapping S is said to be ϕ-nonexpansive, if $\phi(Sx, Sy) \le \phi(x, y)$ for $x, y \in C$. A mapping S is said to be *quasi ϕ-nonexpansive* if $F(S) \neq \emptyset$ and $\phi(p, Sx) \le \phi(p, x)$ for $x \in C$ and $p \in F(S)$.

Let E be a Banach space with norm $\|\cdot\|$, C be a nonempty closed convex subset of E and let E^* be the dual of E. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction, $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function, and $B : C \longrightarrow E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem*, which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.1}$$

The solutions set to (1.1) is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C\}. \tag{1.2}$$

If $B = 0$, the problem (1.1) reduce into the *mixed equilibrium problem for* Θ, denoted by $MEP(\Theta, \varphi)$, which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.3}$$

If $\Theta \equiv 0$, the problem (1.1) reduce into the *mixed variational inequality* of Browder type, denoted by $VI(C, B, \varphi)$, is to find $x \in C$ such that

$$\langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.4}$$

If $B = 0$ and $\varphi = 0$ the problem (1.1) reduce into the *equilibrium problem for* Θ, denoted by $EP(\Theta)$, is to find $x \in C$ such that

$$\Theta(x, y) \ge 0, \quad \forall y \in C. \tag{1.5}$$

The above formulation (1.5) was shown in [5] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational

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inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\Theta)$. In other words, the $EP(\Theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\Theta)$; see, for example [5, 11, 19, 29] and references therein. Some solution methods have been proposed to solve the $EP(\Theta)$; see, for example, [8, 11, 24, 25, 27, 28, 29] and references therein. In 2005, Combettes and Hirstoaga [8] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Theta)$ is nonempty and they also proved a strong convergence theorem.

Let E be a Banach space with norm $\|\cdot\|$, C be a nonempty closed convex subset of E and let E^* denote the dual of E. Let A be a *monotone* operator of C into E^*. The *variational inequality problem* is to find a point $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0 \text{ for all } y \in C. \tag{1.6}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on. An operator A of C into E^* is said to be *inverse-strongly monotone*, if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \tag{1.7}$$

for all $x, y \in C$. In such a case, A is said to be *α-inverse-strongly monotone*. If an operator A of C into E^* is α-inverse-strongly monotone, then A is *Lipschitz continuous*, that is $\|Ax-Ay\| \leq \frac{1}{\alpha}\|x-y\|$ for all $x, y \in C$.

In Hilbert space H, Iiduka et al. [13] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Ax_n), \tag{1.8}$$

where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive real numbers, converges weakly to some element of $VI(C,A)$.

In 2008, Iiduka and Takahashi [12] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A that satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space E:

(C1) A is inverse-strongly monotone,
(C2) $VI(C,A) \neq \emptyset$,
(C3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C,A)$.

Let $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n) \tag{1.9}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to some element of $VI(C,A)$.

Consider the problem of finding:

$$v \in E \text{ such that } 0 \in T(v), \tag{1.10}$$

where T is an operator from E into E^*. Such $v \in E$ is called a *zero point* of T. When T is a maximal monotone operator, a well-know methods for solving (1.10) in a Hilbert space H is the *proximal point algorithm*: $x_1 = x \in H$ and,

$$x_{n+1} = J_{r_n}x_n,\ n = 1, 2, 3, \ldots, \tag{1.11}$$

where $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_nT)^{-1}J$, then Rockafellar [23] proved that the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}(0)$.

In 2000, Kamimura and Takahashi [16] proved the following strong convergence theorem in Hilbert spaces, by the following algorithm

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{r_n}x_n,\ n = 1, 2, 3, \ldots, \tag{1.12}$$

where $J_r = (I + rT)^{-1}J$, then the sequence $\{x_n\}$ converges strongly to $P_{T^{-1}(0)}(x)$, where $P_{T^{-1}(0)}$ is the projection from H onto $T^{-1}(0)$. These results were extended to more general Banach spaces see [15, 17].

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In 2003, Kohsaka and Takahashi [17] introduced the following iterative sequence for a maximal monotone operator T in a smooth and uniformly convex Banach space: $x_1 = x \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n)J(J_{r_n}x_n)),\ n = 1, 2, 3, \dots, \tag{1.13}$$

where J is the duality mapping from E into E^* and $J_r = (I + rT)^{-1}J$.

In 2004, Kamimura et al. [14] considered the algorithm (1.14) in a uniformly smooth and uniformly convex Banach space E, namely

$$x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_{r_n}x_n)),\ n = 1, 2, 3, \dots. \tag{1.14}$$

They proved that the algorithm (1.14) converges weakly to some element of $T^{-1}0$.

In 2008, Li and Song [18] proved a strong convergence theorem in a Banach space, by the following algorithm: $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}x_n)),\\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1-\alpha_n)J(y_n)), \end{aligned} \tag{1.15}$$

with the coefficient sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\longrightarrow\infty}\beta_n = 0$, and $\lim_{n\longrightarrow\infty} r_n = \infty$, where J is the duality mapping from E into E^* and $J_r = (I+rT)^{-1}J$. Then they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_C x$, where Π_C is the generalized projection from E onto C.

In this paper, motivated and inspired by Kamimura et al. [14], Li and Song [18], Iiduka and Takahashi [12] and Zhang [38], we introduce the new hybrid algorithm defined by: $x_1 = x \in C$

$$\begin{cases} u_n \in C \text{ such that}\Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)\\ \qquad\quad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0,\ \forall y \in C,\\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n),\\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}z_n)),\\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x) + (1-\alpha_n)J(y_n)). \end{cases} \tag{1.16}$$

Under appropriate difference conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (1.16) converges strongly to the point $\Pi_{\Omega\cap VI(C,A)\cap T^{-1}(0)}x$ and converges weakly to the point $\lim_{n\longrightarrow\infty} \Pi_{\Omega\cap VI(C,A)\cap T^{-1}(0)}x_n$. The results presented in this paper extend and improve the corresponding ones announced by Kamimura et al. [14], Li and Song [18] and some authors in the literature. Furthermore, in the last section, we will state example for that satisfies the condition (C1)-(C3) and also we consider the minimization problem and the complementarity problem.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{2.1}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [3, 31] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is *2*-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{2.2}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [30]):

(1) if E is smooth, then J is single-valued;
(2) if E is strictly convex, then J is one-to-one and $\langle x-y, x^*-y^*\rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$;
(3) if E is reflexive, then J is surjective;
(4) if E is uniformly convex, then it is reflexive;
(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [10] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.1. ([4, 35]). *If E be a 2-uniformly convex Banach space. Then, for all $x,y \in E$ we have*

$$\|x-y\| \leq \frac{2}{c^2}\|Jx-Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\dfrac{1}{c}$ in Lemma is called the *2*-uniformly convex constant of E; see [3].

Lemma 2.2. ([4, 36]). *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x,y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x-y, Jx-Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x-y\|^p,$$

where J_p is the generalized duality mapping of E and $\dfrac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.3. (*Xu* [35]). *Let E be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \longrightarrow [0,\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda y)\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.3}$$

for all $x,y \in \{z \in E : \|z\| \leq r\}$ and $\lambda \in [0,1]$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x,y \in E. \tag{2.4}$$

Following Alber [1], the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y,x) \tag{2.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function ϕ that (see [1])

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x,y \in E. \tag{2.6}$$

If E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$.

If E is a reflexive, strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.6), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [9, 30] for more details.

Lemma 2.4. (*Kamimura and Takahashi* [15]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\},\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5. (*Alber* [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.6. (*Alber* [1]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Let E be a strictly convex, smooth and reflexive Banach space, let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V : E \times E^* \longrightarrow \mathbb{R}$ as follows (see [17]):

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.7}$$

for all $x \in Ex \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

Lemma 2.7. (*Kohsaka and Takahashi* [17, Lemma 3.2]). *Let E be a strictly convex, smooth and reflexive Banach space, and let V be as in (2.7). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \tag{2.8}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E. Because $\phi(x, y)$ is strictly convex and coercive in the first variable, we know that the minimization problem $\inf_{y \in C} \phi(x, y)$ has a unique solution. The operator $\Pi_C x := \arg\min_{y \in C} \phi(x, y)$ is said to be the generalized projection of x on C.

A set-valued mapping $T : E \longrightarrow E^*$ with domain $D(T) = \{x \in E : T(x) \neq \emptyset\}$ and range $R(T) = \{x^* \in E^* : x^* \in T(x), x \in D(T)\}$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $x^* \in T(x), y^* \in T(y)$. We denote the set $\{s \in E : 0 \in Tx\}$ by $T^{-1}(0)$. T is *maximal monotone* if its graph $G(T)$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}(0)$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is knows that T is a maximal monotone if and only if $R(J + rT) = E^*$ for all $r > 0$.

Define the *resolvent* of T by $J_r x = x_r$. In other words, $J_r = (J + rT)^{-1} J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(T)$. Also, $T^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of T by $T_r = (J - JJ_r)/r$. We know that $T_r x \in T(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 2.8. (*Kohsaka and Takahashi* [17, Lemma 3.1]). *Let E be a smooth, strictly convex and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $T^{-1}(0) \neq \emptyset$, let $r > 0$ and let $J_r = (J + rT)^{-1} J$. Then*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y)$$

for all $x \in T^{-1}0$ and $y \in E$.

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.9}$$

Theorem 2.9. (*Rockafellar* [23]). *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), \ v \in C; \\ \emptyset, \ \textit{otherwise}. \end{cases} \tag{2.10}$$

Then T is maximal monotone and $T^{-1}(0) = VI(C, A)$.

Lemma 2.10. (*Tan and Xu* [33]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} = a_n + b_n \textit{ for all } n \geq 0.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \longrightarrow \infty} a_n$ exists.

Lemma 2.11. (*Xu* [34]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n t_n + r_n \ n \geq 1,$$

where $\{\alpha_n\}$, $\{t_n\}$, and $\{r_n\}$ satisfy $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} t_n \leq 0$ and $r_n \geq 0$, $\sum_{n=1}^{\infty} r_n < \infty$. Then $\lim_{n\longrightarrow\infty} s_n = 0$.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous satisfies the following conditions:

(A1) $\Theta(x,x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x,y);$$

(A4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is convex and lower semi-continuous.

Lemma 2.12. (*Blum and Oettli* [5]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.13. (*Takahashi and Zembayashi* [32]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \longrightarrow C$ as follows:*

$$T_r x = \{z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\} \tag{2.11}$$

for all $x \in E$. Then, the followings hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(\Theta)$;*
(4) *$EP(\Theta)$ is closed and convex.*

Lemma 2.14. (*Takahashi and Zembayashi* [32]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.15. (*Zhang* [38]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \longrightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u,y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C.$$

Define a mapping $K_r : C \longrightarrow C$ as follows:

$$K_r(x) = \{u \in C : \Theta(u,y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C\} \tag{2.12}$$

for all $x \in E$. Then, the followings hold:

(i) *K_r is single-valued;*
(ii) *K_r is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*
(iii) *$F(K_r) = \Omega$;*
(iv) *Ω is closed and convex;*
(v) *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ $\forall p \in F(K_r)$, $z \in E$.*

Remark 2.16. (*Zhang* [38]). It follows from Lemma 2.13 that the mapping $K_r : C \longrightarrow C$ defined by (2.12) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

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3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of mixed equilibrium problems, the set of solution of the variational inequality problem and the zero point of a maximal monotone operators in a Banach space by using the shrinking hybrid projection method.

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, with $F := \Omega \cap VI(C, A) \cap T^{-1}(0) \neq \emptyset$. Assume that A an operator of C into E^* that satisfies the conditions (C1)-(C3). Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x) + (1 - \alpha_n)J(y_n)), \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$.

***Proof*.** Let $H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)$, $y \in C$ and $K_{r_n} = \{u \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C\}$. We first show that $\{x_n\}$ is bounded. Put $v_n = J^{-1}(Ju_n - \lambda_n Au_n)$, let $p \in F := \Omega \cap VI(C, A) \cap T^{-1}(0)$ and $u_n = K_{r_n} x_n$. Since J_{r_n} are relatively nonexpansive mappings. By (3.1)

$$\phi(p, u_n) = \phi(p, K_{r_n} x_n) \leq \phi(p, x_n) \tag{3.2}$$

and Lemma 2.7, the convexity of the function V in the second variable, we have

$$\begin{aligned} \phi(p, z_n) &= \phi(p, \Pi_C v_n) \\ &\leq \phi(p, v_n) = \phi(p, J^{-1}(Ju_n - \lambda_n Au_n)) \\ &\leq V(p, Ju_n - \lambda_n Au_n + \lambda_n Au_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - p, \lambda_n Au_n\rangle \\ &= V(p, Ju_n) - 2\lambda_n\langle v_n - p, Au_n\rangle \\ &= \phi(p, u_n) - 2\lambda_n\langle u_n - p, Au_n\rangle + 2\langle v_n - u_n, -\lambda_n Au_n\rangle. \end{aligned} \tag{3.3}$$

Since $p \in VI(C, A)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned} -2\lambda_n\langle u_n - p, Au_n\rangle &= -2\lambda_n\langle u_n - p, Au_n - Ap\rangle - 2\lambda_n\langle u_n - p, Ap\rangle \\ &\leq -2\alpha\lambda_n\|Au_n - Ap\|^2, \end{aligned} \tag{3.4}$$

and by Lemma 2.1, we obtain

$$\begin{aligned} 2\langle v_n - u_n, -\lambda_n Au_n\rangle &= 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u_n, -\lambda_n Au_n\rangle \\ &\leq 2\|J^{-1}(Ju_n - \lambda_n Au_n) - u_n\|\|\lambda_n Au_n\| \\ &\leq \frac{4}{c^2}\|Ju_n - \lambda_n Au_n - Ju_n\|\|\lambda_n Au_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Au_n\|^2 \\ &\leq \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we get

$$\phi(p, z_n) \leq \phi(p, u_n) - 2\alpha\lambda_n\|Au_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2$$

$$\begin{aligned} &\leq \phi(p,u_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Au_n - Ap\|^2 \\ &\leq \phi(p,u_n) \\ &\leq \phi(p,x_n). \end{aligned} \tag{3.6}$$

By Lemma 2.7, Lemma 2.8 and (3.6), we have

$$\begin{aligned} \phi(p,y_n) &= \phi(p, J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}z_n))) \\ &= V(p, \beta_n J(x_n) + (1-\beta_n)J(J_{r_n}z_n)) \\ &\leq \beta_n V(p, J(x_n)) + (1-\beta_n)V(p, J(J_{r_n}z_n)) \\ &= \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p, J_{r_n}z_n) \\ &\leq \beta_n\phi(p,x_n) + (1-\beta_n)(\phi(p,z_n) - \phi(J_{r_n}z_n, z_n)) \\ &\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,z_n) \\ &\leq \beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,x_n) \\ &= \phi(p,x_n), \end{aligned} \tag{3.7}$$

it follows that

$$\begin{aligned} \phi(p,x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n))) \\ &\leq \phi(p, J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n))) \\ &= V(p, \alpha_n J(x_1) + (1-\alpha_n)J(y_n)) \\ &\leq \alpha_n V(p, J(x_1)) + (1-\alpha_n)V(p, J(y_n)) \\ &= \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,y_n) \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. Hence, by induction, we have that $\phi(p,x_n) \leq \phi(p,x_1)$ for all $n \in \mathbb{N}$. Since $(\|x_n\| - \|p\|)^2 \leq \phi(p,x_n)$. It implies that $\{x_n\}$ is bounded and $\{y_n\}$, $\{z_n\}$, $\{J_{r_n}z_n\}$ are also bounded.
From (3.6), (3.7) and (3.8), we have

$$\begin{aligned} \phi(p,x_{n+1}) &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)(\beta_n\phi(p,x_n) + (1-\beta_n)(\phi(p,x_n) - \phi(J_{r_n}z_n, z_n))) \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - (1-\alpha_n)(1-\beta_n)\phi(J_{r_n}z_n, z_n) \end{aligned}$$

and then

$$(1-\alpha_n)(1-\beta_n)\phi(J_{r_n}z_n, z_n) \leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - \phi(p,x_{n+1})$$

for all $n \in \mathbb{N}$. Since $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\limsup_{n\longrightarrow\infty}\beta_n < 1$, it follows that $\lim_{n\longrightarrow\infty}\phi(J_{r_n}z_n, z_n) = 0$. Applying Lemma 2.4, we have

$$\lim_{n\longrightarrow\infty}\|J_{r_n}z_n - z_n\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty}\|JJ_{r_n}z_n - Jz_n\| = 0. \tag{3.10}$$

By (3.2), (3.6), (3.7) and (3.8) again, we note that

$$\begin{aligned} \phi(p,x_{n+1}) &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)[\beta_n\phi(p,x_n) + (1-\beta_n)[(\phi(p,x_n) - 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Au_n - Ap\|^2]] \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Au_n - Ap\|^2 \end{aligned}$$

and hence

$$2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)\|Au_n - Ap\|^2 \leq \frac{1}{(1-\alpha_n)(1-\beta_n)}(\alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - \phi(p,x_{n+1}))$$

for all $n \in \mathbb{N}$. Since $0 < a \leq \lambda_n \leq b < \dfrac{c^2\alpha}{2}$, $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\limsup_{n\longrightarrow\infty}\beta_n < 1$, we have

$$\lim_{n\longrightarrow\infty}\|Au_n - Ap\| = 0. \tag{3.11}$$

From Lemma 2.6, Lemma 2.7 and (3.5), we get

$$\begin{aligned}
\phi(u_n, z_n) = \phi(u_n, \Pi_C v_n) &\leq \phi(u_n, v_n) \\
&= \phi(u_n, J^{-1}(Ju_n - \lambda_n Au_n)) \\
&= V(u_n, Ju_n - \lambda_n Au_n) \\
&\leq V(u_n, (Ju_n - \lambda_n Au_n) + \lambda_n Au_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u_n, \lambda_n Au_n\rangle \\
&= \phi(u_n, u_n) + 2\langle v_n - u_n, -\lambda_n Au_n\rangle \\
&= 2\langle v_n - u_n, -\lambda_n Au_n\rangle \\
&\leq \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2.
\end{aligned}$$

From Lemma 2.4 and (3.11), we have

$$\lim_{n\longrightarrow\infty} \|u_n - z_n\| = 0. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\longrightarrow\infty} \|Ju_n - Jz_n\| = 0. \tag{3.13}$$

From Lemma 2.6, Lemma 2.7 and (3.5), we obtain

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, \Pi_C J^{-1}(Ju_n - \lambda_n Au_n)) \\
&\leq \phi(x_n, J^{-1}(Ju_n - \lambda_n Au_n)) \\
&= \phi(x_n, Ju_n - \lambda_n Au_n) \\
&\leq V(x_n, (Ju_n - \lambda_n Au_n) + \lambda_n Au_n) - 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u_n, \lambda_n Au_n\rangle \\
&= \phi(x_n, x_n) + 2\langle J^{-1}(Ju_n - \lambda_n Au_n) - u_n, -\lambda_n Au_n\rangle \\
&= \frac{4}{c^2}\lambda_n^2\|Au_n - Ap\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\longrightarrow\infty} \|Au_n - Ap\|^2 = 0$, we have $\lim_{n\longrightarrow\infty} \phi(x_n, z_n) = 0$. Applying Lemma 2.4, we get

$$\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\longrightarrow\infty} \|Jx_n - Jz_n\| = 0. \tag{3.15}$$

So, by the triangle inequality, we get

$$\|x_n - u_n\| \leq \|x_n - z_n\| + \|z_n - u_n\|.$$

By (3.12) and (3.14), we also have

$$\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0. \tag{3.16}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in C$. It follows from (3.9) and (3.14), we have $z_{n_i} \rightharpoonup u$ as $i \longrightarrow \infty$.

Next, we show that $u \in T^{-1}(0)$. Indeed, since $\liminf_{n\longrightarrow\infty} r_n > 0$, it follows from (3.10) that

$$\lim_{n\longrightarrow\infty} \|A_{r_n} z_n\| = \lim_{n\longrightarrow\infty} \frac{1}{r_n}\|Jz_n - J(J_{r_n} z_n)\| = 0. \tag{3.17}$$

If $(z, z^*) \in T$, then it holds from the monotonicity of A that

$$\langle z - z_{n_i}, z^* - A_{r_{n_i}} z_{n_i}\rangle \geq 0$$

for all $i \in \mathbb{N}$. Letting $i \longrightarrow \infty$, we get $\langle z - u, z^*\rangle \geq 0$. Then, the maximality of T implies $u \in T^{-1}(0)$.

Next, we show that $u \in VI(C, A)$. Let $L \subset E \times E^*$ be an operator as follows:

$$Lv = \begin{cases} Av + N_C(v), \ v \in C; \\ \emptyset, \ otherwise. \end{cases}$$

By Theorem 2.9, L is maximal monotone and $L^{-1}(0) = VI(C, A)$. Let $(v, w) \in G(L)$. Since $w \in Lv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $z_n \in C$, we have

$$\langle v - z_n, w - Av\rangle \geq 0. \tag{3.18}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n)$. Then by Lemma 2.5, we have

$$\langle v - z_n, Jz_n - (Ju_n - \lambda_n Au_n)\rangle \geq 0,$$

thus

$$\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n} - Aux_n\rangle \leq 0. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\begin{aligned}
\langle v - z_n, w\rangle &\geq \langle v - z_n, Av\rangle \\
&\geq \langle v - z_n, Av\rangle + \langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n} - Au_n\rangle \\
&= \langle v - z_n, Av - Au_n\rangle + \langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n}\rangle \\
&= \langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Au_n\rangle + \langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n}\rangle \\
&\geq -\|v - z_n\|\frac{\|z_n - u_n\|}{\alpha} - \|v - z_n\|\frac{\|Ju_n - Jz_n\|}{a} \\
&\geq -M(\frac{\|z_n - u_n\|}{\alpha} + \frac{\|Ju_n - Jz_n\|}{a}),
\end{aligned}$$

where $M = \sup_{n\geq 1}\{\|v - z_n\|\}$. From (3.12) and (3.13), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of L, we have $u \in L^{-1}(0)$ and hence $u \in VI(C, A)$.

Next, we show that $u \in \Omega$. From (3.16) and J is uniformly norm-to-norm continuous on bounded set, we obtain

$$\lim_{n\longrightarrow\infty} \|Ju_n - Jx_n\| = 0. \tag{3.20}$$

From the assumption $\liminf_{n\longrightarrow\infty} r_n > a$, we get

$$\lim_{n\to\infty} \frac{\|Ju_n - Jx_n\|}{r_n} = 0.$$

Noticing that $u_n = K_{r_n}x_n$, we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \quad \forall y \in C.$$

Hence,

$$H(u_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jx_{n_i}\rangle \geq 0, \quad \forall y \in C.$$

From the (A2), we note that

$$\|y - u_{n_i}\|\frac{\|Ju_{n_i} - Jx_{n_i}\|}{r_{n_i}} \geq \frac{1}{r_{n_i}}\langle y - u_{n_i}, Ju_{n_i} - Jx_{n_i}\rangle \geq -H(u_{n_i}, y) \geq H(y, u_{n_i}), \quad \forall y \in C.$$

Taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_{n_i} \rightharpoonup u$, we have $H(y, u) \leq 0, \quad \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1 - t)u$. Noticing that $y, u \in C$, we obtains $y_t \in C$, which yields that $H(y_t, u) \leq 0$. It follows from (A1) that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1 - t)H(y_t, \hat{x}) \leq tH(y_t, y).$$

That is, $H(y_t, y) \geq 0$.
Let $t \downarrow 0$, from (A3), we obtain $H(u, y) \geq 0$, $\forall y \in C$. This implies that $u \in \Omega$. Hence $u \in F := \Omega \cap VI(C, A) \cap T^{-1}(0)$.

Finally, we show that $u = \Pi_F x$. Indeed from $x_n = \Pi_{C_n}x$ and Lemma 2.5, we have

$$\langle Jx - Jx_n, x_n - z\rangle \geq 0, \ \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\langle Jx - Jx_n, x_n - p\rangle \geq 0, \ \forall p \in F. \tag{3.21}$$

Taking limit $n \longrightarrow \infty$, we obtain

$$\langle Jx - Ju, u - p\rangle \geq 0, \ \forall p \in F.$$

By again Lemma 2.5, we can conclude that $u = \Pi_F x$. This completes the proof. □

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Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let A be an α-inverse-strongly monotone operator of C into E^*, with $F := VI(C, A) \cap T^{-1}(0) \neq \emptyset$. Assume that A is an operator of C into E^* which satisfies the condition $(C1) - (C3)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n)), \end{cases} \tag{3.22}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$.

4. Weak Convergence Theorem

We next prove a weak convergence theorem under difference condition on data. First we prove the generalized projection sequence $\{\Pi_F x_n\}$ of $\{x_n\}$ is strongly convergent.

Theorem 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let A be an α-inverse-strongly monotone operator of C into E^* and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, with $F := \Omega \cap VI(C, A) \cap T^{-1}(0) \neq \emptyset$. Assume that A is an operator of C into E^* which satisfies the condition $(C1) - (C3)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n)), \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{\Pi_F x_n\}$ converges strongly to an element v of F, which is a unique element of F satisfying

$$\lim_{n\longrightarrow\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\longrightarrow\infty} \phi(y, x_n).$$

***Proof*.** Let $H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)$, $y \in C$ and $K_{r_n} = \{u \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0. \forall y \in C\}$. We first show that $\{x_n\}$ is bounded. Put $v_n = J^{-1}(Ju_n - \lambda_n Au_n)$, let $p \in F := \Omega \cap VI(C, A) \cap T^{-1}(0)$ and $u_n = K_{r_n} x_n$. Since J_{r_n} are relatively nonexpansive mappings. By (3.8), we have that, for all $n \in \mathbb{N}$

$$\phi(p, x_{n+1}) \leq \alpha_n \phi(p, x_1) + (1-\alpha_n)\phi(p, x_n). \tag{4.2}$$

From $\sum_{n=1}^{\infty} \alpha_n < \infty$ and Lemma 2.10, we deduce that $\lim_{n\longrightarrow\infty} \phi(p, x_n)$ exists. This implies that $\{\phi(p, x_n)\}$ is bounded. So $\{x_n\}$ is bounded. Define a function $g : F \longrightarrow [0, \infty)$ as follows:

$$g(p) = \lim_{n\longrightarrow\infty} \phi(p, x_n), \ \forall p \in F.$$

Then, by the same argument as in proof of [14, Theorem 3.1], we obtain g is a continuous convex function and if $\|z_n\| \longrightarrow \infty$ then $g(z_n) \longrightarrow \infty$. Hence, by [30, Theorem 1.3.11], there exists a point $v \in F$ such that

$$g(v) = \min_{y\in F} g(y) (:= l). \tag{4.3}$$

Put $w_n = \Pi_F x_n$ for all $n \geq 0$. We next prove that $w_n \longrightarrow v$ as $n \longrightarrow \infty$. Suppose on the contrary that there exists $\epsilon_0 > 0$ such that, for each $n \in \mathbb{N}$, there is $n' \geq n$ satisfying $\|w_{n'} - v\| \geq \epsilon_0$. Since $v \in F$, we have

$$\phi(w_n, x_n) = \phi(\Pi_F x_n, x_n) \leq \phi(v, \Pi_F x_n) + \phi(\Pi_F x_n, x_n) \leq \phi(v, x_n) \tag{4.4}$$

for all $n \geq 0$. This implies that

$$\limsup_{n\longrightarrow\infty} \phi(w_n, x_n) \leq \lim_{n\longrightarrow\infty} \phi(v, x_n) = l. \tag{4.5}$$

Since $(\|v\| - \|\Pi_F x_n\|)^2 \leq \phi(v, w_n) \leq \phi(v, x_n)$ for all $n \geq 0$ and $\{x_n\}$ is bounded, we get $\{w_n\}$ is also bounded. By Lemma 2.3, there exists a stricly increasing, continuous and convex function $K : [0,\infty) \longrightarrow [0,\infty)$ such that $K(0) = 0$ and

$$\|\frac{w_n + v}{2}\|^2 \leq \frac{1}{2}\|w_n\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}K(\|w_n - v\|), \tag{4.6}$$

for all $n \geq 0$. Now, choose σ satisfying $0 < \sigma < \frac{1}{4}K(\epsilon_0)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\phi(w_n, x_n) \leq l + \sigma,\ \phi(v, x_n) \leq l + \sigma, \tag{4.7}$$

for all $n \geq 0$. Thus there exists $k \geq n_0$ satisfying the following:

$$\phi(w_k, x_k) \leq l + \sigma,\ \phi(v, x_k) \leq l + \sigma,\ \|w_k - v\| \geq \epsilon_0. \tag{4.8}$$

From (4.2), (4.6) and (4.8), we obtain

$$\begin{aligned}
\phi(\frac{w_k + v}{2}, x_{n+k}) &\leq \phi(\frac{w_k + v}{2}, x_k) \\
&= \|\frac{w_k + v}{2}\|^2 - 2\langle\frac{w_k + v}{2}, Jx_k\rangle + \|x_k\|^2 \\
&\leq \frac{1}{2}\|w_k\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}K(\|w_k - v\|) - \langle w_k + v, Jx_k\rangle + \|x_k\|^2 \\
&= \frac{1}{2}\phi(w_k, x_k) + \frac{1}{2}\phi(v, x_k) - \frac{1}{4}K(\|w_k - v\|) \\
&\leq l + \sigma - \frac{1}{4}K(\epsilon_0),
\end{aligned} \tag{4.9}$$

for all $n \geq 0$. Hence

$$l \leq \lim_{n\longrightarrow\infty} \phi(\frac{w_k + v}{2}, x_n) = \lim_{n\longrightarrow\infty} \phi(\frac{w_k + v}{2}, x_{n+k}) \leq l + \sigma - \frac{1}{4}K(\epsilon_0) < l + \sigma - \sigma = l. \tag{4.10}$$

This is a contradiction. So, $\{w_n\}$ converges strongly to $v \in F := \Omega \cap VI(C, A) \cap T^{-1}(0)$. Consequently, $v \in F$ is the unique element of F such that

$$\lim_{n\longrightarrow\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\longrightarrow\infty} \phi(y, x_n). \tag{4.11}$$

This completes the proof. □

Theorem 4.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let A be an α-inverse-strongly monotone operator of C into E^*, with $F := VI(C, A) \cap T^{-1}(0) \neq \emptyset$. Assume that A is an operator of C into E^* which satisfies the condition $(C1) - (C3)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and,*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x_1) + (1 - \alpha_n)J(y_n)), \end{cases} \tag{4.12}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0,\infty)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, $\frac{1}{c}$ is the 2-uniformly convexity

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constant of E. Then the sequence $\{\Pi_F x_n\}$ converges strongly to an element v of F, which is a unique element of F satisfying

$$\lim_{n\longrightarrow\infty}\phi(v,x_n)=\min_{y\in F}\lim_{n\longrightarrow\infty}\phi(y,x_n).$$

Now, we prove a weak convergence theorem for the algorithm (4.13) below under different condition on data.

Theorem 4.3. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C\longrightarrow\mathbb{R}$ be a proper lower semicontinuous and convex function, let $T : E\longrightarrow E^*$ be a maximal monotone operator satisfying $D(T)\subset C$. Let $J_r=(J+rT)^{-1}J$ for $r>0$ and let A be an α-inverse-strongly monotone operator of C into E^* and let $B : C\longrightarrow E^*$ be a continuous and monotone mappings, with $F:=\Omega\cap VI(C,A)\cap T^{-1}(0)\neq\emptyset$. Assume that A is an operator of C into E^* which satisfies the condition $(C1)-(C3)$. Let $\{x_n\}$ be a sequence generated by $x_1=x\in C$ and,*

$$\begin{cases} u_n\in C \text{ such that } \Theta(u_n,y)+\langle Bu_n,y-u_n\rangle+\varphi(y)-\varphi(u_n)\\ \qquad\quad +\frac{1}{r_n}\langle y-u_n,Ju_n-Jx_n\rangle\geq 0,\ \forall y\in C,\\ z_n=\Pi_C J^{-1}(Ju_n-\lambda_n Au_n),\\ y_n=J^{-1}(\beta_n J(x_n)+(1-\beta_n)J(J_{r_n}z_n)),\\ x_{n+1}=\Pi_C J^{-1}(\alpha_n J(x_1)+(1-\alpha_n)J(y_n)),\end{cases}\tag{4.13}$$

for all $n\in\mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\},\{\beta_n\}\subset(0,1]$, $\{r_n\}\subset(0,\infty)$ satisfying $\sum_{n=1}^{\infty}\alpha_n<\infty$, $\limsup_{n\longrightarrow\infty}\beta_n<1$, $\liminf_{n\longrightarrow\infty}r_n>0$ and $\{\lambda_n\}\subset[a,b]$ for some a,b with $0<a<b<\dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges weakly to an element v of F, where $v=\lim_{n\longrightarrow\infty}\Pi_F x_n$.

***Proof*.** As in Proof of Theorem 3.1, we have $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup u\in C$ and hence $u\in F:=\Omega\cap VI(C,A)\cap T^{-1}(0)$. By Theorem 4.1, the $\{\Pi_F x_n\}$ converges strongly to a point $v\in F$ which is a unique element of F such that

$$\lim_{n\longrightarrow\infty}\phi(v,x_n)=\min_{y\in F}\lim_{n\longrightarrow\infty}\phi(y,x_n).\tag{4.14}$$

By the uniform smoothness of E, we also have $\lim_{n\longrightarrow\infty}\|J\Pi_F x_{n_i}-Jv\|=0$.

Finally, we prove $u=v$. From Lemma 2.5 and $u\in F$, we have

$$\langle\Pi_F x_{n_i}-u,Jx_{n_i}-J\Pi_F x_{n_i}\rangle\geq 0.$$

Since J is weakly sequentially continuous, $u_{n_i}\rightharpoonup u$ and $u_n-x_n\longrightarrow 0$, then

$$\langle v-u,Ju-Jv\rangle\geq 0.$$

On the other hand, since J is monotone, we have

$$\langle v-u,Ju-Jv\rangle\leq 0.$$

Hence,

$$\langle v-u,Ju-Jv\rangle=0.$$

Since E is strict convexity, it follows that $u=v$. Therefore the sequence $\{x_n\}$ converges weakly to $v=\lim_{n\longrightarrow\infty}\Pi_F x_n$. This completes the proof. □

Theorem 4.4. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $T : E\longrightarrow E^*$ be a maximal monotone operator satisfying $D(T)\subset C$. Let $J_r=(J+rT)^{-1}J$ for $r>0$ and let A be an α-inverse-strongly monotone operator of C into E^*, with $F:=VI(C,A)\cap T^{-1}(0)\neq\emptyset$. Assume that A is an operator of C into E^* which satisfies the condition $(C1)-(C3)$. Let $\{x_n\}$ be a sequence generated by $x_1=x\in C$ and,*

$$\begin{cases} z_n=\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n),\\ y_n=J^{-1}(\beta_n J(x_n)+(1-\beta_n)J(J_{r_n}z_n)),\\ x_{n+1}=\Pi_C J^{-1}(\alpha_n J(x_1)+(1-\alpha_n)J(y_n)),\end{cases}\tag{4.15}$$

for all $n\in\mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\},\{\beta_n\}\subset(0,1]$, $\{r_n\}\subset(0,\infty)$ satisfying $\sum_{n=1}^{\infty}\alpha_n<\infty$, $\limsup_{n\longrightarrow\infty}\beta_n<1$,

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$\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges weakly to an element v of F, where $v = \lim_{n\longrightarrow\infty} \Pi_F x_n$.

5. Applications

In this section, we consider the convex minimization problem and the problem of finding a zero point of an α-inverse strongly monotone operator. First, we study the problem of finding a minimizer of a continuously Fréchet differentiable, convex functional in the framework of Banach space. We need the following lemma which was proved by Baillon and Haddad [2]:

Lemma 5.1. *(See [2]) Let E be a Banach space, f a continuously Fréchet differentiable, convex functional in E and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse strongly monotone.*

By Lemma (5.1), we can consider the problem of finding minimizer of a continuously Fréchet differentiable, convex functional in a Banach space.

Theorem 5.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : C \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let $B : C \longrightarrow E^*$ be a continuous and monotone mappings, with $F := \Omega \cap VI(C,A) \cap T^{-1}(0) \neq \emptyset$.Assume that f is a functional on E that satisfies the following conditions:*
(1) f is continuously Fréchet differentiable, convex functional on E and ∇f is α-inverse strongly monotone,
(2) $\|\nabla f|_C(y)\| \leq \|\nabla f|_C(y) - \nabla f|_C(u)\|$ for all $y \in C$ and $u \in S$.
Suppose that $x_1 = x \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0,\ \forall y \in C, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n \nabla f|_C u_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n} z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x) + (1-\alpha_n)J(y_n)), \end{cases} \tag{5.1}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1]$, $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} \beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof. By Theorem 3.1, we put $A = \nabla f|_C$ and from Lemma 5.1 and the condition (1) of Theorem 5.2 that $\nabla f|_C$ is an α-inverse strongly monotone operator of C into E^*. So, we obtain that $\{x_n\}$ converges strongly to $\Pi_F x_0$. □

Next, we will consider the zero point of an α-inverse strongly monotone operator of E into E^*. Without loss of generality, we let $C = E$ then the condition $(C3)$ of the operator A in Theorem 3.1 holds.

Theorem 5.3. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let Θ be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A4) let $\varphi : E \longrightarrow \mathbb{R}$ be a proper lower semicontinuous and convex function, let $T : E \longrightarrow E^*$ be a maximal monotone operator satisfying $D(T) \subset C$. Let $J_r = (J+rT)^{-1}J$ for $r > 0$ and let $B : E \longrightarrow E^*$ be a continuous and monotone mappings, with $F := \Omega \cap VI(E,A) \cap T^{-1}(0) \neq \emptyset$.Assume that A is an operator of E into E^* that satisfies the following conditions:*
(1) A is α-inverse strongly monotone,
(2) $A^{-1}(0) = \{u \in E : Au = 0\} \neq \emptyset$.

Suppose that $x_1 = x \in C$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad +\frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C, \\ z_n = J^{-1}(Ju_n - \lambda_n Au_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}z_n)), \\ x_{n+1} = J^{-1}(\alpha_n J(x) + (1-\alpha_n)J(y_n)), \end{cases} \tag{5.2}$$

for all $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from* E *onto* C, J *is the duality mapping on* E. *The coefficient sequence* $\{\alpha_n\}, \{\beta_n\} \subset (0,1]$, $\{r_n\} \subset (0,\infty)$ *satisfying* $\lim_{n\longrightarrow\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\limsup_{n\longrightarrow\infty}\beta_n < 1$, $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \dfrac{c^2\alpha}{2}$, $\dfrac{1}{c}$ *is the 2-uniformly convexity constant of* E. *Then the sequence* $\{x_n\}$ *converges strongly to* $z = \Pi_F x$.

Proof. In Theorem (3.1), we put $C = E$. Therefore $\Pi_E = I$, then we have

$$J^{-1}(Ju_n - \lambda_n Au_n) = \Pi_E J^{-1}(Ju_n - \lambda_n Au_n)$$

for every $n = 1, 2, ...$ and we also have $VI(E, A) = A^{-1}0$ and

$$\|Ay\| = \|Ay - 0\| = \|Ay - Au\|, \ \ \forall y \in E \text{ and } u \in A^{-1}(0).$$

Hence, by Theorem 3.1, $\{x_n\}$ converges strongly to some element z in $\Omega \cap A^{-1}(0) \cap T^{-1}(0)$ ∎

6. Examples and Numerical results

In this section, we give examples and numerical results which support our strong convergence theorem.

Example 6.1. Let $E = \mathbb{R}$ and $C = [-1, 1]$. Let $\Theta(x, y) = -5x^2 + xy + 4y^2$, $Bx = 4x$ and $\varphi x = x^2$. Find $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) \geq \varphi(x), \ \ \forall y \in [-1, 1].$$

Solution. We can see that Θ, B and φ are satisfied all conditions in Theorem 3.1. For each $r > 0$ and $x \in [-1, 1]$, by Lemma(2.12), we can guarantee that there exists $z \in [-1, 1]$ such that for each $y \in [-1, 1]$,

$$\begin{aligned} &\Theta(z, y) + \langle Bz, y - z\rangle + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z) \\ &\Leftrightarrow -5z^2 + zy + 4y^2 + \langle 4z, y - z\rangle + y^2 + \frac{1}{r}\langle y - z, z - x\rangle \geq z^2 \\ &\Leftrightarrow 5ry^2 + (5rz + z - x)y + (-10z^2 r - z^2 + zx) \geq 0 \end{aligned}$$

Put $H(y) = 5ry^2 + (5rz + z - x)y + (-10z^2r - z^2 + zx)$. We can see that H is a quadratic function of y with the coefficient $a = 5r$, $b = (5rz + z - x)$ and $c = (-10z^2r - z^2 + zx)$. Next, we will compute the discriminant $\triangle$ of H as shown in the following:

$$\begin{aligned} \triangle &= b^2 - 4ac \\ &= (5rz + z - x)^2 - 4(5r)(-10z^2r - z^2 + zx) \\ &= x^2 - 2(15rz + z)x + (15rz + z)^2 \\ &= (x - (15rz + z))^2. \end{aligned}$$

We know that $H(y) \geq 0$ for all $y \in [-1, 1]$ if it has at most one solution in $[-1, 1]$. So $\triangle \leq 0$ and hence $x = 15rz + z$. Now we have $z = K_r x = \dfrac{x}{15r + 1}$.

Next, we consider our main algorithm in the strong convergence theorem. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by $x_1 = x \in [-1, 1]$ and

$$\begin{cases} u_n \in C \text{ such that } \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad +\frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \ \forall y \in C, \\ z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n), \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}z_n)), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x) + (1-\alpha_n)J(y_n)), \end{cases} \tag{6.1}$$

Algorithm 1. Let $Ax = x, J = I$ and $J_{r_n} = I$ then the above algorithm is equivalent to the following:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)(1-\lambda_n)K_{r_n} x_n]. \tag{6.2}$$

Algorithm 2. Let $Ax = \frac{x}{1+x}, J = I$ and $J_{r_n} = I$, then we have

$$x_{n+1} = \alpha_n x + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)(\frac{x_n(n+1)}{16n+1} - \lambda_n \frac{x_n(16n+1)}{x_n(n+1)+16n+1})]. \tag{6.3}$$

Next, we give the numerical test for both algorithms (6.2,6.3).
Let $\alpha_n = \frac{1}{100n}$, $\beta_n = \lambda_n = \frac{1}{2}$ and $r_n = \frac{n}{n+1}$. Choose $x_1 = x = 1$ then the algorithm (6.2) becomes

$$x_{n+1} = \frac{1}{100n} + (1 - \frac{1}{100n})[\frac{33n+3}{64n+4}]x_n, \quad \forall n \geq 1$$

and the algorithm (6.3) becomes

$$x_{n+1} = \frac{1}{100n} + (1 - \frac{1}{100n})[\frac{1}{2}x_n + \frac{1}{2}(\frac{x_n(n+1)}{16n+1} - \frac{x_n(n+1)}{2x_n(n+1)+32n+2})], \quad \forall n \geq 1.$$

Numerical Result

n	x_n by Algorithm 1.	n	x_n by Algorithm 2.
1	1.0000	1	1.0000
2	0.5341	2	0.5770
3	0.2828	3	0.3245
4	0.1500	4	0.1815
5	0.0802	5	0.1017
6	0.0435	6	0.0575
7	0.0242	7	0.0329
8	0.0139	8	0.0193
9	0.0084	9	0.0117
$\vdots$	$\vdots$	$\vdots$	$\vdots$
413	0.0001	435	0.0001
414	0.0001	436	0.0001
415	0.0000	437	0.0000

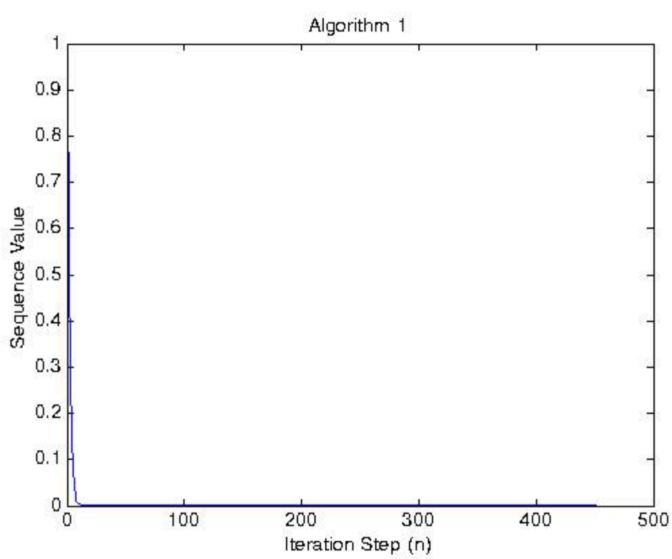

Figure 1. This table shows the value of sequence $\{x_n\}$ on each iteration steps.

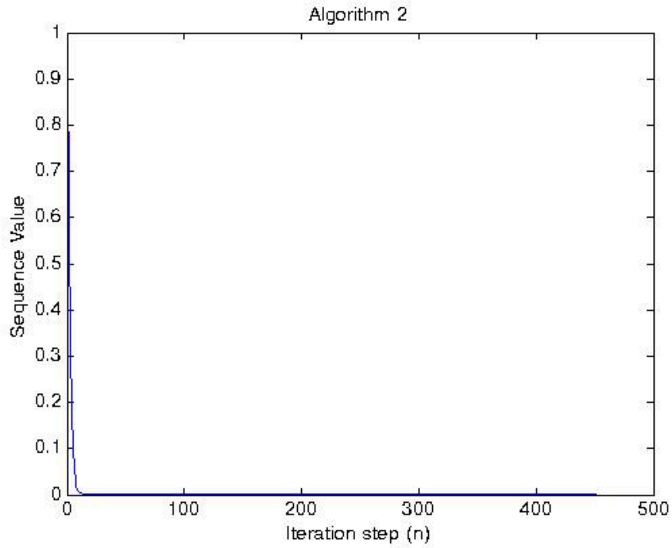

Figure 2. This figure shows the graph of the above table, we can see that x_n converges to zero.

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References

1. Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, A.G. Kartsatos (Ed), Marcel Dekker, New York., **178** (1996) 15–50.
2. J.B. Baillon, G. Haddad, Quelques propriétés des opérateurs angle-bornés et η-cycliquement monotones, Israel J. Math. **26**(1977) 137–150.
3. K. Ball, E. A. Carlen and E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norm*, Invent. Math. **115** (1994) 463–482.
4. B. Beauzamy, *Introduction to Banach spaces, and their Geometry*, 2nd ed., Noth Holland, 1995.
5. E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, *Math. Student.*, **63** (1994) 123–145.
6. D. Butnariu, S. Reich and A.J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7** (2001) 151–174.
7. Y. Cens or, S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996) 323–339.
8. P. L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005) 117–136.
9. I. Cioranescu, *Geometry of Banach spaces*, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990.
10. J. Diestel, *Geometry of Banach spaces-selected topics*, vol. 485 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1975.
11. S. D. Flam and A. S. Antipin, *Equilibrium progamming using proximal-link algolithms*, Math. Program., **78** (1997) 29–41.
12. H. Iiduka and W. Takahashi, *Weak convergence of a projection algorithm for variational inequalities in a Banach space*, J. Math. Anal. Appl. **339** (2008) 668–679.
13. H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, Panamer. Math. J. **14** (2004) 49–61.
14. S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence of a maximal monotone operators in a Banach space*, Set-Valued Analysis, **12** no.4 (2004) 417–429.
15. S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** no.3 (2002) 938–945.
16. S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory. **160** (2000) 226–240.
17. F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl.Anal. **3** (2004) 239–249.

18. L. Li and W. Song, *Modified Proximal-Point Algorithm for maximal monotone operators in Banach spaces*, J. Optim. Theory Appl. **138** (2008) 45–64.
19. A. Moudafi and M. Thera, *Proximal and dynamical approaches to equilibrium problems.* in: Lecture note in Economics and Mathematical Systems, Springer-Verlag, New York, **477** (1999), 187–201.
20. S. Mutsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134** (2005), 257–266.
21. W. Nilsrakoo and S. Saejung, *Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings*, Fixed Point Theory and Appl. (2008), Article ID 312454, 19 pages.
22. S. Reich, *A weak convergence theorem for the alternating method with Bregman distance*, in: A.G. Kartsatos (Ed.), Theory and Applicationsof Nonlinear Operatorsof Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313–318.
23. R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976) 877–898.
24. S. Saewan and P. Kumam, *A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems*, Abstract and Applied Analysis, Volume 2010, Article ID 123027, 31 pages
25. S. Saewan, P. Kumam and K. Wattanawitoon, *Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces*, Abstract and Applied Analysis, Volume 2010, Article ID 734126, 26 pages.
26. Y. Su, D. Wang and M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,* Fixed Point Theory and Appl., (2008) Article ID 284613, 8 pages.
27. A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, *Yokohama Publishers*, Yokohama, 2006, pp. 609-617.
28. A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mappings and an equilibrium problem*, J. Optim. Theory Appl., **133** (2007),359–370.
29. S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331** (2007), 506–515.
30. W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and Its Application*, Yokohama-Publishers, Yokohama, Japan 2000.
31. Y. Takahashi, K. Hashimoto and M. Kato, *On shap uniform convexity, smoothness, and strong type, cotype inequalities*, J. Nonlinear Convex Anal. **3** (2002) 267–281.
32. W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal. **70** (2009), 45–57.
33. K. K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J.Math.Anal.Appl. **178** (1993) 301–308.
34. H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc., **66** (2002), 240–256.
35. H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (2009) 1127–1138.
36. C. Zalinescu, *On uniformly convex functions*, J.Math.Anal.Appl. **95** (1983) 344–374.
37. H. Zegeye and N. Shahzad, *Strong convergence for monotone mappings and relatively weak nonexpansive mappings,* Nonlinear Anal., **70** (2009) 2707–2716.
38. S. Zhang, *Generalized mixed equlibrium problem in Banach spaces,* Appl. Math. Mech. -Engl. Ed., 30 (2009) 1105–1112.

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On a class of two dimensional (w,q)-Bernoulli and (w,q)-Euler polynomials: Properties and location of zeros

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Abstract

The main purpose of this paper is to introduce and investigate two dimensional (w,q)-Bernoulli and (w,q)-Euler polynomials. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava–Pintér addition theorem is obtained. Furthermore we explore the shapes of the q-Bernoulli numbers and the q-Bernoulli polynomials. We describe the structure of the roots of the q-Bernoulli polynomials for values of the index n using a computer.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n\in\mathbb{N}, \quad (a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q|<1, \quad a\in\mathbb{C}.$$

The q-number and q-factorial is defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q\neq 1); \quad [0]_q! = 1; \quad [n]_q! = [1]_q [2]_q ... [n]_q \qquad n\in\mathbb{N}, \quad a\in\mathbb{C}$$

respectively. The q-polynomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n\in\mathbb{N}_0.$$

The q-binomial formula is known as

$$(1-a)_q^n = (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.$$

In the standard approach to the q-calculus two exponential functions are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0<|q|<1,\ |z|<\frac{1}{|1-q|},$$
$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1+(1-q)q^k z\right), \qquad 0<|q|<1,\ z\in\mathbb{C}.$$

From this form we easily see that $e_q(z)E_q(-z)=1$. Moreover,

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(qz),$$

where D_q is defined by

$$D_q f(z) := \frac{f(qz)-f(z)}{qz-z}, \quad 0<|q|<1,\ 0\neq z\in\mathbb{C}.$$

The above q-standard notation can be found in [1].

Over 70 years ago, Carlitz extended the classical Bernoulli and Euler numbes and polynomials and introduced the q-Bernoulli and the q-Euler numbers and polynomials (see [2], [3] and [4]). There are numerous recent investigations on this subject by, among many other authors, Cenki et al. ([12], [13], [14]), Choi et al. ([15] and [16]), Kim et al. ([17]-[24]), Ozden and Simsek [26], Ryoo et al. [29], Simsek ([30], [31] and [32]), and Luo and Srivastava [11], Srivastava et al. [33], Mahmudov [25].

We first give here the definitions of the (w,q)-Bernoulli and the (w,q)-Euler polynomials as follows.

Definition 1 *Let $q\in\mathbb{C},\ 0<|q|<1$. The (w,q)-Bernoulli numbers $\mathfrak{B}_{n,q}$ and polynomials $\mathfrak{B}_{n,q}(x,y)$ in x,y are defined, in a suitable neighborhood of $t=0$, by means of the generating function functions:*

$$\frac{t}{we_q(t)-1} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)} \frac{t^n}{[n]_q!},$$
$$\frac{t}{we_q(t)-1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!}.$$

Definition 2 *Let $q\in\mathbb{C},\ 0<|q|<1$. The (w,q)-Euler numbers $\mathfrak{E}_{n,q}$ and polynomials $\mathfrak{E}_{n,q}(x,y)$ in x,y are defined, in a suitable neighborhood of $t=0$, by means of the generating function functions:*

$$\frac{2}{we_q(t)+1} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)} \frac{t^n}{[n]_q!},$$
$$\frac{2}{we_q(t)+1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!}.$$

It is obvious that

$$\mathfrak{B}_{n,q}^{(w)} = \mathfrak{B}_{n,q}^{(w)}(0), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(w)}(x,y) = B_n^{(w)}(x+y), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q} = B_n^{(w)}.$$

Here $B_n^{(w)}(x)$ denote the w-Bernoulli polynomials which are defined by

$$\frac{t}{we^t-1} e^{tx} = \sum_{n=0}^{\infty} B_n^{(w)}(x) \frac{t^n}{[n]_q!}.$$

2 Properties

In this section we shall provide some basic formulas for the (w,q)-Bernoulli and (w,q)-Euler polynomials in order to obtain the main results of this paper in the next section. The following result is q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3 (Addition Theorems) *For all $x,y\in\mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{(w)}(x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{B}_{k,q}^{(w)}x^{n-k},\quad \mathfrak{E}_{n,q}^{(w)}(x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}^{(w)}x^{n-k}.$$

Lemma 4 (Differential relation) *We have*

$$D_{q,x}\mathfrak{B}_{n,q}^{(w)}(x)=[n]_q\,\mathfrak{B}_{n-1,q}^{(w)}(x),\ D_{q,x}\mathfrak{E}_{n,q}^{(w)}(x)=[n]_q\,\mathfrak{E}_{n-1,q}^{(w)}(x)\,..$$

Lemma 5 (Difference Equations) *We have*

$$w\mathfrak{B}_{n,q}^{(w)}(x,1)-\mathfrak{B}_{n,q}^{(w)}(x,0)=[n]_q\,x^{n-1},$$
$$w\mathfrak{E}_{n,q}^{(w)}(x,1)+\mathfrak{E}_{n,q}^{(w)}(x,0)=2x^n.$$

Proof. The proof is based on the following identities

$$\frac{wt}{we_q(t)-1}e_q(tx)\,e_q(t)-\frac{t}{we_q(t)-1}e_q(tx)=\frac{t}{we_q(t)-1}e_q(tx)\,(we_q(t)-1)=te_q(tx)\,,$$

$$\frac{2w}{we_q(t)+1}e_q(tx)\,e_q(t)+\frac{2}{we_q(t)+1}e_q(tx)=\frac{2}{we_q(t)+1}e_q(tx)\,(we_q(t)+1)=2e_q(tx)\,.$$

■

Lemma 6 (Theorem of complement) *For all $x\in\mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{(w)}(x)=\frac{1}{w}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q(-1)^k\,q^{\frac{1}{2}k(k-1)}\mathfrak{B}_{k,1/q}^{(1/w)}(1)\,x^{n-k}$$

Proof. The proof is based on the following identity

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x,0)\,\frac{t^n}{[n]_q!}&=\frac{tE_q(-t)}{w-E_q(-t)}e_q(tx)=\frac{-t}{e_{1/q}(-t)-w}e_{1/q}(-t)\,e_q(tx)\\&=\frac{1}{w}\sum_{n=0}^{\infty}q^{\frac{1}{2}n(n-1)}\mathfrak{B}_{n,1/q}^{(1/w)}(1)\,\frac{(-t)^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{x^nt^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\frac{1}{w}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q(-1)^k\,q^{\frac{1}{2}k(k-1)}\mathfrak{B}_{k,1/q}^{(1/w)}(1)\,x^{n-k}\frac{t^n}{[n]_q!}.\end{aligned}$$

■

3 q-analoques of the addition theorems

In this section we shall investigate some explicit relationships between the (w,q)-Bernoulli and (w,q)-Euler polynomials. Here some q-analogues of known results will be given. We also obtain new formulas and their some special cases below. These formulas are some extensions of the formulas of Srivastava and Á. Pintér, Cheon and others.

We present natural q-extensions of th main results of the papers [10], [8], see Theorems 7 and 11.

Theorem 7 *For $n \in \mathbb{N}_0$, the following relationship*

$$\mathfrak{B}_{n,q}^{(w)}(x,y) = \frac{1}{2m^n}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q \left[m^k \mathfrak{B}_{k,q}^{(w)}(x) + w\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q m^j \mathfrak{B}_{j,q}^{(w)}(x)\right]\mathfrak{E}_{n-k,q}^{(w)}(my).$$

holds true between the (w,q)-Bernoulli polynomials and (w,q)-Euler polynomials.

Proof. Using the following identity

$$\frac{t}{we_q(t)-1}e_q(tx)\,e_q(ty) = \frac{2}{we_q\left(\frac{t}{m}\right)+1}\cdot e_q\left(\frac{t}{m}my\right)\cdot\frac{we_q\left(\frac{t}{m}\right)+1}{2}\cdot\frac{t}{we_q(t)-1}e_q(tx)$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{wt^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!}\\ &+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!}\\ &=: I_1 + I_2.\end{aligned}$$

It is clear that

$$I_2 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q m^{k-n}\mathfrak{B}_{k,q}^{(w)}(x)\,\mathfrak{E}_{n-k,q}^{(w)}(my)\frac{t^n}{[n]_q!}.$$

On the other hand

$$\begin{aligned}I_1 &= \frac{1}{2}w\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q m^{-n}\mathfrak{E}_{j,q}^{(w)}(0,my)\frac{t^n}{[n]_q!}\\ &= \frac{1}{2}w\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{B}_{k,q}^{(w)}(x)\sum_{j=0}^{n-k}\left[\begin{array}{c} n-k \\ j \end{array}\right]_q m^{k-n}\mathfrak{E}_{j,q}^{(w)}(my)\frac{t^n}{[n]_q!}\\ &= \frac{1}{2}w\sum_{n=0}^{\infty}m^{-n}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q \mathfrak{E}_{j,q}^{(w)}(my)\sum_{k=0}^{j}\left[\begin{array}{c} j \\ k \end{array}\right]_q m^k\mathfrak{B}_{k,q}^{(w)}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x,y)\frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q m^{k-n}\left[\mathfrak{B}_{k,q}^{(w)}(x) + m^{-k}w\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q m^j\mathfrak{B}_{j,q}^{(w)}(x)\right]\mathfrak{E}_{n-k,q}^{(w)}(my)\frac{t^n}{[n]_q!}.$$

■

Next we discuss some special cases of Theorem 7

Corollary 8 *[8] For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.*

$$B_n(x+y) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left(B_k(y) + \frac{k}{2}y^{k-1}\right)E_{n-k}(x),$$

$$B_n(x+y) = \frac{1}{2m^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left[m^kB_k(x) + m^kB_k\left(x-1+\frac{1}{m}\right) + km\left(1+m(x-1)\right)^{k-1}\right]E_{n-k}(my).$$

Corollary 9 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\mathfrak{B}_{k,q}(0,y) + q^{\frac{1}{2}(k-1)(k-2)} \frac{[k]_q}{2} y^{k-1} \right) \mathfrak{E}_{n-k,q}(x,0). \tag{1}$$

Corollary 10 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{B}_{n,q}(x,0) = \sum_{\substack{k=0 \\ (k\neq 1)}}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(x,0) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(x,0), \tag{2}$$

$$\mathfrak{B}_{n,q}(0,y) = \sum_{\substack{k=0 \\ (k\neq 1)}}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(0,y) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(0,y). \tag{3}$$

The formulas (1)-(3) are q-extension of the Cheon's main result [5]. Notice that $\mathfrak{B}_{1,q} = -\frac{1}{[2]_q}$, see [27], and the extra term becomes zeo for $q \to 1^-$.

Theorem 11 *For $n \in \mathbb{N}_0$, the following relationship*

$$\mathfrak{E}_{n,q}^{(w)}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{[k+1]_q} m^{k+1-n} \left(\mathfrak{E}_{k+1,q}^{(w)}\left(x, \frac{1}{m}\right) - \mathfrak{E}_{k+1,q}^{(w)}(x) \right) \mathfrak{B}_{n-k,q}(mx).$$

holds true between the (w,q)-Bernoulli polynomials and (w,q)-Euler polynomials.

Proof. The proof is based on the following identity

$$\frac{2}{we_q(t)+1} e_q(tx) e_q(ty) = \frac{2}{we_q(t)+1} e_q(tx) \cdot \frac{e_q\left(\frac{t}{m}\right)-1}{t/m} \cdot \frac{t/m}{e_q\left(\frac{t}{m}\right)-1} e_q\left(\frac{t}{m} mx\right).$$

Indeed

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!} &= \frac{m}{t} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)}(x) \frac{t^n}{[n]_q!} \left(\sum_{n=0}^{\infty} \frac{t^n}{m^n [n]_q!} - 1 \right) \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\
&= m \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q m^{k-n} \mathfrak{E}_{k,q}^{(w)}(x) - \mathfrak{E}_{n,q}^{(w)}(x) \right) \frac{t^{n-1}}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\
&= m \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q m^{k-n-1} \mathfrak{E}_{k,q}^{(w)}(x) - \mathfrak{E}_{n+1,q}^{(w)}(x) \right) \frac{t^n}{[n+1]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\
&= m \sum_{n=0}^{\infty} \left(\mathfrak{E}_{n+1,q}^{(w)}\left(x, \frac{1}{m}\right) - \mathfrak{E}_{n+1,q}^{(w)}(x) \right) \frac{t^n}{[n+1]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{[k+1]_q} \left[\begin{array}{c} n \\ k \end{array} \right]_q m^{k+1-n} \left(\mathfrak{E}_{k+1,q}^{(w)}\left(x, \frac{1}{m}\right) - \mathfrak{E}_{k+1,q}^{(w)}(x) \right) \mathfrak{B}_{n-k,q}(mx) \frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Next we discuss some special cases of Theorem 11.

Corollary 12 *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship*

$$\begin{aligned}
\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{m^{-n}}{[k+1]_q} &\left[2 \sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q m^j (x-1)_q^j \right. \\
&\left. - \sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q m^j \mathfrak{E}_{j,q}(x,-1) - m^{k+1} \mathfrak{E}_{k+1,q}(x,0) \right] \mathfrak{B}_{n-k,q}(0,my)
\end{aligned}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Corollary 13 *[8] For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.*

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1} \left(\begin{array}{c} n \\ k \end{array} \right) \left(y^{k+1} - E_{k+1}(y) \right) B_{n-k}(x),$$

$$E_n(x+y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{m^{k-n+1}}{k+1} \left[2\left(x + \frac{1-m}{m} \right)^{k+1} - E_{k+1}\left(x + \frac{1-m}{m} \right) - E_{k+1}(x) \right] B_{n-k}(my).$$

Corollary 14 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{2}{[k+1]_q} \left(q^{\frac{1}{2}k(k+1)} y^{k+1} - \mathfrak{E}_{k+1,q}(0,y) \right) \mathfrak{B}_{n-k,q}(x,0).$$

Corollary 15 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,0) = -\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(x,0),$$

$$\mathfrak{E}_{n,q}(0,y) = -\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(0,y).$$

4 Location of zeros of the q-Bernoulli polynomials

In this section, we display the shapes of the q-Bernoulli numbers and polynomials. Next, we investigate the zeros of the q-Bernoulli polynomials using a computer.

The shapes of the q-Bernoulli numbers $\mathfrak{B}_{n,q}$ for $n = 1, ..., 10$. $\frac{1}{2} \leq q \leq 1$ are shown in figures 1, 2 and 3.

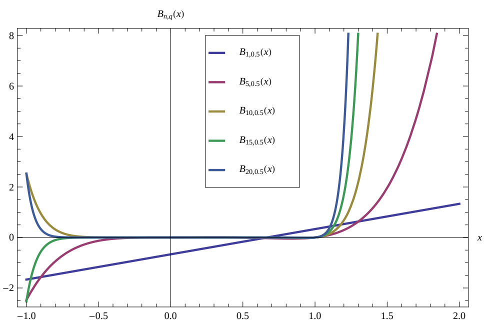

Figure 1: Shape of $B_{n,0.5}(x)$

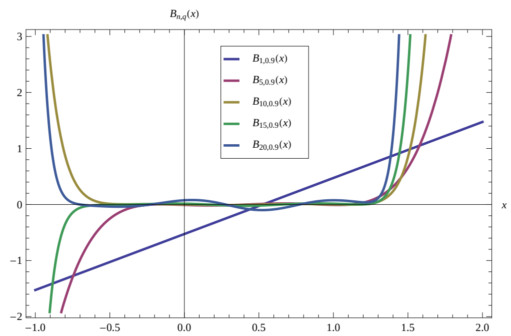

Figure 2: Shape of $\mathfrak{B}_{n,0.9}(x)$

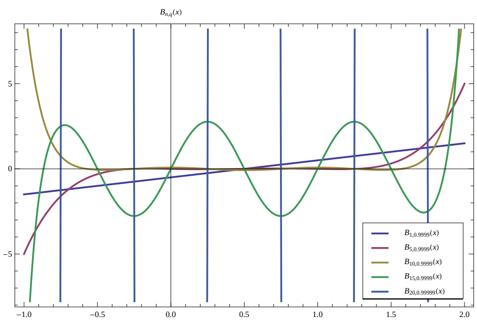

Figure 3: Shape of $\mathfrak{B}_{n,0.9999}$

The shapes of the q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x)$ for $n = 1, ..., 20$. $-0.5 \leq x \leq 1$, are shown in figures 3-6. The zeros of the q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x)$, $x \in C$ are plotted in figures 7-12 ($q = \frac{1}{2}, \frac{9}{10}, 0.\overline{9}$).

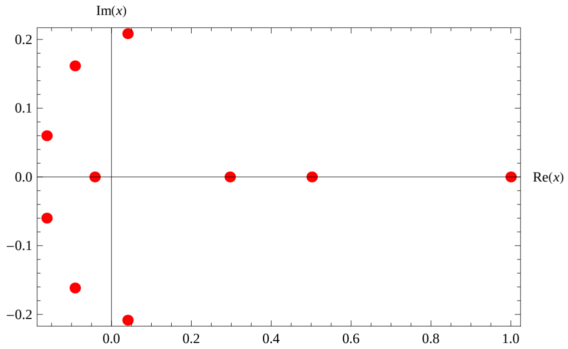

Figure 4: Zeros of $\mathfrak{B}_{10,0.5}(x)$

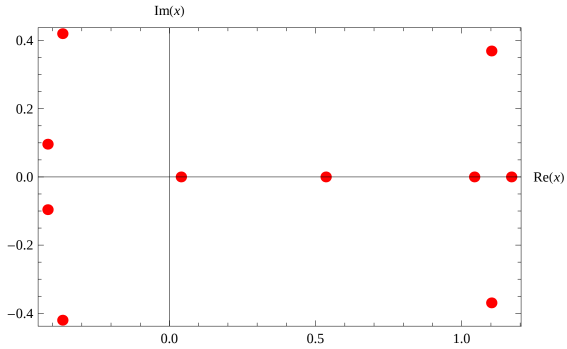

Figure 5: Zeros of $\mathfrak{B}_{10,0.9}(x)$

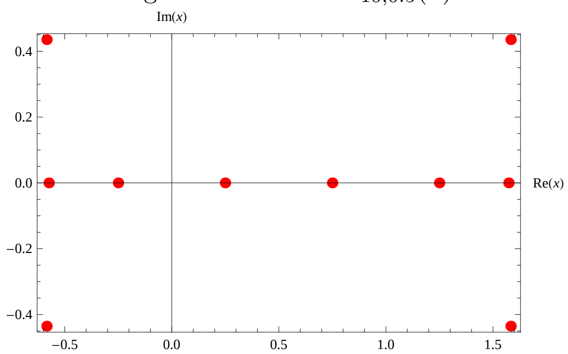

Figure 6: Zeros of $\mathfrak{B}_{10,0.9999}(x)$

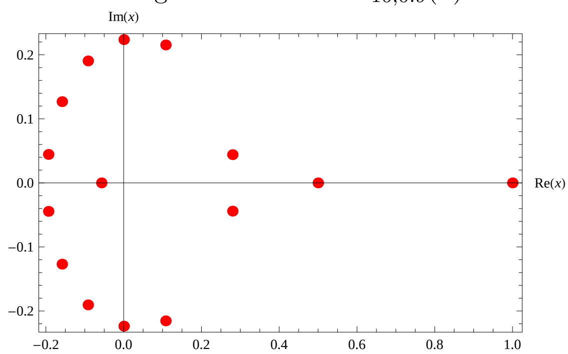

Figure 7: Zeros of $\mathfrak{B}_{15,0.5}(x)$

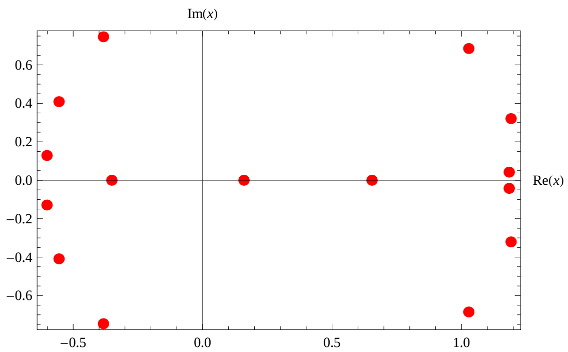

Figure 8: Zeros of $\mathfrak{B}_{15,0.9}(x)$

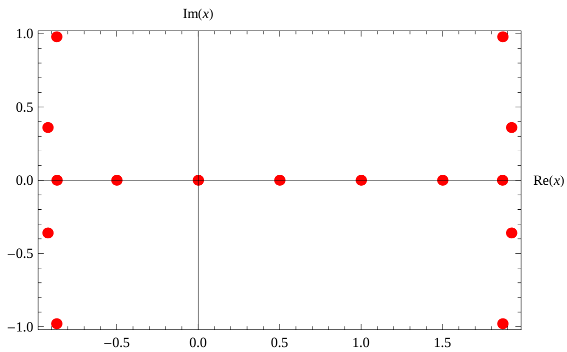

Figure 9: Zeros of $\mathfrak{B}_{15,0.9999}(x)$

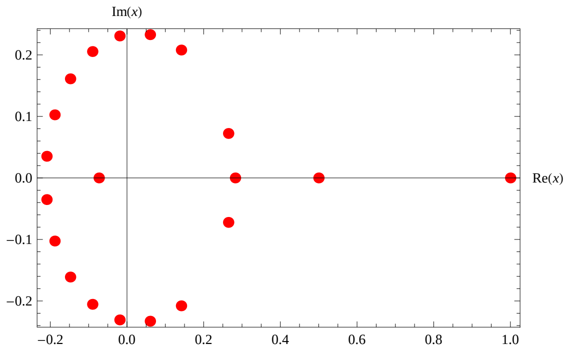

Figure 10: Zeros of $\mathfrak{B}_{20,0.5}(x)$

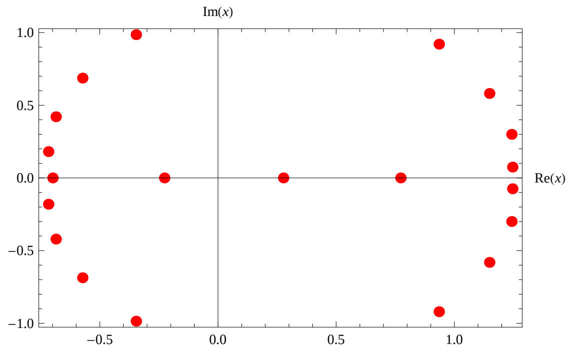

Figure 11: Zeros of $\mathfrak{B}_{20,0.9}(x)$

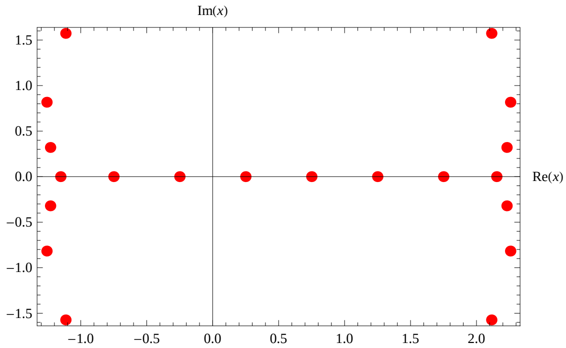

Figure 12: Zeros of $\mathfrak{B}_{20,0.9999}(x)$

In figures 7-12, $\mathfrak{B}_{n,q}(x)$, $x \in C$ have $\text{Im}(x) = 0$ reflection symmetry. This translates to the following open problem: prove that $\mathfrak{B}_{n,q}(x)$, $x \in C$, has $\text{Im}(x) = 0$ reflection symmetry.

Table 1: Approximate solutions of $\mathfrak{B}_{n,q}(x) = 0$

Degree n	Real Zeros
10	-0.0416672, 0.296755,0.501855, 1.0
15	-0.0569424, 0.49992, 1.0
20	-0.0730929, 0.282403, 0.500003, 1.0

Table 2: Approximate solutions of $\mathfrak{B}_{n,q}(x) = 0,\ x \in \mathbb{R}$

Degree n	Real Zeros
10	-0.369208,-0.210514, 0.290877, 0.789895, 1.29165, 1.61001
15	-0.562566, -0.394066, 0.105868, 0.60595, 1.10587, 1.6059,
20	-0.4876, -0.249796, 0.250325, 0.750206, 1.25032

Our numerical results for the approximate solutions of the real zeros of $\mathfrak{B}_{n,q}(x)$, $q = 0.\overline{9}$, are shown in tables 1 and 2. The results were obtained using the Mathematica®software.

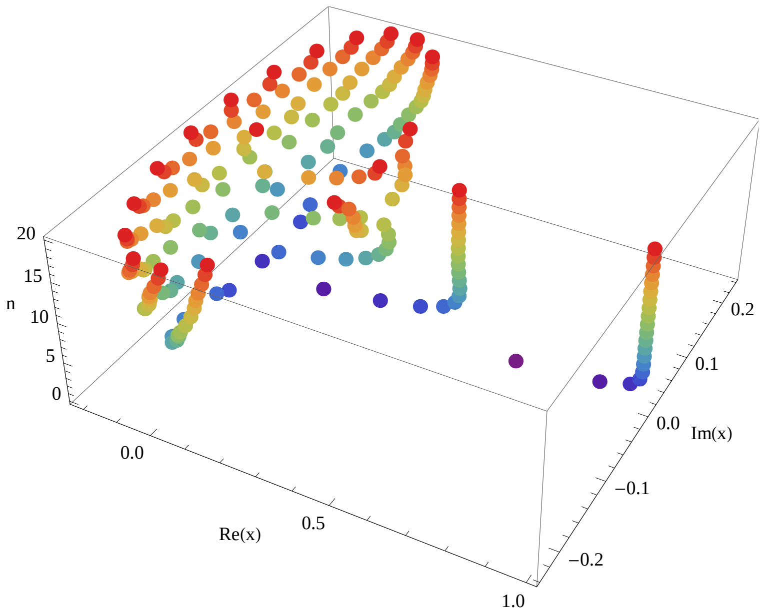

Figure 13: 3D shape of $B_{n=20,0.5}(x)$

References

[1] G. E. Andrews, R. Askey and R. Roy Special functions, volume 71 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1999.

[2] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.

[3] L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954) 332–350.

[4] L. Carlitz, Expansions of q-Bernoulli numbers, Duke Math. J. 25 (1958) 355–364.

[5] G.-S. Cheon, A note on the Bernoulli and Euler polynomials, Appl. Math. Lett. 16 (3) (2003) 365–368.

[6] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions (translated from French by J.W. Nienhuys), Reidel, Dordrecht, Boston, 1974.

[7] Q.-M. Luo, H.M. Srivastava, Some relationships between the Apostol–Bernoulli and Apostol–Euler polynomials, Comput. Math. Appl. 51 (2006) 631–642.

[8] Q.-M. Luo, Some results for the q-Bernoulli and q-Euler polynomials, J. Math. Anal. Appl. 363 (2010) 7–18.

[9] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[10] H.M. Srivastava, Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett. 17 (2004) 375–380.

[11] Q.-M. Luo, H.M. Srivastava, q-extensions of some relationships between the Bernoulli and Euler polynomials, Taiwanese Journal Math., 15, No. 1, pp. 241-257, 2011.

[12] M. Cenkci and M. Can, Some results on q-analogue of the Lerch Zeta function, Adv. Stud. Contemp. Math., 12 (2006), 213-223.

[13] M. Cenkci, M. Can and V. Kurt, q-extensions of Genocchi numbers, J. Korean Math. Soc., 43 (2006), 183-198.

[14] M. Cenkci, V. Kurt, S. H. Rim and Y. Simsek, On (i, q)-Bernoulli and Euler numbers, Appl. Math. Lett., 21 (2008), 706-711.

[15] J. Choi, P. J. Anderson and H. M. Srivastava, Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz Zeta function, Appl. Math. Comput., 199 (2008), 723-737.

[16] J. Choi, P. J. Anderson and H. M. Srivastava, Carlitz's q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions, Appl. Math. Comput., 215 (2009), 1185-1208.

[17] T. Kim, Some formulae for the q-Bernoulli and Euler polynomial of higher order, J. Math. Anal. Appl., 273 (2002), 236-242.

[18] T. Kim, q-Generalized Euler numbers and polynomials, Russian J. Math. Phys. 13 (2006), 293-298.

[19] T. Kim, On the q-Extension of Euler numbers and Genocchi numbers, J. Math. Anal. Appl., 326 (2007), 1458-1465.

[20] T. Kim, q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients, Russian J. Math. Phys., 15 (2008), 51-57.

[21] T. Kim, The modified q-Euler numbers and polynomials, Adv. Stud. Contemp. Math., 16 (2008), 161-170.

[22] T. Kim, L. C. Jang and H. K. Pak, A note on q-Euler numbers and Genocchi numbers, Proc. Japan Acad. Ser. A Math. Sci., 77 (2001), 139-141.

[23] T. Kim, Y.-H. Kim and K.-W. Hwang, On the q-extensions of the Bernoulli and Euler numbers. related identities and Lerch zeta function, Proc. Jangjeon Math. Soc., 12 (2009), 77-92.

[24] T. Kim, S.-H. Rim, Y. Simsek and D. Kim, On the analogs of Bernoulli and Euler numbers, related identities and zeta and L-functions, J. Korean Math. Soc., 45 (2008), 435-453.

[25] N. I. Mahmudov, q-Analogues of the Bernoulli and Genocchi Polynomials and the Srivastava-Pinter Addition Theorems, Discrete Dynamics in Nature and Society Volume 2012 (2012), Article ID 169348, 8 pages, doi:10.1155/2012/169348.

[26] H. Ozden and Y. Simsek, A new extension of q-Euler numbers and polynomials related to their interpolation functions, Appl. Math. Lett., 21 (2008), 934-939.

[27] O-Yeat Chan, D. Manna, A new q-analogue for bernoulli numbers, Preprint, oyeat.com/papers/qBernoulli-20110825.pdf

[28] G. M. Phillips, On generalized Bernstein polynomials. Numerical analysis, 263–269, World Sci. Publ., River Edge, NJ, 1996.

[29] C. S. Ryoo, J. J. Seo and T. Kim, A note on generalized twisted q-Euler numbers and polynomials, J. Comput. Anal. Appl., 10 (2008), 483-493.

[30] Y. Simsek, q-Analogue of the twisted l-series and q-twisted Euler numbers, J. Number Theory, 110 (2005), 267-278.

[31] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl., 324 (2006), 790-804.

[32] Y. Simsek, Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions, Adv. Stud. Contemp. Math., 16 (2008), 251-278.

[33] H. M. Srivastava, T. Kim and Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-Zeta functions and basic L-series, Russian J. Math. Phys., 12 (2005), 241-268.

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SOME IDENTITIES OF POLYNOMIALS ARISING FROM UMBRAL CALCULUS

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ABSTRACT. In this paper, we study some properties of associated sequences in umbral calculus. From these properties, we derive new and interesting identities of several kinds of polynomials.

1. Introduction

We recall that the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (see[7,8]),$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$.
In the special case, $x=0$, $B_n(0)=B_n$ are called the n-th Bernoulli numbers $(see[1-14])$.
For $r\in\mathbb{Z}_+$, the higher order Bernoulli polynomials are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\underbrace{\left(\frac{t}{e^t-1}\right)\cdots\left(\frac{t}{e^t-1}\right)}_{r-times}e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}.$$

In the special case, $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$ are called the n-th Bernoulli numbers of order r $(see\,[5,6])$. From the definition of Bernoulli numbers, we note that

$$B_0=1,\quad (B+1)^n-B_n=\delta_{1,n},\quad (see[7,8,10]),$$

where $\delta_{n,k}$ is the Kronecker's symbol.
As is well Known, the Euler and higher-order Euler polynomials are also defined by the generating functions as follows:

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},$$

and

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{r-times}e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},$$

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with the usual convention about replacing $\left(E^{(r)}(x)\right)^n$ by $E_n^{(r)}(x)$ $(see[3,4,5,6])$. Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\{f(t)=\sum_{n=0}^{\infty}\frac{a_k}{k!}t^k \mid a_k\in\mathbb{C}\}.$$

Let $\mathbb{P}=\mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functional on $\mathbb{P}$. Now, we use the notation $< L \mid p(x) >$ to denote the action of a linear functional L on a polynomial $p(x)$ $(see[4,11])$.
The formal power series

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\in\mathcal{F},\quad (see[4,11]),$$

define a linear functional on $\mathbb{P}$ by setting

$$< f(t) \mid x^n >= a_n,\quad for\quad all\quad n\geq 0,\quad (see[4,11]).$$

Thus, we have

$$< t^k \mid x^n >= n!\delta_{n,k},\quad (see[4]).$$

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{<L \mid x^k>}{k!}t^k$. Then, we note that $< f_L(t) \mid x^n >=< L \mid x^n >$ and so as linear functionals $L=f_L(t)$. It is known in [11] that the map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ on to $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$ and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra and modern classical umbral calculus can be described as a systematic study of the class of Sheffer sequences $(see[11])$. The order $O(f(t))$ of the power series $f(t)\neq 0$ is the smallest integer k for which a_k does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$ if and only if $O(f(t))=0$. Such a series is called invertible series. A series $f(t)$ for which $O(f(t))=1$ is called a delta series $(see[4,11])$. Let $f(t),g(t)\in\mathcal{F}$. Then, we see that

$$< f(t)g(t) \mid p(x) >=< f(t) \mid g(t)p(x) >=< g(t) \mid f(t)p(x) >,\quad (see[11]).$$

In [11], we note that for all $f(t)$ in $\mathcal{F}$

$$f(t)=\sum_{k=0}^{\infty}\frac{< f(t) \mid x^k >}{k!}t^k,$$

and for all polynomials $p(x)$

$$p(x)=\sum_{k=0}^{\infty}\frac{< t^k \mid p(x) >}{k!}x^k.$$

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Thus, we get

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{1}{(l-k)!} < t^l \mid p(x) > x^{l-k},$$

and

$$p^{(k)}(0) =< t^k \mid p(x) > \quad and < 1 \mid p^{(k)}(x) >= p^{(k)}(0).$$

From this, we have $t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, (k \geq 0)$. It is not difficult to show that $e^{yt}p(x) = p(x+y)$ $(see[4, 11])$. Let $S_n(x)$ be a polynomial with deg $S_n(x) = n$, $f(t)$ a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $< g(t)f(t)^k \mid S_n(x) >= n!\delta_{n,k}, (n, k \geq 0)$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequences for $f(t)$, or $S_n(x)$ is associated to $f(t)$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Applell for $g(t)$ $(see[4, 11])$. For $p(x) \in \mathbb{P}$, we have $< \frac{e^{yt}-1}{t} \mid p(x) >= \int_0^y p(u)du$ $(see[4, 11])$
In this paper, we study some properties of associated sequences in umbral algebra. From these properties, we derive new and interesting identities of several kinds of polynomials.

2. **Some identities of polynomials arising from umbral calculus**

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then, for $n \geq 1$, we have

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x), \quad (see[11]). \tag{1}$$

Let us take $p_n(x) = (x)_n$ and $q_n(x) = x^n$. Then we see that $(x)_n \sim (1, e^t-1)$ and $x^n \sim (1, t)$.
It is easy to show that

$$\left(\frac{e^t-1}{t}\right)^n = \underbrace{\left(\frac{e^t-1}{t}\right) \cdots \left(\frac{e^t-1}{t}\right)}_{n-times} = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l+n, n) t^l, \tag{2}$$

where $S_2(n, k)$ is the stirling number of the second kind.
For $n \geq 1$, by (1), we get

$$\begin{aligned} x^n &= x \left(\frac{e^t-1}{t}\right)^n x^{-1} (x)_n \\ &= x \left(\frac{e^t-1}{t}\right)^n (x-1)_{n-1} \\ &= x \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l+n, n) t^l (x-1)_{n-1}, \end{aligned} \tag{3}$$

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where $(x)_n = x(x-1)...(x-n+1)$.
The stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (see[6,11]). \tag{4}$$

By (3) and (4), we get

(5)

$$\begin{aligned}(x+1)^n &= \sum_{l=0}^{n} \frac{(n+1)!}{(l+n+1)!} S_2(l+n+1,n+1) \sum_{m=0}^{n-l} S_1(n,l+m)x^m(l+m)_l \\ &= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{l+n+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m)x^m,\end{aligned}$$

and

$$(x+1)^m = \sum_{m=0}^{n} \binom{n}{m} x^m. \tag{6}$$

Therefore , by (5) and (6), we obtain the following theorem.

Theorem 1 . For $m,n \in \mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$ with $n \geq m \geq 0$, we have

$$\binom{n}{m} = \sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{l+n+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m).$$

It is known that

$$x^n \sim (1,t), \quad (x)_n \sim (1, e^t-1), \quad (see[11]). \tag{7}$$

By (1) and (7), we get

$$\begin{aligned}(x)^n &= x\left(\frac{t}{e^t-1}\right)^n x^{-1}x^n \\ &= x\left(\frac{t}{e^t-1}\right)^n x^{n-1} = xB_{n-1}^{(n)}(x).\end{aligned} \tag{8}$$

Thus, by (8), we have

$$B_{n-1}^{(n)}(x) = (x-1)_{n-1}, (n \in \mathbb{N}). \tag{9}$$

Therefore, by(9), we obtain the following lemma.

Lemma 2 . For $n \in \mathbb{N}$, we have

$$B_n^{(n+1)}(x+1) = (x)_n.$$

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Note that

$$\begin{aligned}
(10) \qquad \sum_{l=0}^{\infty}\left(\frac{e^t-1}{t}\right)B_l^{(n+1)}(x)\frac{t^l}{l!} &= \left(\frac{e^t-1}{t}\right)\left(\frac{t}{e^t-1}\right)^{n+1}e^{xt} \\
&= \left(\frac{t}{e^t-1}\right)^{n}e^{xt} \\
&= \sum_{l=0}^{\infty}B_l^{(n)}(x)\frac{t^l}{l!}.
\end{aligned}$$

By comparing the coefficients on the both sides of (10), we get

$$(11) \qquad \left(\frac{e^t-1}{t}\right)B_l^{(n+1)}(x) = B_l^{(n)}(x), \quad (l \geq 0).$$

From (11), we have

$$(12) \qquad \left(\frac{e^t-1}{t}\right)B_n^{(n+1)}(x) = B_n^{(n)}(x), \quad (n \geq 0).$$

By Lemma 2, (11) and (12), we get

$$\begin{aligned}
(13) \qquad B_n^{(n)}(x+1) &= \left(\frac{e^t-1}{t}\right)B_n^{(n+1)}(x+1) \\
&= \left(\frac{e^t-1}{t}\right)(x)_n \\
&= \int_x^{x+1}(u)_n du.
\end{aligned}$$

From (4) and (13), we have

$$\begin{aligned}
(14) \qquad \int_x^{x+1}(u)_n du &= \sum_{l=0}^{n}S_1(n,l)\int_{x-1}^{x}u^l du \\
&= \sum_{l=0}^{n}\frac{S_1(n,l)}{l+1}(x^{l+1}-(x-1)^{l+1}).
\end{aligned}$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 3 . For $n \geq 1$, we have

$$B_n^{(n)}(x+1) = \sum_{l=0}^{n}S_1(n,l)\frac{1}{l+1}(x^{l+1}-(x-1)^{l+1}).$$

For $a \neq 0$, Abel sequence is defined by $A_n(x;a) = x(x-an)^{n-1}$. In [11], we note that $A_n(x;a) \sim (1, te^{at})$.

Let us consider the following associated sequences:

$$A_n(x;a) = x(x-an)^{n-1} \sim (1, te^{at}), \quad a \neq 0, \tag{15}$$
$$(\frac{x}{b})_n \sim (1, e^{bt}-1), \quad (b \neq 0).$$

For $n \geq 1$, by(1), we get

$$\begin{aligned} (\frac{x}{b})_n &= x\left(\frac{te^{at}}{e^{bt}-1}\right)^n x^{-1}A_n(x;a) \\ &= \frac{x}{b^n}\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant}(x-an)^{n-1}, \end{aligned} \tag{16}$$

where

$$\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant} = \sum_{k=0}^{\infty} b^k B_k^{(n)}\left(\frac{an}{b}\right)\frac{t^k}{k!}. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} (\frac{x}{b})_n &= \frac{x}{b^n}\left(\sum_{k=0}^{\infty} b^k B_k^{(n)}\left(\frac{an}{b}\right)\frac{t^k}{k!}\right)(x-an)^{n-1} \\ &= x\sum_{k=0}^{n-1} b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)\frac{(n-1)_k}{k!}(x-an)^{n-1-k} \\ &= \sum_{k=0}^{n-1} b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)\binom{n-1}{k}x(x-an)^{n-1-k} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)A_{n-k}(x-ak;a). \end{aligned} \tag{18}$$

On the other hand, by (16), we get

$$\begin{aligned} (\frac{x}{b})_n &= \left(\frac{1}{b}\right)^n x\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant}(x-an)^{n-1} \\ &= \left(\frac{1}{b}\right)^n x\left(\frac{bt}{e^{bt}-1}\right)^n x^{n-1} \\ &= \left(\frac{1}{b}\right)^n x\sum_{l=0}^{n-1}\frac{B_l^{(n)}}{l!}b^l(n-1)_l x^{n-l-1} \\ &= \frac{x}{b}\sum_{l=0}^{n-1}\binom{n-1}{l}\left(\frac{1}{b}\right)^{n-l-1}x^{n-l-1}B_l^{(n)} \\ &= \frac{x}{b}B_{n-1}^{(n)}(\frac{x}{b}). \end{aligned} \tag{19}$$

Therefore, by (18) and (19), we obtain the following theorem.

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Theorem 4 . For $n \geq 1$, we have

$$
\begin{aligned}
(\frac{x}{b})_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} b^{k-n} B_k^{(n)} \left(\frac{an}{b}\right) A_{n-k}(x-ak;a) \\
&= \frac{x}{b} B_{n-1}^{(n)}(\frac{x}{b}).
\end{aligned}
$$

Moreover,

$$
xB_{n-1}^{(n)}(\frac{x}{b}) = \sum_{k=0}^{n-1} \binom{n-1}{k} b^{k-n+1} B_k^{(n)} \left(\frac{an}{b}\right) A_{n-k}(x-ak;a).
$$

Remark . For $b=1$, $n \geq 1$, we have

$$
(x)_n = xB_{n-1}^{(n)}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k^{(n)}(an) A_{n-k}(x-ak;a).
$$

Let $\phi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k$ be exponential polynomial.
Then, we note that

$$
\phi_n(x) \sim (1, \log(1+t)), \quad x^n \sim (1,t). \tag{20}
$$

It is well known that

$$
\left(\frac{\log(1+t)}{t}\right)^n = n \sum_{k=0}^{\infty} \frac{B_k^{(n+k)}}{n+k} \frac{t^k}{k!}, (see[5,6]). \tag{21}
$$

By (1) and (20), we get

$$
\begin{aligned}
x^n &= x \left(\frac{\log(1+t)}{t}\right)^n x^{-1}\phi_n(x) \\
&= x\{n \sum_{k=0}^{\infty} \frac{B_k^{(n+k)}}{k+n} \frac{t^k}{k!}\} x^{-1}\phi_n(x) \\
&= n \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} \frac{\binom{l-1}{k}}{n+k} B_k^{(n+k)} S_2(n,l) x^{l-k} \\
&= n \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \frac{\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m) x^m \\
&= n \sum_{m=1}^{n} \{\sum_{k=0}^{n-m} \frac{\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m)\} x^m.
\end{aligned} \tag{22}
$$

Thus, by (22), we obtain the following theorem.

Theorem 5 . For $n \geq 1$ with $1 \leq m \leq n$, we have

$$
\sum_{k=0}^{n-m} \frac{n\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m) = \delta_{m,n}.
$$

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Let $M_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n-1)_{n-k}2^k(x)_k$ be Mittag-Leffler polynomials with $M_n(x) \sim (1, \frac{e^t-1}{e^t+1})$. Then, let us consider the following associated sequence:

$$M_n(x) \sim (1, \frac{e^t-1}{e^t+1}), \quad (x)_n \sim (1, e^t-1). \tag{23}$$

For $n \geq 1$, by (1) and (23), we get

$$\begin{aligned}(x)_n &= x\left(\frac{1}{e^t+1}\right)^n x^{-1}M_n(x) \\ &= \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k x\left(\frac{1}{e^t+1}\right)^n (x-1)_{k-1} \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^k S_1(k-1,l)\frac{1}{2^n}x\left(\frac{2}{e^t+1}\right)^n (x-1)^l \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)xE_l^{(n)}(x-1).\end{aligned} \tag{24}$$

Thus, by (24), we obtain the following proposition.

Proposition 6 . For $n \geq 1$, we have

$$(x)_n = \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)xE_l^{(n)}(x-1).$$

For $n \geq 1$, by (1) and (23), we get

$$\begin{aligned}M_n(x) &= x(e^t+1)^n x^{-1}(x)_n = x(e^t+1)^n(x-1)_{n-1} \\ &= x\sum_{k=0}^{n}\binom{n}{k}e^{kt}(x-1)_{n-1} = x\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}.\end{aligned} \tag{25}$$

The equation (25) is different from the expression

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k.$$

Therefore, by (25), we obtain the following corollary.

Corollary 7 . For $n \geq 1$, we have

$$M_n(x) = x\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}.$$

Let $L_n^{(\alpha)}(x)$ be the Laguerre polynomials of order $\alpha(\in \mathbb{R})$. Then we note that $L_n^{(\alpha)}(x) \sim ((1-t)^{-\alpha-1}, \frac{t}{t-1})$. Especially, $L_n(x) \sim (1, \frac{t}{t-1})$. By the

definition of associated sequences, we see that

$$< \left(\frac{t}{t-1}\right)^n \mid \quad L_n(x) >= n!\delta_{n,k}, \quad (n,k \geq 0). \tag{26}$$

From (26), we have

$$< \left(\frac{t}{t+1}\right)^n \mid \quad L_n(-x) >= n!\delta_{n,k}. \tag{27}$$

Thus, by (27), we get

$$L_n(-x) \sim (1, \frac{t}{t+1}). \tag{28}$$

As it is shown in Roman [11], one can find an explicit expression for $L_n(x)$ by using the transfer formula

$$L_n(-x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} x^k, \quad (n \geq 1), (see[11]).$$

It is well known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{k=0}^{\infty} B_k^{(k)} \frac{t^k}{k!}, (see[5,6]). \tag{29}$$

Thus, by (29), we get

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^n = \sum_{k=0}^{\infty} \left(\sum_{l_1+...+l_n=k} \binom{k}{l_1,...,l_n} B_{l_1}^{(l_1)} ... B_{l_n}^{(l_n)} \right) \frac{t^k}{k!}. \tag{30}$$

From (1),(20) and (28), we have

$$\begin{aligned} \phi_n(x) &= x \left(\frac{t}{(1+t)\log(1+t)}\right)^n x^{-1} L_n(-x) \\ &= x \left(\frac{t}{(1+t)\log(1+t)}\right)^n x^{-1} \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x^m. \end{aligned} \tag{31}$$

By (30) and (31), we get

(32)

$$\begin{aligned} &\phi_n(x) \\ &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x \{ \sum_{k=0}^{m-1} \sum_{l_1+...+l_n=k} \binom{k}{l_1,...,l_n} B_{l_1}^{(l_1)} ... B_{l_n}^{(l_n)} \} \frac{t^k}{k!} x^{m-1} \\ &= \sum_{m=1}^{n} \sum_{k=0}^{m-1} \sum_{l_1+...+l_n=k} \binom{n-1}{m-1} \binom{m-1}{k} \frac{n!}{m!} \binom{k}{l_1,...,l_n} B_{l_1}^{(l_1)} ... B_{l_n}^{(l_n)} x^{m-k} \\ &= \sum_{m=1}^{n} \{ \sum_{l=1}^{m} \sum_{l_1+...+l_n=m-l} \binom{n-1}{m-1} \binom{m-1}{l-1} \frac{n!}{m!} \binom{m-l}{l_1,...,l_n} B_{l_1}^{(l_1)} ... B_{l_n}^{(l_n)} \} x^l. \end{aligned}$$

From (20), we have

$$\phi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k = \sum_{k=1}^{n} S_2(n,k)x^k, \quad (n \geq 1). \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 8 . For $n \geq 1$ with $1 \leq l \leq n$, we have

$$S_2(n,l) = \sum_{l \leq m \leq n} \sum_{l_1+\ldots+l_n=m-l} \binom{n-1}{m-1}\binom{m-1}{l-1}\frac{n!}{m!}\binom{m-l}{l_1,\ldots,l_n} B_{l_1}^{(l_1)} \ldots B_{l_n}^{(l_n)}.$$

It is well known in [5,6] that

$$\frac{(e^t-1)^n}{e^{tx}t^n} = (n!)^2 \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}x^{k-j}\right)\frac{t^k}{k!}. \tag{34}$$

From (35), we have

(35)

$$\left(\frac{e^{bt}-1}{te^{at}}\right)^n = (n!)^2 b^n \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}(an)^{k-j}b^j\right)\frac{t^k}{k!},$$

where $a, b \neq 0$.
By (1) and (15), we get

(36)

$$\begin{aligned} A_n(x;a) &= x\left(\frac{e^{bt}-1}{te^{at}}\right)^n x^{-1}(\frac{x}{b})_n \\ &= x\left(\frac{e^{bt}-1}{te^{at}}\right)^n \frac{1}{b}(\frac{x}{b}-1)_{n-1} \\ &= (n!)^2 b^{n-1} x \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(-an)^{k-j}b^j\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}\right)\frac{t^k}{k!}(\frac{x}{b}-1)_{n-1}, \end{aligned}$$

where

$$\begin{aligned} t^k(\frac{x}{b}-1)_{n-1} &= \sum_{l=0}^{n-1} S_1(n-1,l)t^k(\frac{x}{b}-1)^l \\ &= \sum_{l=k}^{n-1} S_1(n-1,l)(\frac{1}{b})^k(l)_k(\frac{x}{b}-1)^{l-k}. \end{aligned} \tag{37}$$

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From (36) and (37), we have

(38)

$$\begin{aligned}A_n(x;a) &= x(x-an)^{n-1}\\ &= (n!)^2 b^{n-1}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\sum_{l=k}^{n-1}\left(-\frac{an}{b}\right)^{k-j}\frac{\binom{k}{j}\binom{l}{k}S_2(j+n,n)S_1(n-1,l)}{\binom{j+n}{j}}\right)x\left(\frac{x}{b}-1\right)^{l-k}.\end{aligned}$$

Therefore, by (39), we obtain the following lemma.

Lemma 9 . For $n \geq 1$, we have

$$\begin{aligned}&A_n(x;a)\\ &= (n!)^2 b^{n-1}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\sum_{l=k}^{n-1}\left(-\frac{an}{b}\right)^{k-j}\frac{\binom{k}{j}\binom{l}{k}S_2(j+n,n)S_1(n-1,l)}{\binom{j+n}{j}}\right)x\left(\frac{x}{b}-1\right)^{l-k}.\end{aligned}$$

Remark. Let $b = 1$. Then we have

$$\begin{aligned}A_n(x;a) &= x(x-an)^{n-1}\\ &= (n!)^2\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\sum_{l=k}^{n-1}(-an)^{k-j}\frac{\binom{k}{j}\binom{l}{k}S_2(j+n,n)S_1(n-1,l)}{\binom{j+n}{j}}\right)x(x-1)^{l-k}.\end{aligned}$$

It is well known in[5,6] that

$$\frac{(1+t)^{x-1}t^n}{(\log(1+t))^n} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\frac{t^k}{k!}. \tag{39}$$

From (39), we have

$$\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(an+1)\frac{t^k}{k!}. \tag{40}$$

Let us consider the following associated sequences:

$$S_n(x) \sim (1, t(t+1)^a), (a \neq 0), \quad x^n \sim (1,t). \tag{41}$$

Then, for $n \geq 1$, by (1) and (41), we get

$$\begin{aligned}S_n(x) &= x\left(\frac{t}{t(1+t)^a}\right)^n x^{-1}x^n\\ &= x(1+t)^{-an}x^{n-1}\\ &= x\sum_{k=0}^{n-1}\binom{-an}{k}(n-1)_k x^{n-1-k}\\ &= \sum_{k=1}^{n}\binom{-an}{n-k}(n-1)_{n-k}x^k.\end{aligned} \tag{42}$$

Therefore, by (42), we obtain the following proposition.

Proposition 10 . For $n \geq 1$, let $S_n(x) \sim (1, t(1+t)^a)$, $(a \neq 0)$. Then, we have

$$S_n(x) = \sum_{k=1}^{n} \binom{-an}{n-k} (n-1)_{n-k} x^k.$$

By (1), (20) and (41), we get

$$\phi_n(x) = x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n x^{-1} S_n(x) = x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} x^{l-1}. \tag{43}$$

From (40) and (43), we get

(44)

$$\begin{aligned}
\phi_n(x) &= \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n x^{l-1} \\
&= \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} \sum_{k=0}^{l-1} B_k^{(k-n+1)}(an+1) \frac{1}{k!} x t^k x^{l-1} \\
&= \sum_{l=1}^{n} \sum_{k=0}^{l-1} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{k} B_k^{(k-n+1)}(an+1) x^{l-k} \\
&= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-n-m+1)}(an+1) x^m \\
&= \sum_{m=1}^{n} \{ \sum_{l=m}^{n} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-n-m+1)}(an+1) \} x^m.
\end{aligned}$$

Therefore, by (20) and (44), we obtain the following theorem.

Theorem 11 . For $n \geq 1$ and $m \geq 0$, we have

$$S_2(n, m) = \sum_{m \leq l \leq n} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-m-n+1)}(an+1).$$

Let us consider the following associated sequences:

$$S_n^*(x) \sim (1, \frac{t(e^t+1)}{2}), \quad x^n \sim (1, t) \tag{45}$$

Then, by (1) and (45), we get

$$S_n^*(x) = x \left(\frac{2}{e^t+1} \right)^n x^{n-1} = x E_{n-1}^{(n)}(x). \tag{46}$$

For $n \geq 1$, by (1), (15) and (46), we get

$$xE_{n-1}^{(n)}(x) = x\left(\frac{2}{e^t+1}\right)^n e^{ant}(x-an)^{n-1} = x\sum_{k=0}^{n-1}\frac{E_k^{(n)}(an)}{k!}t^k(x-an)^{n-1} = \sum_{k=0}^{n-1}\binom{n-1}{k}E_k^{(n)}(an)x(x-an)^{n-1-k}. \tag{47}$$

Therefore, by (47) , we obtain the following theorem.

Theorem 12 . For $n \geq 1$, we have

$$xE_{n-1}^{(n)}(x) = \sum_{k=0}^{n}\binom{n-1}{k}E_k^{(n)}(an)A_{n-k}(x-ak;a).$$

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References

[1] S. Araci, M. Acikgoz, H. Jolany, J. J. Seo,*A unified generating function of the q-Genocchi polynomials with their interpolation functions*, Proc. Jangjeon Math. Soc. 15 (2012), no. 2, 227-233.

[2] L. Carlitz,*Some remarks on the multiplication theorems for the Bernoulli and Euler polynomials*, Glas. Mat. Ser. III 16(36) (1981), no. 1, 3-23.

[3] L. Carlitz,*Some polynomials related to the Bernoulli and Euler polynomials*, Utilitas Math. 19 (1981), 81-127.

[4] D. S. Kim, T. Kim,*Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Diff. Equ. 2012, 2012:196 doi:10.1186/1687-1847-2012-196.

[5] H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly 79 (1972), 44-51.

[6] H. W. Gould, *Note on recurrence relations for Stirling numbers*, Publ. Inst. Math. 6 (20) (1966), 115-119.

[7] D. S. Kim, D.V. Dolgy, T. Kim, S.-H. Rim,*Some formulae for the product of two Bernoulli and Euler polynomials*, Abstr. Appl. Anal. 2012, Article ID 784307, 15 p. (2012).

[8] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. 16, No. 1, 93-96 (2009).

[9] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. 14 (2008), no. 12, 1267-1277.

[10] S.-H. Rim, J. Jeong, On the modified q-Euler numbers of higher order with weight, Adv. Stud. Contemp. Math. 22 (2012), no. 1, 93-98.

[11] S. Roman,*The umbral calculus*, Dover Publ. Inc. New York, 2005.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 16, NO.2, 2014, COPYRIGHT 2014 EUDOXUS PRESS, LLC

[12] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on* $\mathbb{Z}_p$, Rocky Mountain J. Math. 41 (2011), no. 1, 239-247.
[13] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory 132 (2012), no. 12, 2854-2865.
[14] C. S. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math. 21 (2011), no. 2, 217-223.

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An incomplete discontinuous Galerkin finite element method for second order elliptic problem

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Abstract

In this paper, we propose an incomplete discontinuous Galerkin (iDG) finite element method for solving second linear elliptic equations, This method, called iDG-FEM, is designed by choosing an incomplete discontinuous finite element space in place of the classical finite element space. Optimal order error estimate in a discrete H^1 norm is established for the corresponding iDG-FEM solutions. Numerical results are presented to demonstrate the robustness, reliability, and accuracy of the iDG-FEM.
Keywords: Galerkin FEMs, incomplete discontinuous , elliptic problem, error estimate, numerical experiments

1 Introduction.

The goal of this paper is to introduce a numerical approximation technique for partial differential equations based on a new incomplete discontinuous finite element space. To illustrate the main idea, we consider the Dirichlet problem for second-order elliptic equation which seeks an unknown functions $u = u(x)$ satisfying

$$-\nabla \cdot (A\nabla u) = f, \quad \text{in} \quad \Omega, \tag{1a}$$
$$u = g, \quad \text{on} \quad \partial\Omega, \tag{1b}$$

where Ω is a bounded region in R^2, with a Lipschitz continuous boundary $\partial\Omega$, $A = (a_{ij}(x))_{2\times 2} \in [L^\infty(\Omega)]^{2^2}$ is a symmetric matrix-valued function. Assume that the matrix A is a positive definite matrix, i.e., there exists a positive constant α_0 such that

$$\xi^T A\xi \geq \alpha_0 \xi^T \xi, \quad \forall \xi \in R^2. \tag{2}$$

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The standard variational weak form for (1) is: Find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$a(u, v) = (f, v), \quad v \in H_0^1(\Omega), \tag{3}$$

where

$$a(u, v) = (A\nabla u, \nabla v), \tag{4}$$

(φ, ψ) represents the standard L^2-inner product of $\varphi = \varphi(x)$ and $\psi = \psi(x)$ which are either vector-valued or scalar-valued functions. Here ∇u denotes the gradient of the function $u = u(x)$, and ∇ is known as the gradient operator. In the standard Galerkin method (e.g., see [3, 8]), the trial space $H^1(\Omega)$ and the test space $H_0^1(\Omega)$ in (3) are replaced by finite dimensional subspaces defined properly, respectively.

Many FEMs have been developed for such problem (1). The existing FEMs can be classified into two categories [14]: (1) methods based on the primary unknown function u, and (2) methods based on the primary unknown u and a flux unknown $\sigma = -\nabla u$ with/without coefficient (mixed formulation). The standard Galerkin FEMs ([3, 8]) and various interior penalty type discontinuous Galerkin methods ([2, 12]) are typical examples of the first category. The standard mixed FEMs ([5, 6, 4, 13]) and various discontinuous Galerkin methods based on both unknowns ([7, 9, 10, 11]) are representatives of the second category.

In the above references, finite element spaces are built based on complete continuous function in whole domain or complete discontinuous in whole domain but continuous in separate sub-domains. The purpose of this paper is to consider the incomplete discontinuous Galerkin finite element procedures, based on incomplete discontinuous finite element spaces in which basis functions are continuous in separate belonging elements and on middle points of each elemental boundary, for elliptic problem. The corresponding finite element space is called incomplete discontinuous finite element space. Details can be found in Section 2.

The outline of this article is organized as follows. Section 1 is introduction. In Section 2, we present an iDG-FEM for problem (1) by introducing an incomplete discontinuous Galerkin finite element space. In Section 3, we analyze and present optimal order discrete H^1-norm error estimates for the deriving iDG-FEM (10). Finally in Section 4 we give some numerical examples to verify the theory results.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted as those in [1, 8].

2 The iDG-FEM

2.1 Incomplete discontinuous finite element space

We place a shape regular triangle grid $\mathcal{T}_h$ on Ω with mesh size h. Obviously, the routine inverse inequality in the finite element analysis holds true [8]. Define

$(u,v)_T = \int_T uv\mathrm{d}x$ and $< u, v >_{\partial T} = \int_{\partial T} uv\mathrm{d}s$. We construct a trial function space $V_h \in H^1(T)$, which is called incomplete discontinuous Galerkin finite element space, as follows

$$V_h := \{v \in L^2(\Omega) : v|_T \in \mathbb{P}_1(T), v \,\text{is continuous at}\, M, \forall T \in \mathcal{T}_h\}, \tag{5}$$

where $\mathbb{P}_k(T)$ denotes the space of polynomials in element T of degree at most k and M is a middle point of edge e which belongs to boundary ∂T of element T. Note that the functions in V_h are allowed to have discontinuities across element interfaces except for middle points Ms of edges of element. For V_h, we define V_h^0 as a subspace of V_h with zero boundary values on $\partial\Omega$; i.e.,

$$V_h^0 := \{v \in V_h, v\,|_{\partial T \bigcap \partial\Omega} = 0, \forall T \in \mathcal{T}_h\}. \tag{6}$$

V_h and V_h^0 are called incomplete discontinuous finite element spaces.

In this paper we will consider a local projection. Let g be a given function, whose restriction in each triangular element belongs to $H^1(T)$. The so-called L^2-projection $\mathcal{P}_0 g$ is defined as the unique function in V_h such that in each triangular element T there holds

$$(\mathcal{P}_0 g - g, v)_T = 0, \quad \forall\, v \in \mathbb{P}_0(T), \tag{7}$$

2.2 iDG-FEM

Based on test function space V_h^0, the variational weak form for problem (1) is: Find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$a_h(u,v) - \sum_{T\in\mathcal{T}_h} \int_{\partial T} A\nabla u \cdot \mathbf{n} v\mathrm{d}s = (f,v), \quad v \in V_h^0, \tag{8}$$

where the bilinear form $a_h(u,v)$ is defined as

$$a_h(u,v) := \sum_{T\in\mathcal{T}_h} \int_T A\nabla u \cdot \nabla v\mathrm{d}x. \tag{9}$$

This is because test function $v \in V_h^0$ is C^1-continuous only on each element $T \in \mathcal{T}_h$ but no longer on global domain Ω. The case is different from that of traditional FEMs but not complete same as existing discontinuous Galerkin FEMs.

Making use of (8), we define the iDG finite element scheme for (1): Find the approximation solution $u_h \in V_h$ such that $u_h = g_I$ on $\partial\Omega$ and

$$a_h(u_h, v_h) = (f, v_h), \tag{10}$$

$\forall\, v_h \in V_h^0$, where bilinear form $a_h(\cdot,\cdot)$ has been defined in (9), and g_I is some approximation to g. In fact, we can choose g_I is the L^2 projection g onto $P_1(\partial\Omega)$.

2.3 Existence and uniqueness for iDG-FEM

Assume that u_h is an iDG-FEM approximation for the problem (1) arising from (10) by using the trial function space V_h and the test function space V_h^0. The goal of this section is to derive a uniqueness and existence result for u_h.

Theorem 2.1 *For $f \in L^2(\Omega)$ in problem (1), the iDG-FEM defined in (10), with numerical boundary condition $u_h = g_I$, has a unique solution in the trial function finite element space V_h if the meshsize h is sufficiently small, but a fixed positive constant.*

Proof. In fact, note that uniqueness is equivalent to existence for the finite element solution of (10) since the number of unknowns is the same as the number of equations. To prove the uniqueness, suppose u_h^1 and u_h^2 are finite element solutions of (10), i.e., for $\forall v_h \in V_h^0$, such that $u_h^1, u_h^2 \in V_h$; $u_h^1 = u_h^2 = g_I$ on $\partial\Omega$, and

$$a_h(u_h^1, v_h) = (f, v_h), \tag{11}$$

meanwhile

$$a_h(u_h^2, v_h) = (f, v_h). \tag{12}$$

Subtract (12) from (11), we have

$$a_h(u_h^1 - u_h^2, v_h) = 0. \tag{13}$$

Obviously, $u_h^1 - u_h^2$ belongs to test function finite element space V_h^0. So after choosing $v_h = u_h^1 - u_h^2$ in (13), we know

$$a_h(u_h^1 - u_h^2, u_h^1 - u_h^2) = \sum_{T \in \mathcal{T}_h} \int_T A\nabla(u_h^1 - u_h^2) \cdot \nabla(u_h^1 - u_h^2)\mathrm{d}x = 0,$$

we can deduce that $u_h^1 - u_h^2$ must be constant on each element $T \in \mathcal{T}_h$. Further, combining with one property of finite element spaces V_h and V_h^0, i.e., all functions belong to V_h or V_h^0 are continuous on middle points of edges of the boundary ∂T of element $T \in \mathcal{T}_h$, we are sure that $u_h^1 - u_h^2$ must be zero in the domain Ω. Till now we complete the proof of this theorem. $\square$

3 Error estimate

In this section we will derive some error estimates for the iDG-FEM (10) for smooth solution of (1). Below we denote C (maybe with indicates) a positive constant depending solely on the exact solution, which may have a different value in each occurrence.

Firstly, we introduce an important lemma which is called trace inequality ([8, 14]).

Lemma 3.1 *Let T be an element in triangle partition $\mathcal{T}_h$, with $e \in \partial T$ as an edge. For any function $\phi \in H^1(T)$, the following trace inequality is valid:*

$$\|\phi\|_e^2 \leq C(h^{-1}\|\phi\|_T^2 + h\|\nabla\phi\|_T^2). \tag{14}$$

Theorem 3.1 *Let u and u_h be the solutions to the problem (1) and the iDG-FEM (10), respectively. Assume that the exact solution is so regular that $u \in H^2(\Omega)$. Then, there exists a constant C such that*

$$\|u - u_h\|_h \leq Ch\|u\|_2, \tag{15}$$

where $\|w\|_h = (\sum\limits_{T\in\mathcal{T}_h} \|\nabla w\|_T^2)^{1/2}$ for $\forall w \in H^1(T)$.

Proof Let $\rho = u - u_I, e = u_I - u_h$. where u_I is an approximation to exact u such that $u_I = g_I$ on boundary $\partial\Omega$ and

$$\|u - u_I\|_h \leq Ch\|u\|_2. \tag{16}$$

We pay attention to estimate to e. Since u and u_h satisfy (8) and (10) respectively, we have

$$\|e\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\nabla e\|_T^2 = a_h(u_I - u_h, u_I - u_h),$$

further,

$$\begin{aligned}\|e\|_h^2 &= a_h(u_I - u, u_I - u_h) + a_h(u - u_h, u_I - u_h)\\ &= a_h(u_I - u, u_I - u_h) + \sum_{T\in\mathcal{T}_h} \int_{\partial T} A\nabla u \cdot \mathbf{n}(u_I - u_h)\mathrm{d}s\\ &= S_1 + S_2,\end{aligned}$$

where $\mathbf{n}$ is unit outer normal vector of boundary ∂T of element T.

$$\begin{aligned}S_1 &= \sum_{T\in\mathcal{T}_h} (\nabla(u_I - u), \nabla(u_I - u_h))_T\\ &\leq \sum_{T\in\mathcal{T}_h} \|(\nabla(u_I - u)\|_T \|\nabla(u_I - u_h))\|_T\\ &\leq (\sum_{T\in\mathcal{T}_h} \|(\nabla(u_I - u)\|_T^2)^{1/2} (\sum_{T\in\mathcal{T}_h} \|\nabla(u_I - u_h))\|_T^2)^{1/2}\\ &\leq (\sum_{T\in\mathcal{T}_h} Ch^2\|u\|_{2,T}^2)^{1/2} (\sum_{T\in\mathcal{T}_h} \|\nabla(u_I - u_h))\|_T^2)^{1/2}\\ &\leq Ch\|u\|_2 (\sum_{T\in\mathcal{T}_h} \|\nabla(u_I - u_h))\|_T^2)^{1/2}\end{aligned} \tag{17}$$

In what follows, we will analyze the second term S_2.

$$\begin{aligned}|S_2| &= |\sum_{T\in\mathcal{T}_h} \int_{\partial T} A\nabla u \cdot \mathbf{n}(u_I - u_h)\mathrm{d}s|\\ &\leq C|\sum_e \int_e \nabla u \cdot [u_I - u_h]\mathrm{d}s|,\end{aligned} \tag{18}$$

where e is an edge of triangle element $T \in \mathcal{T}_h$ and the symbol $[v]$ is a jump of v between two triangle elements T_1 and T_2 which have the common edge e. Using L^2-projection operator $\mathcal{P}_0$ which is defined in (7), further, we rewrite the above inequality to

$$|S_2| \leq C|\sum_e \int_e (\nabla u - \mathcal{P}_0\nabla u)([u_I - u_h] - \mathcal{P}_0[u_I - u_h])\mathrm{d}s|, \tag{19}$$

Making use of Hölder inequality and trace inequality, we have

$$\begin{aligned} |S_2| &\leq C\sum_e \|\nabla u - \mathcal{P}_0\nabla u\|_e \|[u_I - u_h] - \mathcal{P}_0([u_I - u_h])\|_e \\ &\leq C(\sum_e \|S_2^1\|_e^2)^{1/2}(\sum_e \|S_2^{21}\|_e^2 + \sum_e \|S_2^{22}\|_e^2)^{1/2} \end{aligned} \tag{20}$$

where $S_2^1 = \nabla u - \mathcal{P}_0\nabla u$, $S_2^{21} = (u_I^{(1)} - u_h^{(1)}) - \mathcal{P}_0(u_I^{(1)} - u_h^{(1)})$, and $S_2^{22} = (u_I^{(2)} - u_h^{(2)}) - \mathcal{P}_0(u_I^{(2)} - u_h^{(2)})$. For the term S_2^0, it follows from trace inequality of lemma 3.1 that

$$\begin{aligned} \|S_2^1\|_e^2 &\leq C(h^{-1}\|\nabla u - \mathcal{P}_0\nabla u\|_T^2 + h\|\nabla(\nabla u - \mathcal{P}_0\nabla u\|_T^2 \\ &\leq C(h^{-1}h^2\|u\|_{2,T}^2 + h\|u\|_{2,T}^2) \\ &\leq Ch\|u\|_{2,T}^2 \end{aligned} \tag{21}$$

For S_2^{21}, making use of trace inequality of lemma 3.1, we know that

$$\begin{aligned} \|S_2^{21}\|_e^2 &\leq C(h^{-1}h^2\|u_I^1 - u_h^1\|_{1,T_1}^2 + h\|u_I^1 - u_h^1\|_{1,T_1}^2) \\ &\leq Ch\|u_I^1 - u_h^1\|_{1,T_1}^2 \end{aligned} \tag{22}$$

Similar estimate to S_2^{22}.

$$\|S_2^{22}\|_e^2 \leq Ch\|u_I^2 - u_h^2\|_{1,T_2}^2 \tag{23}$$

Here T_1 and T_2 denote two triangle elements which have common edge. u_I^1 and u_h^1 denote functions u_I and u_h are restricted in element T_1.

So far, we can get bound for $\|e\|_h = \|u_I - u_h\|_h$ as follows

$$\|e\|_h = \|u_I - u_h\|_h = (\sum_{T\in\mathcal{T}_h} \|\nabla(u_I - u_h)\|_T^2)^{1/2} \leq Ch\|u\|_2 \tag{24}$$

Combining (24) with (16), we complete the proof of this theorem. □

4 Numerical examples

In this section, we list four numerical examples using scheme (10) constructed in Section 2 to verify the error estimate in Theorem 3.1.

We consider the following elliptic problem

$$-\Delta u = f, \quad \text{in} \quad \Omega, \tag{25}$$

with zero boundary condition. In all three numerical examples, for simplicity, we let Ω be an unit square, i.e., $\Omega = (0,1)\times(0,1)$. We choose source term $f(x)$ according to the corresponding analytical solution of each examples.

We construct triangular mesh as follows. Firstly, we divide the square domain $\Omega = (0,1)\times(0,1)$ into $N \times N$ sub-squares uniformly to obtain the square mesh. Secondly, we divide each square element into two triangles by the diagonal line with a negative slope so that we construct of triangular mesh.

The analytical solutions of the four examples are:

Example 1. $u = \sin(\pi x)\sin(\pi y)$;
Example 2. $u = x(1-x)y(1-y)$;
Example 3. $u = x(1-x)y(1-y)\exp(x+y)$;
Example 4. $u = x(1-x)y(1-y)\exp(x-y)$.

For a set of simulations, different mesh sizes $h = 1/N(N = 2, 4, 8, 16, 32)$ are taken, and their corresponding discrete H^1-norms errors, defined in Theorem 3.1, and convergence rates are listed in Table 1, 2, respectively.

Table 1: Discrete H^1-norm error and convergence rate for examples 1 and 2.

	Example 1		Example 2	
N	$\|\|u-u_h\|\|_h$	order	$\|\|u-u_h\|\|_h$	order
2	1.2887E-00		8.8388E-02	
4	6.4860E-01	0.9905	4.6366E-02	0.9308
8	3.2490E-01	0.9973	2.3528E-02	0.9787
16	1.6253E-01	0.9993	1.1810E-02	0.9944
32	8.1274E-02	0.9998	5.9110E-03	0.9985

Table 2: Discrete H^1-norm error and convergence rate for examples 3 and 4.

	Example 3		Example 4	
N	$\|\|u-u_h\|\|_h$	order	$\|\|u-u_h\|\|_h$	order
2	3.2061E-01		1.2194E-01	
4	1.7834E-01	0.8462	6.6755E-02	0.8692
8	9.2917E-02	0.9406	3.4353E-02	0.9584
16	4.6991E-02	0.9836	1.7309E-02	0.9889
32	2.3564E-02	0.9958	8.6717E-03	0.9971

All these four numerical examples given above are good agreement with the theoretical analysis in Section 3, which show that the iDG-FEM (10) is stable and first order convergent in discrete H^1-norm.

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References

[1] R. A. Adams. Sobolev Spaces. New York: Academic Press, 1975.

[2] D. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. *Unified analysis of discontinuous Galerkin methods for elliptic problem*, SIAM J. Numer.Anal., 39(2002) 1749-1779

[3] S. Brenner, R. Scott. The mathematical theory of finite element methods, New York: Springer-Verlag, 1994

[4] F. Brezzi, M. Fortin, *Mixed and Hybrid finite elements*, New York: Springer-Verlag, 1991.

[5] F. Brezzi, J. Douglas, Jr., R. Durn, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987) 237-250.

[6] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985) 217-235.

[7] P. Castillo, B. Cockburn, I. Perugia, and D. Schotzau, *An a priori error analysis of the local discontinuous Galerkin method for elliptic problems*, SIAM J. Numer. Anal., 38 (2000) 1676-1706.

[8] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland, 1978.

[9] B. Cockburn and C. W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440-2463.

[10] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second- order elliptic problems*, SIAM J. Numer. Anal. 47 (2009) 1319-1365.

[11] Y. Jeon, E. Park, *A Hybrid Discontinuous Galerkin Method for Elliptic Problems*, SIAM J. Numer. Anal. 48 (2010) 1968-1983.

[12] B. Riviere, M. F. Wheeler, and V.Girault. *A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems,* SIAM J. Numer. Anal., 39 (2001) 902-931

[13] J. Wang, *Mixed finite element methods, Numerical Methods in Scientific and Engineering Computing*, Eds: W. Cai, Z. Shi, C-W. Shu, and J. Xu, Academic Press.

[14] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comput. Appl. Math., 241 (2013) 103-115.

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Leader-following consensus of multi-agent systems with memory*

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Abstract

The sufficient conditions which consensus under leader-following protocol with memory (by introducing the outdated states) can reach for both continuous time and discrete time single order multi-agent system are obtained. A sufficient condition which consensus under leader-following protocol with memory can reach for discrete time single order multi-agent system is also obtained, which can make system obtain fastest convergence (this also solved the problem how to choose parameter α to obtain the maximal consensus speed for protocol in Li, Xu, Chu and Wang [Distributed average consensus control in networks of agents using outdated states, *IET Control Theory* & *Appl.*, 4(5)(2010):746-758]). Likewise, for a leader-following protocol with memory of second order continuous time multi-agent systems, the sufficient conditions which consensus can reach are also obtained. Finally, numerical examples illustrate our theoretical results.

Keywords: Consensus protocols; Graph theory; Multi-agent systems; Time-delay; Continuous time system

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1 Introduction

Recently, multi-agent systems have received significant attention due to their potential impacts in numerous civilian, homeland security, and military applications, etc. Consensus plays an important role in achieving distributed coordination. The basic idea of consensus is that a team of vehicles reaches an agreement on a common value by negotiating with their neighbors. Consensus algorithms are studied for both single-integrator kinematics [2,3,4] and high-order-integrator dynamics [5-9].

Formal study of consensus problems in groups of experts originated in management science and statistics in 1960s. Distributed computation over networks has a tradition in system and control theory starting with the pioneering work of Borkar, Varaiya [10] and Tsitsiklis [11]. Vicsek et al. provided a formal analysis of emergence of alignment in the simplified model of flocking [12]. This paper have an important influence on developing of the multi-agent systems consensus theory. On the study of consensus of continnuous-time system, classical model of consensus is provided by Olfati-Saber and Murray [13].

Two major approaches to accelerate the convergence of consensus algorithms can be identified: optimizing the weight matrix [14,15,16], and incorporating memory into the distributed averaging algorithm [1,17,18]. For discrete time systems, the spectral radius of the weight matrix governs the asymptotic convergence rate, so we can accelerate convergence by changing the entries of the weight matrix. Likewise, a more promising research direction is based on using local node memory. In [17] T. Aysal et al. propose an approach to accelerate local, linear iterative network algorithms asymptotically achieving distributed average consensus by employing a linear predictor to predict future node values. In [18] B. Oreshkin et al. provide the first theoretical demonstration that adding a local prediction component to the update rule can significantly improve the convergence rate of distributed averaging algorithms. In [1] J. Li et al. propose some consensus protocols which use both current states and outdated states. It is shown that the use of the outdated information can accelerate the consensus if the consensus protocols are chosen properly.

In this paper, we considered leader-following problems with memory for a multi-agent system. Firstly, the sufficient conditions of consensus of leader-following protocols with memory are obtained for both continuous and discrete single order multi-agent systems. Secondly, for a discrete single order multi-agent system, a sufficient condition of consensus is also obtained to achieve fastest convergence speed. Finally, the sufficient conditions which consensus can reach for a leader-following protocol of second order continuous time multi-agent systems are obtained.

The paper is organized as follows. In Section 2, we introduce basic concepts and preliminary results, while Section 3, we derive some sufficient conditions on consensus for single order dynamics protocols. In Section 4, we derive some sufficient conditions on consensus of second order dynamics protocols which is presented in

this paper. Moreover, in Section 5, numerical examples are presented to illustrate our theoretical results. Conclusions are drawn in Section 6.

2 Preliminaries

A directed graph (digraph) $G = (V, E)$ of order n consists of a set of nodes $V = \{1, \cdots, n\}$ and a set of edges $E = V \times V$. (i, j) is a edge of G if and only if $(i, j) \in G$. Accordingly, agent i is a neighbor of agent j. The set of all neighbors of agent i is denoted by $\mathcal{N}_i(t)$. Suppose that there are n nodes in the graph. The adjaency matrix $A \in \mathbb{R}^{n\times n}$ is defined as $a_{ii} = 0, a_{ij} = 1$, if $(j, i) \in E$ and 0 otherwise. A graph with the property that $(i, j) \in E$ implies $(j, i) \in E$ is said to be undirected. The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as $l_{ii} = \sum_{j\neq i} a_{ij}, l_{ij} = -a_{ij}$, for $i \neq j$. Moreover, matrix L is symmetric if an undirected graph has symmetric weights, i.e., $a_{ij} = a_{ji}$. A directed path is a sequence of edges in a directed graph with the form $(v_1, v_2), (v_2, v_3), \cdots$, where $v_i \in V$. A directed graph has a directed spanning tree if there exists at least one agent that has a directed path to all other agents.

First, we consider a continuous-time multi-agent system consisting of n follower-agents and a leader:

$$\dot{x}_i(t) = u_i(t),\ i = 1, \cdots, n, \tag{2.1}$$

where $x_i(t),\ u_i(t)$ denote the position and control of agent i. The consensus protocol is given as

$$u_i(t) = (1-\alpha)(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) - b_i(x_i(t) - x_0))$$

$$+\alpha(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau) - x_i(t-\tau)) - b_i(x_i(t-\tau) - x_0), \tag{2.2}$$

where x_0 denotes the position of leader, $\alpha(0 \leq \alpha \leq 1),\ b_i,\ \tau(b_i \geq 0,\ \tau > 0)$ are parameters, and $x_i(t-\tau) \triangleq x_i(t)$ when $0 \leqslant t < \tau$. Then (2.1) and (2.2) can be rewritten in the following compact form:

$$\dot{x}(t) = -(1-\alpha)(L+B)x(t) - \alpha(L+B)x(t-\tau) + B(1_n \otimes x_0), \tag{2.3}$$

where $B = diag(b_1, \cdots, b_n)$. Let $\bar{x}_i = x_i - x_0$. Then (2.3) can be rewritten as

$$\dot{\bar{x}}(t) = -(1-\alpha)(L+B)\bar{x}(t) - \alpha(L+B)\bar{x}(t-\tau). \tag{2.4}$$

The other consensus protocol is given as

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) - b_i(x_i(t) - x_0)). \tag{2.5}$$

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Then (2.1) and (2.5) can be rewritten in the following compact form:

$$\dot{\bar{x}}(t) = -(L+B)\bar{x}(t). \tag{2.6}$$

Similarly, we consider a discrete-time multi-agent system consisting of n follower-agents and a leader:

$$x_i(k+1) = x_i(k) + \epsilon u_i(k),\ i = 1, \cdots, n, \tag{2.7}$$

where $\epsilon,\ u_i(k)$ denote the step-size and control of multi-agent i. The consensus protocol is given as

$$u_i(k) = (1-\alpha)(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) - b_i(x_i(k) - x_0))$$

$$+\alpha(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k-1) - x_i(k-1)) - b_i(x_i(k-1) - x_0). \tag{2.8}$$

Corresponding, we consider the other consensus protocol

$$u_i(k) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) - b_i(x_i(k) - x_0)). \tag{2.9}$$

Lemma 2.1. [13] *(i). All the eigenvalues of Laplacian matrix L have nonnegative real parts; (ii). Zero is an eigenvalue of L with 1_n (where 1_n is the $n \times 1$ column vector of all ones) as the corresponding right eigenvector. Furthermore, zero is a simple eigenvalue of L if and only if graph G has a directed spanning tree.*

Definition 2.1 [1] *We say system (2.1) reaches average consensus with a speed faster or equal to (faster than) γ if $x(t)$ converges faster or equal to (faster than) $\delta(t)$, where $\delta(t)$ satisfies $\delta(0) = x(0)$ and $\dot{\delta}(t) = -\gamma\delta(t)$. Correspondingly, we say system (2.7) reaches average consensus with a speed faster or equal to (faster than) γ, if $x(k)$ converges faster or equal to (faster than)$\delta(k)$, where $\delta(k)$ satisfies $\delta(0) = x(0)$ and $\delta(k+1) = -\gamma\delta(k)$.*

Definition 2.2 [1] *We say system (2.1) [system (2.7)] reaches average consensus under protocol A faster or equal to (faster than) protocol B, if under protocol A, $x(t)(x(k))$ converges faster or equal to (faster than) under protocol B.*

In Definition 2.1 and 2.2 of [1], the author only considered average consensus problem. But similar definition can be obtained for the general consensus problem of multi-agent systems, which is also main problem considered in this paper.

3 Leader-following consensus of first-order multi-agent systems

In the following section we give a result of system (2.4).

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Theorem 3.1. *Suppose that the undirected graph consisting of follower-agents is connected and entire graph (which consists of follower-agents and leader) is also connected, then:*

1) system (2.1) reaches consensus asymptotically under protocol (2.2), if $\tau\lambda_i(L+B)\alpha < \zeta_i \sin\zeta_i - \tau\lambda_i(L+B)(1-\alpha)\cos\zeta_i, i = 1,\cdots,n,$ *where* ζ_i *is the root of* $\zeta = -\tau\lambda_i(L+B)(1-\alpha)\tan\zeta(0<\zeta<\pi)$ *if* $\alpha \neq 1$ *and* $\zeta_i = \frac{\pi}{2}$ *if* $\alpha = 1$*;*

2) system (2.1) reaches consensus faster under protocol (2.2) than protocol (2.5) if $\alpha > 0$*,* $\tau\lambda_i(L+B)\alpha < 1$*, and* $e^{\tau\lambda_i(L+B)} < \frac{1}{\cos\zeta_i}, i = 1,\cdots,n,$ *where* ζ_i *is the root of* $\zeta = \tau\lambda_i(L+B)\alpha\tan\zeta,\ 0<\zeta<\pi$*;*

3) for any $\gamma > 0$*, system (2.1) reaches consensus with a speed faster than* γ *if the following conditions hold (for* $i = 1,\cdots,n$*): (a)* $\tau\gamma - \tau\lambda_i(L+B)(1-\alpha) < 1$; *(b)* $\gamma - \lambda_i(L+B)\alpha e^{\tau\lambda_i(L+B)}$*; (c)* $\tau\lambda_i(L+B)\alpha e^{\tau\gamma} < \zeta_i\sin\zeta - [\tau\lambda_i(L+B)(1-\alpha)-\tau\gamma]\cos\zeta_i$*, where* ζ_i *is the root of* $\zeta = [\tau\gamma - \tau\lambda_i(L+B)(1-\alpha)]\tan\zeta, 0<\zeta<\pi$*, if* $\gamma \neq \lambda_i(L+B)(1-\alpha)$ *and* $\zeta_i = \pi/2$ *if* $\gamma = \lambda_i(L+B)(1-\alpha)$*.*

Proof. Similar to the proof of Theorem 1 of [1]. □

Theorem 3.2. *Suppose that the undirected graph consisting of follower-agents is connected and entire graph is also connected. Consensus is reached under protocol (2.8) if* $0 \le \alpha < 1/\varepsilon\lambda_n(L+B)$*. In addition, system (2.7) reaches consensus under protocol (2.8) faster or equal to protocol (2.9) if* $0 \le \alpha \le \min\{\varepsilon\lambda_1(L+B)+1/\varepsilon\lambda_1(L+B)-2, \varepsilon\lambda_n(L+B)+1/\varepsilon\lambda_n(L+B)-2\}$*.*

Proof. Similar to the proof of Theorem 3 of [18]. □

Theorem 3.3. *Suppose that the undirected graph consisting of follower-agents is connected and entire graph is also connected. The fastest consensus can be achieved under protocol (2.8) when*

1) $\alpha = 1$*, if* $\varepsilon\lambda_i(L+B) \le 1/4\ , i = 1,\cdots,n$*.*

2) $\alpha \le 1$ *is equal to the solution of the following equation*

$$\sqrt{\underline{a}\alpha} = \frac{1-\bar{a}(1-\alpha)+\sqrt{(1-\bar{a}(1-\alpha))^2-4\bar{a}\alpha}}{2},$$

where $\bar{a} = \min_i \varepsilon\lambda_i(L+B)$*,* $\underline{a} = \max_i \varepsilon\lambda_i(L+B)$ *and* $\frac{1}{4} < \underline{a} \le 1$*. Moreover,* α *is equal to 1, if* $\bar{a} < \frac{1}{4}$*,* $\frac{1}{4} < \underline{a} \le 1$ *and solution of the above equation is greater than 1.*

3) $\alpha = 1 + \frac{1-2\sqrt{\underline{a}}}{\underline{a}}$*, where* $\underline{a} = \max_i \varepsilon\lambda_i(L+B) > 1$*.*

Proof. (2.7) and (2.8) can be rewritten as

$$\bar{x}(k+1) = \bar{x}(k) - \varepsilon(1-\alpha)(L+B)\bar{x}(k) - \varepsilon\alpha(L+B)\bar{x}(k-1), \tag{3.1}$$

where $\bar{x} = x(k) - x_0$. The Z transform of system (3.1) is

$$\bar{x}(z)z = \bar{x}(z) - \varepsilon(1-\alpha)(L+B)\bar{x}(z) - \varepsilon\alpha(L+B)\bar{x}(z)z^{-1}, \tag{3.2}$$

which can be rewritten as

$$[z^2I - (I-\varepsilon(1-\alpha)(L+B))z + \varepsilon\alpha(L+B)]\bar{x}(z) = 0. \tag{3.3}$$

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So the roots of (3.3) satisfy

$$\det[z^2 I - (I - \varepsilon(1-\alpha)(L+B))z + \varepsilon\alpha(L+B)] =$$

$$\prod_{i=1}^{n}[z^2 - (1 - \varepsilon(1-\alpha)\lambda_i(L+B))z + \varepsilon\alpha\lambda_i(L+B)] = 0. \tag{3.4}$$

Solving (3.4), for $i = 1, \cdots, n$, we obtain

$$z_{2i-1} = \frac{1 - \varepsilon\lambda_i(L+B)(1-\alpha) + \sqrt{(1 - \varepsilon\lambda_i(L+B)(1-\alpha))^2 - 4\varepsilon\lambda_i(L+B)\alpha}}{2},$$

$$z_{2i-2} = \frac{1 - \varepsilon\lambda_i(L+B)(1-\alpha) - \sqrt{(1 - \varepsilon\lambda_i(L+B)(1-\alpha))^2 - 4\varepsilon\lambda_i(L+B)\alpha}}{2}.$$

Let $\Delta = (1 - \varepsilon\lambda_i(L+B)(1-\alpha))^2 - 4\varepsilon\lambda_i(L+B)\alpha$.

Case 1. $1 - \varepsilon\lambda_i(L+B)(1-\alpha) > 0$. We know $|z_{2i-1}| \geq |z_{2i-2}|$. Thus, we only need to consider z_{2i-1}.

When $\Delta \geq 0$, by $0 \leq \alpha \leq 1$ and $0 < \varepsilon\lambda_i(L+B) < 2$, we get

$$0 \leq \alpha \leq 1 + \frac{1 - 2\sqrt{\varepsilon\lambda_i(L+B)}}{\varepsilon\lambda_i(L+B)}.$$

Take the derivatives of z_{2i-1} about α, we get

$$z'_{2i-1} = \frac{\varepsilon\lambda_i(L+B)}{2}[1 + \frac{1 - \varepsilon\lambda_i(L+B)(1-\alpha) - 2}{\sqrt{(1 - \varepsilon\lambda_i(L+B)(1-\alpha))^2 - 4\varepsilon\lambda_i(L+B)\alpha}}] < 0,$$

So z_{2i-1} is decreasing monotonically function about α in $[0, 1 + \frac{1-2\sqrt{\varepsilon\lambda_i(L+B)}}{\varepsilon\lambda_i(L+B)}]$.

When $\Delta \leq 0$ and $\varepsilon\lambda_i(L+B) \geq 1/4$, we have $1 + \frac{1-2\sqrt{\varepsilon\lambda_i(L+B)}}{\varepsilon\lambda_i(L+B)} \leq \alpha \leq 1$, and

$$|z_{2i-1}| = \sqrt{\varepsilon\lambda_i(L+B)\alpha}.$$

So z_{2i-1} is increasing monotonically function about α in $[1 + \frac{1-2\sqrt{\varepsilon\lambda_i(L+B)}}{\varepsilon\lambda_i(L+B)}, 1]$. Therefore, $|z_{2i-1}|$ reach the minimum value when $\alpha = 1 + \frac{1-2\sqrt{\varepsilon\lambda_i(L+B)}}{\varepsilon\lambda_i(L+B)}$. Else, if $\varepsilon\lambda_i(L+B) \leq 1/4$, $i = 1, \cdots, n$, then $|z_{2i-1}|$ reach the minimum value when $\alpha = 1$.

Replace $\varepsilon\lambda_i(L+B))$ by a in the expression of z_{2i-1}, and take the derivatives of z_{2i-1} about a, we get

$$\acute{z}_{2i-1} = \frac{\alpha - 1 + \frac{(1-a(1-\alpha))(\alpha-1)-2\alpha}{\sqrt{(1-a(1-\alpha))^2-4a\alpha}}}{2} < 0.$$

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Again let $a_i = \varepsilon\lambda_i(L+B)$. By $1 - a_i(1-\alpha) > 0$ and $\alpha \leq 1 + \frac{1-2\sqrt{a_i}}{a_i}$, we get $a_i < 1$. Take the derivatives of $1 + \frac{1-2\sqrt{a}}{a}$ about a, we get

$$(1 + \frac{1-2\sqrt{a}}{a})' = \frac{\sqrt{a}-1}{a^2} < 0.$$

So $1 + \frac{1-2\sqrt{a}}{a}$ is decreasing monotonically function about a in $[0, 1)$. Therefore, we get that the maximal one among modulus of roots of (3.3) reach minimum when α is equal to the solution of the following equation

$$\sqrt{\underline{a}\alpha} = \frac{1 - \bar{a}(1-\alpha) + \sqrt{(1-\bar{a}(1-\alpha))^2 - 4\bar{a}\alpha}}{2},$$

where $\bar{a} = \min_i \varepsilon\lambda_i(L+B)$, $\underline{a} = \max_i \varepsilon\lambda_i(L+B), \bar{a}, \underline{a} < 1$, and $0 < \alpha \leq 1$. Moreover, if solutions of the above equation are greater than 1, by the monotonicity of function, then the maximal one among modulus of roots of (3.3) reach minimum when α is equal to 1.

Case 2. $1-\varepsilon\lambda_i(L+B)(1-\alpha) \leq 0$. Then $a_i = \varepsilon\lambda_i(L+B) \geq 1$ and $|z_{2i-2}| > |z_{2i-1}|$. So we only need to consider z_{2i-2}. When $\Delta \geq 0$, Take the derivatives of $|z_{2i-2}|$ about α, we get

$$(|z_{2i-2}|)' = \frac{a_i}{2}[-1 + \frac{-1 - a_i(1-\alpha)}{\sqrt{(1-a_i(1-\alpha))^2 - 4a_i\alpha}}] < 0.$$

So $|z_{2i-2}|$ is decreasing monotonically function about α in $[0, 1 + \frac{1-2\sqrt{a_i}}{a_i}]$.

When $\Delta \leq 0$ and $a_i \geq 1/4$,

$$|z_{2i-2}| = \sqrt{a_i\alpha}.$$

So $|z_{2i-2}|$ is increasing monotonically function about α in $[1 + \frac{1-2\sqrt{a_i}}{a_i}, 1]$. Therefore, $|z_{2i-2}|$ reach the minimum value when $\alpha = 1 + \frac{1-2\sqrt{a_i}}{a_i}$.

Replace $\varepsilon\lambda_i(L+B))$ by a in the expression of $|z_{2i-2}|$, and take the derivatives of $|z_{2i-2}|$ about a, we get

$$(|z_{2i-2}|)' = \frac{1 - \alpha + \frac{(1-a(1-\alpha))(\alpha-1)-2\alpha}{\sqrt{(1-a(1-\alpha))^2-4a\alpha}}}{2} > 0.$$

By $(1+\frac{1-2\sqrt{a}}{a})' = \frac{\sqrt{a}-1}{a^2} > 0$ about a and the monotonicity of function, we get that the maximal one among modulus of roots of (3.3) reach minimum when $\alpha = 1 + \frac{1-2\sqrt{\underline{a}}}{\underline{a}}$ and $\underline{a} = \max_i \varepsilon\lambda_i(L+B) \geq 1$.

According to Case 1 and Case 2, by monotonicity of $|z_{2i-1}|$ and $|z_{2i-2}|$ about a and α, we get that the maximal one among modulus of roots of (3.3) reach minimum when:

1) $\alpha = 1$, if $\varepsilon\lambda_i(L+B) \le 1/4$, $i = 1, \cdots, n$.

2) $\alpha \le 1$ is equal to the solution of the following equation

$$\sqrt{\underline{a}\alpha} = \frac{1-\bar{a}(1-\alpha)+\sqrt{(1-\bar{a}(1-\alpha))^2-4\bar{a}\alpha}}{2},$$

where $\bar{a} = \min_i \varepsilon\lambda_i(L+B)$, $\underline{a} = \max_i \varepsilon\lambda_i(L+B)$ and $\frac{1}{4} < \underline{a} \le 1$. Moreover, α is equal to 1, if $\bar{a} < \frac{1}{4}$, $\frac{1}{4} < \underline{a} \le 1$ and solution of the above equation is greater than 1.

3) $\alpha = 1 + \frac{1-2\sqrt{\bar{a}}}{\underline{a}}$, where $\underline{a} = \max_i \varepsilon\lambda_i(L+B) > 1$. □

Remark 3.1. Theorem 3.3 has solved the problem how to choose parameter α to obtain the maximal consensus speed for protocol (31) in [1], which can be got by the process of proof of Theorem 3.3.

4 Leader-following consensus of second-order multi-agent systems

In the following section, we consider a second-order continuous-time multi-agent system consisting of n follower-agents and a leader:

$$\dot{x}_i(t) = v_i(t), \tag{4.1}$$

$$\dot{v}_i(t) = u_i(t),\ i = 1, \cdots, n, \tag{4.2}$$

where $x_i(t), v_i(t),\ u_i(t)$ denote the position, velocity and control of multi-agent i. The consensus protocol is given as

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t)-x_i(t)) - b_i(x_i(t)-x_0) + (1-\alpha)(\sum_{j\in\mathcal{N}_i} a_{ij}(v_j(t)-v_i(t)) - b_i(v_i(t)-v_0))$$

$$+\alpha(\sum_{j\in\mathcal{N}_i} a_{ij}(v_j(t-\tau)-v_i(t-\tau)) - b_i(v_i(t-\tau)-v_0), \tag{4.3}$$

where x_0 and v_0 denote the position and velocity of leader, respectively, $\alpha(0 \le \alpha \le 1)$, b_i, $\tau(b_i \ge 0,\ \tau > 0)$ are parameters, and $x_i(t) = x_i(0)$, $v_i(t) = 0, t \in [-\tau, 0)$. Then (2.1) and (2.2) can be rewritten in the following compact form:

$$\dot{x}(t) = v(t) \tag{4.4}$$

$$\dot{v}(t) = -(L+B)x(t) + B(1_n \otimes x_0) - (1-\alpha)(L+B)v(t) - \alpha(L+B)v(t-\tau) + B(1_n \otimes v_0), \tag{4.5}$$

Let

$$\bar{x} = x - x_0,\ \ \bar{v} = v - v_0 \text{ and } \xi^T = \begin{pmatrix} \bar{x}^T & \bar{v}^T \end{pmatrix}^T.$$

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Then (4.4) and (4.5) can be rewritten in the following form:

$$\dot{\xi}(t) = \begin{pmatrix} 0_{n\times n} & I_n \\ -L-B & -(1-\alpha)(L+B) \end{pmatrix} \xi(t) + \begin{pmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -\alpha(L+B) \end{pmatrix} \xi(t-\tau). \tag{4.6}$$

For convenience of discussion, denote

$\Gamma = \begin{pmatrix} 0_{n\times n} & I_n \\ -L-B & -(1-\alpha)(L+B) \end{pmatrix}$ and $\Phi = \begin{pmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \alpha(L+B) \end{pmatrix}$.

Then a result of system (4.6) is given as follows.

Theorem 4.1. *Suppose that the undirected graph consisting of follower-agents is connected and entire graph is also connected. For system (4.6), consensus can be achieved if there exist symmetric positive definite matrices $P,\ Q,\ R \in \mathbb{R}^{2n\times 2n}$ such that*

$$\Xi = \begin{pmatrix} \Xi_1 & -\tau\Gamma^T R\Phi & \tau P\Phi \\ * & -Q+\tau\Phi^T R\Phi & 0_{n\times n} \\ * & * & -\tau R \end{pmatrix} < 0, \tag{4.7}$$

where $\Xi_1 = P(\Gamma-\Phi) + (\Gamma-\Phi)^T P + Q + \tau\Gamma^T R\Gamma$.

Proof. Difine a Lyapunov function for system (4.6) as follows

$$V = \xi^T(t)P\xi(t) + \int_{t-\tau}^{t} \xi^T(s)Q\xi(s)ds + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s)R\dot{\xi}(s)dsd\theta. \tag{4.8}$$

Calculating $\dot{V}$, we have

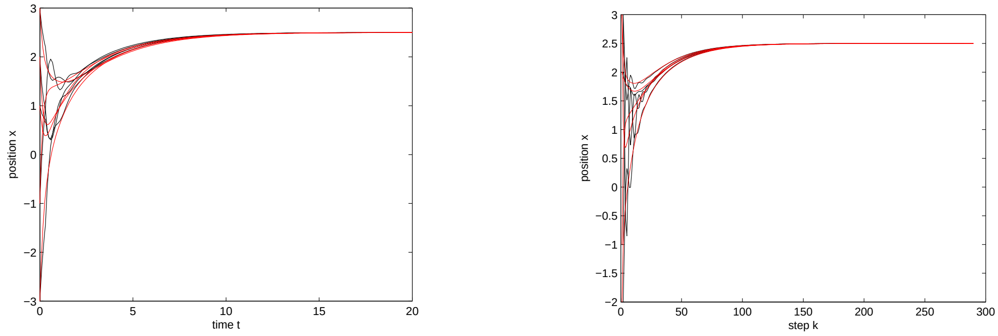
$$\begin{aligned} \dot{V} &= 2\xi^T(t)P\Gamma\xi(t) - 2\xi^T(t)P\Phi\xi(t-\tau) + \xi^T(t)Q\xi(t) - \xi^T(t-\tau)Q\xi(t-\tau) \\ &\quad + \tau\dot{\xi}^T(t)R\dot{\xi}(t) - \int_{t-\tau}^{t} \dot{\xi}^T(s)R\dot{\xi}(s)ds. \end{aligned} \tag{4.9}$$

By the Newton-Leibniz formula, $\int_{t-\tau}^{t} \dot{\xi}^T(s)ds = \xi(t) - \xi(t-\tau)$, we have

$$-2\xi^T(t)P\Phi\xi(t-\tau) = -2\xi^T(t)P\Phi\xi(t) + \int_{t-\tau}^{t} 2\xi^T(t)P\Phi\dot{\xi}(s)ds.$$

Consequently,

$$\begin{aligned} \dot{V} &= 2\xi^T(t)P(\Gamma-\Phi)\xi(t) + \int_{t-\tau}^{t} 2\xi^T(t)P\Phi\dot{\xi}(s)ds + \xi^T(t)Q\xi(t) - \xi^T(t-\tau)Q\xi(t-\tau) \\ &\quad + \tau\dot{\xi}^T(t)R\dot{\xi}(t) - \int_{t-\tau}^{t} \dot{\xi}^T(s)R\dot{\xi}(s)ds = \frac{1}{\tau}\int_{t-\tau}^{t} \eta^T(s)\Xi\eta(t)ds, \end{aligned} \tag{4.10}$$



(a) Under protocols (2.2) and (2.5)

(b) Under protocols (2.8) and (2.9)

Figure 1: State trajectories of multi-agent system

where $\eta^T(t) = \begin{pmatrix} \xi^T(t) & \xi^T(t-\tau) & \dot{\xi}^T(s) \end{pmatrix}^T$ and

$$\Xi = \begin{pmatrix} \Xi_1 & -\tau\Gamma^T R\Phi & \tau P\Phi \\ * & -Q+\tau\Phi^T R\Phi & 0_{n\times n} \\ * & * & -\tau R \end{pmatrix}$$

with $\Xi_1 = P(\Gamma-\Phi)+(\Gamma-\Phi)^T P + Q + \tau\Gamma^T R\Gamma$. Then condition (4.7) guarantees $\dot{V} < 0$ and by Lyapunov theory, we have

$$\lim_{t\to+\infty} \xi(t) = 0. \tag{4.11}$$

Therefore, consensus can be achieved under condition (4.7). □

5 Simulation

In this section, several simulation results are presented to illustrate the proposed consensus protocols introduced in section 3 and section 4.

In the following examples we consider a system consisting of five follower-agents guarded by a leader. The corresponding Laplacian matrices L, B and initial condition are given as

$$L = \begin{pmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ -1 & -1 & 4 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } x(0) = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 3 \\ -2 \end{pmatrix}.$$

Example 4.1 Figure 1 (a) is the state trajectories of multi-agent system under protocol (2.2) and (2.5), where the position of leader is 2.5. The red line is the trajectories of follower-agents under protocol (2.5), and the black line is the trajectories

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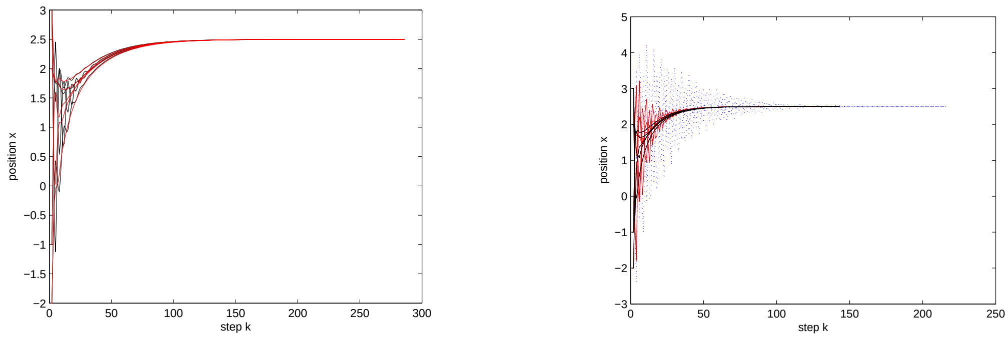


(a) Under protocol (2.8) with $\epsilon = 0.1$ (b) Under protocol (2.8) with $\epsilon = 0.2$

Figure 2: State trajectories of multi-agent system

Table 1: Comparision between Iterative steps of protocol (2.8) with $\epsilon = 0.1$ and $\| x - x_0 \|_2 < 10^{-4}$.

parameter α	0.3	0.4	0.5	0.65	0.8	1
step k	287	286	285	283	282	279

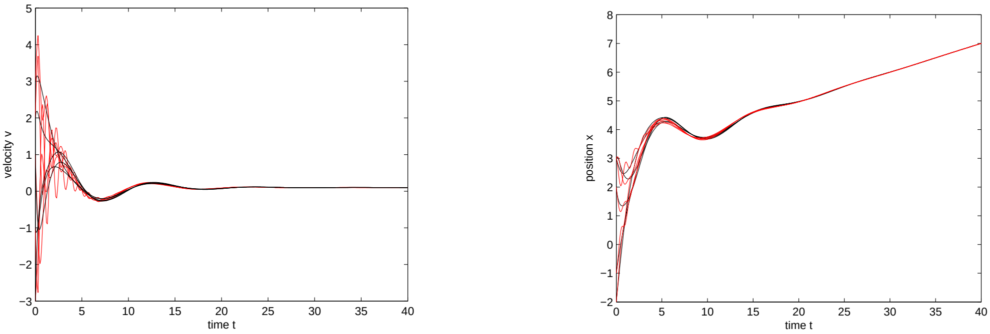
Table 2: Comparision between Iterative steps of protocol (2.8) with $\epsilon = 0.2$ and $\| x - x_0 \|_2 < 10^{-4}$.

parameter α	0.1	0.3	0.4	0.542	0.6	0.65
step k	142	140	139	137	140	215

of follower-agents under protocol (2.2). From the Figure 1(a), we can obtain that consensus is reached with a faster speed under protocol (2.2) than protocol (2.5) by choosing suitable parameter α, though the improved speed is not obviously. The final states of the follower-agents are consistent with Theorem 3.1.

Example 4.2 Figure 1 (b) is the state trajectories of multi-agent system under protocol (2.8) and (2.9), where the position of leader is 2.5. The red line is the trajectories of follower-agents under protocol (2.9), and the black line is the trajectories of follower-agents under protocol (2.8). From the Figure 1(b), we can also obtain that consensus is reached with a faster speed under protocol (2.8) than protocol (2.9) by choosing suitable parameter α, though the improved speed is not obviously. The final states of the follower-agents are consistent with Theorem 3.2.

Example 4.3 Figure 2 is the state trajectories of multi-agent system under



(a) Velocity trajectories under protocol (4.3) (b) Position trajectories under protocol (4.3)

Figure 3: State trajectories of multi-agent system (4.6)

protocol (2.8) with different parameters α and ϵ. where the position of leader is 2.5.

In Figure 2(a), the red line is the trajectories of agents with $\alpha = 0.3$ and $\epsilon = 0.1$, and the black line is the trajectories of follower-agents with $\alpha = 1$ and $\epsilon = 0.1$. Moreover, iterative steps of protocol (2.8) with different parameter α is listed in Table 1. When $\epsilon = 0.1$, consensus is reached with a fastest speed under protocol (2.8) at $\alpha = 1$, which is consistent with 1) of Theorem 3.3.

In Figure 2(b), the red line is the trajectories of follower-agents with $\alpha = 0.542$ and $\epsilon = 0.2$, the black line is the trajectories of follower-agents with $\alpha = 1$ and $\epsilon = 0.2$ and the blue line is the trajectories of follower-agents with $\alpha = 0.65$ and $\epsilon = 0.2$. Moreover, iterative steps of protocol (2.8) with different parameter α is listed in Table 2. When $\epsilon = 0.1$, consensus is reached with a fastest speed under protocol (2.8) at $\alpha = 0.542$, which is consistent with 3) of Theorem 3.3.

Example 4.4 Figure 3 is the state trajectories of multi-agent system under protocol (4.3) with different parameters α. where the initial position of leader is 2.5, velocity of leader is 0.1 and initial velocities of follower-agents is $V(0)^T = \begin{pmatrix} 1 & -3 & 2 & -1 & 3 \end{pmatrix}^T$. The red line is the trajectories of follower-agents with $\alpha = 0.55$, and the black line is the trajectories of follower-agents with $\alpha = 0$. From the Figure 3, we can also obtain that consensus is reached with a speed faster under protocol (4.3) by choosing suitable parameter α than parameter $\alpha = 0$, though the improved speed is not obviously. The final states of the follower-agents are consistent with Theorem 4.1.

6 Conclusions

In this paper, the leader-following consensus problems with memory are considered for both discrete time and continuous time single order multi-agent system. And the

sufficient conditions which consensus can reach are obtained with parameter α. The consensus problem of protocol (31) in [1] how to obtain fastest speed has been solved in our paper. Moreover, for a leader-following protocol with memory of second order continuous time multi-agent systems, the sufficient condition which consensus can reach is also obtained. Likewise, we find parameter α play an important role in multi-agent system's convergence speed. For continuous time multi-agent systems how to choose parameter α to obtain the fastest convergence speed is what we need to do in the future. Finally, numerical examples illustrate our theoretical results.

References

[1] J. Li, S. Xu, Y. Chu, H. Wang, Distributed average consensus control in networks of agents using outdated states, *IET Control Theory & Appl.*, 4(5)(2010): 746-758.

[2] N. A. Lynch, *Distributed Algorithms*, San Francisco, CA: Morgan Kaufmann, 1997.

[3] Y. P. Tian, C. L. Liu, Consensus of multi-agent systems with diverse input and communication delays, *IEEE Trans. Autom. Control*, 53(9)(2008): 2122-2128.

[4] W. Ren, Collective motion from consensus with cartesian coordinate coupling-Part I: Single-integrator Kinematics, *Proceedings of the 47th IEEE Conference on Decision and Control Cancun*, Mexico, Dec. 9-11, 2008: 1006-1011.

[5] S. C. Weller, N. C. Mann, Bassessing rater performance without a F gold standard using consensus theory, *Med. Decision Making*, 17(1)(1997): 71-79.

[6] J. D. Zhu, Y. P. Tian, J. Kuang, On the general consensus protocol of multi-agent systems with double integrator dynamics, *Lin. Alg. Appl.*, 431(2009): 701-715.

[7] P. Lin, Y. Jia, Further results on decentralised coordination in networks of agents with second-order dynamics, *IET Control Theory & Applications*, 3(7)(2009): 957 - 970.

[8] W. Ren, Collective motion from consensus with cartesian coordinate coupling - Part II: Double-integrator dynamics. *Proceedings of the 47th IEEE Conference on Decision and Control Cancun*, Mexico, Dec. 9-11, 2008: 1012-1017.

[9] H. Su, X. Wang, Second-order consensus of multiple agents with coupling delay, *Proceedings of the 7th world Congress on Intelligent Control and Automation*, 2008: 7181-7186

[10] V. Borkar and P. Varaiya, Asymptotic agreement in distributed estimation, *IEEE Trans. Autom. Control*, vol. AC-27(3)(1982): 650-655.

[11] J. N. Tsitsiklis, *Problems in Decentralized Decision Making and Computation*, Ph. D. Dissertation, Dept. Electr. Eng. Comput. Sci., Lab. Inf. Decision Syst., Massachusetts Inst. Technol., Cambridge, MA, Nov. 1984.

[12] T. Vicsek, A. Cziroo k, E. Ben-Jacob, I. Cohen, and O. Shochet, B, Novel type of phase transition in a system of self-deriven particles, *Phys. Rev. Lett.*, 75(6)(1995): 1226-1229.

[13] R.Olfati-Saber, R. Murray, Consensus problems in the networks of agents with switching topology and time delays, *IEEE Trans. Automatic Control*, 49(9)(2004): 1520-1533.

[14] L. Xiao and S. Boyd, Fast linear iterations for distributed averaging, *Syst. Control Letters*, 53(1)(2004): 65-78.

[15] L. Xiao, S. Boyd, and S. J. Kim, Distributed average consensus with least-mean-square deviation, *J. Paralle and Distributed Computing*, 67(1)(2007): 33-46.

[16] E. Kokiopoulou and P. Frossard, Polynomial filtering for fast convergence in distributed consensus, *IEEE Trans. Sig. Proc.*, 57(1)(2009): 342-354.

[17] T. Aysal, B. Oreshkin, and M. Coates, Accelerated distributed average consensus via localized node state prediction, *IEEE Trans. Sig. Proc.*, 57(4)(2009): 1563-1576.

[18] B. Oreshkin, M. Coates and M. Rabbat, Optimization and analysis of distributed averaging with short node memory, *IEEE Trans. Sig. Proc.*, 58(5)(2010): 2850-2865.

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Tripled coincidence points for mixed comparable mappings in partially ordered metric spaces *

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Abstract

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space, $g : X \to X$ be a mapping. In the paper, we introduce a new concept of mixed comparable property with respect to g, and obtain some tripled coincidence point theorems for such a class of mappings with this property. Our results generalize and extend the work of V. Berinde and M. Borcut [V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897]. Moreover, some examples are given to support our results.

Keywords: Metric space; Mixed comparable property; g- continuous; Tripled coincidence point

MSC 2000: 47H10

1 Introduction

Most recently, G. Bhaskar and Lakshmikantham [1] introduced the concepts of coupled fixed point and mixed monotone property for contractive operators of the form $F : X \times X \to X$, where X is a partially ordered metric space, and established some existence and uniqueness coupled fixed point theorems. F. Sabetghadam et al. [2] extended the results of Gnana Bhaskar and Lakshmikantham [1]. Based on the works of G. Bhaskar and Lakshmikantham and F. Sabetghadam, V. Berinde and M. Borcut [3] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces, and obtain existence, and existence and uniqueness theorems for contractive type mappings. The results given by V. Berinde and M. Borcut generalized and extended the works of G. Bhaskar and Lakshmikantham and F. Sabetghadam. Firstly, we review some concepts given by G. Bhaskar, Lakshmikantham and F. Sabetghadam.

Definition 1 ([1]). Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Further, endow the product space $X \times X$ with the following partial order: for

$(x, y), (u, v) \in X, (u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v.$

Definition 2 ([1]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone non increasing in y, that is, for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$

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Definition 3 ([1]). Call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping F if

$$F(x, y) = x, F(y, x) = y.$$

Consider on the product space $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Rightarrow x \leq u, y \geq v, z \leq w.$$

Definition 4 ([3]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x and z, and is monotone non-increasing in y, that is, for any $x, y, z \in X$,

$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z)$,

$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$, and

$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2)$.

Definition 5 ([3]). An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if $F(x, y, z) = x, F(y, x, y) = y$, and $F(z, y, x) = z$.

Let (X, d) be a metric space. The mapping $\widetilde{d} : X \times X \times X \to \mathbb{R}$ given by

$$d[(x, y, z), (u, v, w)] = d(x, u) + d(y, v) + d(z, w)$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d, too.

G. Bhaskar and Lakshmikantham established some interesting coupled fixed point theorems in [1]. Now, we state their main results.

Theorem 1 ([1]). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \quad \text{for each } x \geq u, y \leq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2 ([1]). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all n.

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume there exists $k \in [0, 1)$ such that (1.1) is satisfied for each $x \geq u, y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Also the uniqueness of coupled fixed points were considered, see [1] for more details.

F. Sabetghadam et al. [2] extended the results of G. Bhaskar and Lakshmikantham [1] by replacing the contractive condition (1.1) by a more general one, i.e., by considering the condition

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \tag{1.2}$$

where k, l are nonnegative constants with $k + l < 1$. Their main result reads as follows.

Theorem 3 ([2]). Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:

$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v)$,

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.
The concepts used in Theorem 3 we refer to [2].

In [3], V. Berinde and M. Borcut extended and generalized both the results of G. Bhaskar and Lakshmikantham [1] and F. Sabetghadam et al.[2] to the case of contractive operators of the form $F : X \times X \times X \to X$ in the presence of a contraction condition similar to (1.2). Their main results read as follows.

Theorem 4 ([3]). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1.3}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$.

By replacing the continuity of F by the conditions (i) and (ii) in the following theorem, V. Berinde and M. Borcut obtained the following theorem.

Theorem 5 ([3]). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that (1.3) is satisfied for each $x \geq u, y \leq v, z \geq w$. Assume that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all n,

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$.

To ensure the uniqueness of coupled fixed points, the following theorems complete the previous ones.

Theorem 6 ([3]). By adding to the hypothesis of Theorem 4 the condition: for every $(x, y, z), (x_1, y_1, z_1) \in X \times X \times X$, there exists a $(u, v, w) \in X \times X \times X$ that is comparable to (x, y, z) and (x_1, y_1, z_1), we obtain the uniqueness of the tripled fixed point of F.

Theorem 7 ([3]). In addition to the hypothesis of Theorem 4 (resp. Theorem 5) suppose that every triple of elements of X has an upper bound or lower bound in X. Then $x = y = z$.

Theorem 8 ([3]). In addition to the hypothesis of Theorem 4(resp. Theorem 5) suppose that $x_0, y_0, z_0 \in X$ are comparable. Then $x = y = z$.

Our main aim in this note is to extend and generalize the results of V. Berinde and M. Borcut [3]. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$, $g : X \to X$. We introduce the new concepts of g- continuous mapping and mixed comparable property with respect to g, and consider the existence, existence and uniqueness of the tripled coincidence points of g and F. It is notable that our methods could be done for other fixed point theorems, see [4–6] or for coupled fixed point, coupled common fixed point and coupled coincidence point results, e.g., [7–12].

2 Main results

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Let $x, y \in X$, we say that x, y are comparable if $x \le y$ or $y \le x$ holds. Let $F : X \times X \times X \to X$, $g : X \to X$ be two mappings. Following the basic concepts and results established in [3] and as generalizations, we introduce the new concepts of g- continuous mapping and mixed comparable property with respect to g and obtain some tripled coincidence point results for g and F, where F has the mixed comparable property with respect to g.

Definition 6. We say that F has the mixed comparable property with respect to g if $F(x, y, z)$ and $F(u, v, w)$ are comparable for any pair (x, y, z) and (u, v, w) in $X \times X \times X$ for which $g(x)$ and $g(u)$, $g(y)$ and $g(v)$, and $g(z)$ and $g(w)$ are comparable.

Remark 1. Obviously, the mixed comparable property with respect to g is a generalization of the mixed monotone property.

Definition 7. An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of g and F if $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$. If there exists $x \in X$ such that $g(x) = F(x, x, x)$, then we say x is a coincidence point of g and F.

A tripled fixed point of F can be looked as a tripled coincidence point of g and F if we take g as the identity mapping on X. But converse, in general, it is not true.

The mapping $\widetilde{d} : X \times X \times X \to X$, given by

$$\widetilde{d}[(x, y, z), (u, v, w)] = d(x, u) + d(y, v) + d(z, w),$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d, too.

Definition 8. If $g(x_n) \to g(x)$ together with $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$ implies $F(x_n, y_n, z_n) \to F(x, y, z)$ as $n \to \infty$ for any sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ of X, then F is said to be a g- continuous mapping.

Example 1. Let $\mathbb{R}$ be the set of all real numbers with the usual metric d, that is, $d(x, y) = |x - y|$ for any $x, y \in \mathbb{R}$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a mapping defined as follows: for any $x \in \mathbb{R}$, if $x \neq 0$, then $g(x) = \sin \frac{1}{x}$ and $g(x) = 0$ if $x = 0$. Let $F(x, y, z) = (2g(x), g(y), g(z))$ for any $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Obviously, F is g- continuous but not continuous.

Remark 2. The concept of g- continuous mapping generalizes that of continuous mapping, because if g is just taken as the identity mapping on X, then each g- continuous mapping is continuous.

The following theorem is our first main result.

Theorem 9. Let g be a surjection and F be a g- continuous mapping having the mixed comparable property with respect to g. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which

$$d(F(x, y, z), F(u, v, w)) \le jd(g(x), g(u)) + kd(g(y), g(v)) + ld(g(z), g(w)), \tag{2.1}$$

for any (x, y, z) and $(u, v, w) \in X \times X \times X$ satisfying that $g(x)$ and $g(u)$, $g(y)$ and $g(v)$, and $g(z)$ and $g(w)$ are comparable. If there exist x_0, y_0, $z_0 \in X$ such that $g(x_0)$ and $F(x_0, y_0, z_0)$, $g(y_0)$ and $F(y_0, x_0, y_0)$, and $g(z_0)$ and $F(z_0, y_0, x_0)$ are comparable, then there exists a tripled coincidence point of g and F.

Proof. Since g is a surjection, there exists $x_1 \in X$ such that $g(x_1) = F(x_0, y_0, z_0)$ and $g(x_1)$ and $g(x_0)$ are comparable. Similarly, there exist $y_1, z_1 \in X$ such that $g(y_1) = F(y_0, x_0, y_0)$ and $g(z_1) = F(z_0, y_0, x_0)$, furthermore $g(y_1)$ and $g(y_0)$ are comparable and $g(z_1)$ and $g(z_0)$ are comparable. Continuing this process and noting that F has the mixed comparable property with respect to g, for $n \ge 1$, we obtain that there exist $x_n, y_n, z_n \in X$, such that $g(x_n) = F(x_{n-1}, y_{n-1}, z_{n-1})$ and $g(x_n)$, $g(x_{n-1})$ are comparable, $g(y_n) = F(y_{n-1}, x_{n-1}, y_{n-1})$ and $g(y_n)$, $g(y_{n-1})$ are comparable, $g(z_n) = F(z_{n-1}, y_{n-1}, x_{n-1})$ and $g(z_n)$, $g(z_{n-1})$ are comparable. To simplify the writing, denote

$$D_n^x = d(g(x_{n-1}), g(x_n)), D_n^y = d(g(y_{n-1}), g(y_n)), D_n^z = d(g(z_{n-1}), g(z_n)).$$

Then by (2.1) we have

$$\begin{aligned} D_2^x &= d(g(x_1), g(x_2)) = d(F(x_0, y_0, z_0), F(x_1, y_1, z_1)) \\ &\leq jd(g(x_0), g(x_1)) + kd(g(y_0), g(y_1)) + ld(g(z_0), g(z_1)) \\ &= jD_1^x + kD_1^y + lD_1^z. \end{aligned}$$

Similarly, we have

$$D_2^y \leq (j+l)D_1^y + kD_1^x + 0 \cdot D_1^z,$$

$$D_2^z \leq jD_1^z + kD_1^y + lD_1^x$$

and

$$D_3^x \leq (j^2 + k^2 + l^2)D_1^x + (2jk + 2kl)D_1^y + 2jlD_1^z,$$
$$D_3^y \leq (kl + 2jk)D_1^x + ((j+l)^2 + k^2)D_1^y + klD_1^z,$$
$$D_3^z \leq (2jl + k^2)D_1^x + (2kj + 2kl)D_1^y + (j^2 + l^2)D_1^z.$$

For simplicity, we also consider the matrix

$$\mathbf{A} = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \text{denoted by} \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix}$$

and further denote

$$\mathbf{A^2} = \begin{pmatrix} j^2 + k^2 + l^2 & 2jk + 2kl & 2jl \\ kl + 2jk & (j+l)^2 + k^2 & kl \\ 2jl + k^2 & 2jk + 2kl & j^2 + l^2 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix},$$

where

$$a_2 + b_2 + c_2 = d_2 + e_2 + f_2 = g_2 + b_2 + h_2 = (j + k + l)^2 < j + k + l < 1. \tag{2.2}$$

Now we prove by induction that

$$\mathbf{A^n} = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix},$$

where

$$a_n + b_n + c_n = d_n + e_n + f_n = g_n + b_n + h_n = (j + k + l)^n < j + k + l < 1. \tag{2.3}$$

In fact, if we assume that (2.3) is true for n, then since

$$\begin{aligned} \mathbf{A^{n+1}} = \mathbf{A^n A} &= \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \\ &= \begin{pmatrix} ja_n + kb_n + lc_n & ka_n + (j+l)b_n + kc_n & la_n + jc_n \\ jd_n + ke_n + lf_n & kd_n + (j+l)e_n + kf_n & ld_n + jf_n \\ jg_n + kb_n + lh_n & kg_n + (j+l)b_n + kh_n & lg_n + jh_n \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned}
a_{n+1}+b_{n+1}+c_{n+1} &= a_nj+b_nk+c_nl+a_nk+b_nj+c_nk+a_nl+b_nl+c_nj \\
&= a_n(j+k+l)+b_n(k+j+l)+c_n(l+k+j) \\
&= (a_n+b_n+c_n)(j+k+l)=(j+k+l)^n(j+k+l) \\
&= (j+k+l)^{n+1}<j+k+l<1.
\end{aligned}$$

Similarly one has

$$d_{n+1}+e_{n+1}+f_{n+1}=g_{n+1}+b_{n+1}+h_{n+1}=(j+k+l)^{n+1}<j+k+l<1.$$

Therefore, we have

$$\begin{pmatrix} D^x_{n+1} \\ D^y_{n+1} \\ D^z_{n+1} \end{pmatrix} \le \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}^n \begin{pmatrix} D^x_1 \\ D^y_1 \\ D^z_1 \end{pmatrix}$$

that is

$$D^x_{n+1}\le a_nD^x_1+b_nD^y_1+c_nD^z_1, \tag{2.4}$$

$$D^y_{n+1}\le d_nD^x_1+e_nD^y_1+f_nD^z_1, \tag{2.5}$$

$$D^z_{n+1}\le g_nD^x_1+b_nD^y_1+h_nD^z_1. \tag{2.6}$$

Following from (2.4-2.6), we can easily to show that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences. In fact, for $m>n$, we obtain

$$\begin{aligned}
d(g(x_m),g(x_n)) &\le d(g(x_m),g(x_{m-1}))+\cdots+d(g(x_{n+1},g(x_n))=D^x_m+D^x_{m-1}+\cdots+D^x_{n+1} \\
&\le (a_{m-1}D^x_1+b_{m-1}D^y_1+c_{m-1}D^z_1)+\cdots+(a_nD^x_1+b_nD^y_1+c_nD^z_1) \\
&= (a_n+\cdots+a_{m-1})D^x_1+(b_n+\cdots+b_{m-1})D^y_1+(c_n+\cdots+c_{m-1})D^z_1 \\
&\le (\alpha^n+\alpha^{n+1}+\alpha^{m-1})D^x_1+(\alpha^n+\alpha^{n+1}+\alpha^{m-1})D^y_1+(\alpha^n+\alpha^{n+1}+\alpha^{m-1})D^z_1 \\
&= (\alpha^n+\alpha^{n+1}+\alpha^{m-1})\cdot(D^x_1+D^y_1+D^z_1) \\
&= \alpha^n\frac{1-\alpha^{m-n}}{1-\alpha}(D^x_1+D^y_1+D^z_1),
\end{aligned}$$

where $\alpha=j+k+l<1$, which shows that $\{g(x_n)\}$ is a Cauchy sequence.

Similarly one can verify that $\{g(y_n)\}$ and $\{g(z_n)\}$ are also Cauchy sequences.

Since X is a complete metric space and $g:X\to X$ is a surjection, there exist $x,y,z\in X$ such that

$$\lim_{n\to\infty}g(x_n)=g(x),\ \lim_{n\to\infty}g(y_n)=g(y),\ \lim_{n\to\infty}g(z_n)=g(z). \tag{2.7}$$

Finally, we prove $F(x,y,z)=g(x),F(y,x,y)=g(y),F(z,y,x)=g(z)$. By using the g- continuity of F and noting (2.7), we have, as $n\to\infty$,

$$g(x_{n+1})=F(x_n,y_n,z_n)\to F(x,y,z). \tag{2.8}$$

Thus $g(x)=F(x,y,z)$. Similarly, we have $g(y)=F(y,x,y)$ and $g(z)=F(z,y,x)$. Namely, (x,y,z) is a tripled coincidence point of g and F.

In the following theorem we replace the g- continuity of F by an additional property. We discuss this in the following theorem.

Theorem 10. Let g be a surjection and F be a mapping having the mixed comparable property with respect to g. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that (2.1) is satisfied for any (x, y, z) and $(u, v, w) \in X \times X \times X$ for which $g(x)$ and $g(u)$, $g(y)$ and $g(v)$, and $g(z)$ and $g(w)$ are comparable. Assume that X has the following property:

(i) if a sequence $\{x_n\} \subset X$ satisfying that $g(x_n)$ and $g(x_{n+1})$ are comparable for all n and $g(x_n)$ converges to $g(x)$, then $g(x_n)$ and $g(x)$ are comparable for all n.

Furthermore, if there exist x_0, y_0, $z_0 \in X$ such that $g(x_0)$ and $F(x_0, y_0, z_0)$, $g(y_0)$ and $F(y_0, x_0, y_0)$, and $g(z_0)$ and $F(z_0, y_0, x_0)$ are comparable. Then there exists a tripled coincidence point of g and F.

Proof. From the proof of Theorem 9, we get, for any $n > 0$, $g(x_n)$ and $g(x_{n+1})$ are comparable, the same argument holds for $\{g(y_n)\}$ and $\{g(z_n)\}$, and $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$. By condition (i), we obtain $g(x_n)$ and $g(x)$ are comparable for all n. Similarly, $g(y_n)$ and $g(y)$, and $g(z_n)$ and $g(z)$ are comparable for all n. Next we only have to prove that $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$.

Let $\varepsilon > 0$, Since $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$, there exists $N > 0$ such that for all $n \geq N$, we have $d(g(x_n), g(x)) < \frac{\varepsilon}{4}$, $d(g(y_n), g(y)) < \frac{\varepsilon}{4}$ and $d(g(z_n), g(z)) < \frac{\varepsilon}{4}$.

Taking $n > N$, we get

$$\begin{aligned} d(F(x,y,z), g(x)) &\leq d(F(x,y,z), g(x_{n+1})) + d(g(x_{n+1}), g(x)) \\ &= d(F(x,y,z), F(x_n, y_n, z_n)) + d(g(x_{n+1}), g(x)) \\ &\leq jd(g(x), g(x_n)) + kd(g(y), g(y_n)) + ld(g(z), g(z_n)) + d(g(x_{n+1}), g(x)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This implies that $F(x, y, z) = g(x)$. Similarly, we can show that

$$d(F(y,x,y), g(y)) < \varepsilon, \quad d(F(z,y,x), g(z)) < \varepsilon$$

which implies that $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$.

Example 2. Let $X = \mathbb{R}$, $d(x, y) = |x-y|$ and $g : X \to X$ defined as follows: for any $x \in X$, if $x \neq 0$, then $g(x) = \frac{1}{x^3}$ and $g(x) = 0$ if $x = 0$, and $F : X \times X \times X \to X$ defined by $F(x, y, z) = \frac{1}{8}g(x)+\frac{1}{4}g(y)-\frac{3}{8}g(z)+\frac{1}{8}$. It is easy to check that F satisfies (2.1) with $j = \frac{1}{8}, k = \frac{1}{4}$ and $l = \frac{3}{8}$ and $(2, 2, 2)$ is the unique tripled coincidence point of g and F.

3 Uniqueness of tripled coincidence point of g and F

In this section, we consider some additional conditions to ensure the uniqueness of the tripled coincidence point of g and F and appropriate conditions to ensure that for the tripled coincidence point (x, y, z) of g and F we have $x = y = z$.

Theorem 11. Suppose the hypothesis of Theorem 9 (resp. Theorem 10) and the following conditions hold:

(i) for every $(a, b, c), (a_1, b_1, c_1) \in X \times X \times X$, there exists a $(u, v, w) \in X \times X \times X$ such that $(g(u), g(v), g(w))$is comparable to $(g(a), g(b), g(c))$ and $(g(a_1), g(b_1), g(c_1))$,

(ii) g is an injection.

Then there exists a unique tripled coincidence point of g and F.

Proof. By the proof of Theorem 9 (resp. Theorem 10), we obtain that there exists $(x, y, z) \in X \times X \times X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$. If g and F have another tripled fixed point (u, v, w), then we prove that $g(x) = g(u)$, $g(y) = g(v)$ and $g(z) = g(w)$ as follows.

In fact, by the conditions of Theorem 11, there exists $(r, s, t) \in X \times X \times X$ such that $g(r)$ is comparable to $g(x)$ and $g(u)$, $g(s)$ is comparable to $g(y)$ and $g(v)$ and $g(t)$ is comparable to $g(z)$ and $g(w)$. Let $r_0 = r, s_0 = s, t_0 = t$. By a proof similar to that of Theorem 9 (resp. Theorem 10), we can prove that there exist sequences $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ such that, for any n,

$$g(r_{n+1}) = F(r_n, s_n, t_n), g(s_{n+1}) = F(s_n, r_n, s_n), g(t_{n+1}) = F(t_n, s_n, r_n).$$

Since F has the mixed comparable property with respect to g and by induction, we can prove easily that $g(r_n)$ is comparable to $g(x)$ for any n. Similarly, $g(s_n)$ is comparable to $g(y)$ and $g(t_n)$ is comparable to $g(z)$ for any n. Hence, from the conditions of Theorem 9 (resp. Theorem 10), we have

$$d(g(r_{n+1}), g(x)) = d(F(r_n, s_n, t_n), F(x, y, z)) \leq jd(g(r_n), g(x)) + kd(g(s_n), g(y)) + ld(g(t_n), g(z)). \quad (3.1)$$

$$d(g(s_{n+1}), g(y)) = d(F(s_n, r_n, s_n), F(y, z, y)) \leq jd(g(s_n), g(y)) + kd(g(r_n), g(x)) + ld(g(s_n), g(y)). \quad (3.2)$$

$$d(g(t_{n+1}), g(z)) = d(F(t_n, s_n, r_n), F(z, y, x)) \leq jd(g(t_n), g(z)) + kd(g(s_n), g(y)) + ld(g(r_n), g(x)). \quad (3.3)$$

Let $p_n = \max\{d(g(r_n), g(x)), d(g(s_n), g(y)), d(g(t_n), g(z))\}$. From (3.1), (3.2) and (3.3), we obtain that, for any n,

$$p_{n+1} \leq \alpha p_n, \quad (3.4)$$

where $\alpha = j + k + l < 1$, which shows that $\lim_{n\to\infty} g(r_n) = g(x)$, $\lim_{n\to\infty} g(s_n) = g(y)$ and

$$\lim_{n\to\infty} g(t_n) = g(z).$$

Similarly, we can prove that $\lim_{n\to\infty} g(r_n) = g(u)$, $\lim_{n\to\infty} g(s_n) = g(v)$ and $\lim_{n\to\infty} g(t_n) = g(w)$. Thus $g(x) = g(u), g(y) = g(v)$ and $g(z) = g(w)$. Noting that g is an injection, we have $x = u$, $y = v$ and $z = w$. Hence g and F have the unique tripled coincidence point.

Theorem 12. Suppose the hypothesis of Theorem 9 (resp. Theorem 10) and the following conditions hold: if g is an injection and x, y and z are mutually g- comparable. Then $x = y = z$, thus x is a coincidence point of g and F.

Proof. Suppose x, y and z are mutually comparable and g is an injection, by the mixed comparable property with respect to g of F, we have

$$d(g(x), g(z)) = d(F(x, y, z), F(z, y, x)) \leq (j + l)d(g(x), g(z)). \quad (3.5)$$

Due to $j + l < 1$, we have $g(x) = g(z)$. Since g is an injection, we get $x = z$. By the mixed comparable property with respect to g of F again, we obtain

$$d(g(x), g(y)) = d(F(x, y, z), F(y, x, y)) \leq jd(g(x), g(y)) + kd(g(y), g(x)) + ld(g(z), g(y)). \quad (3.6)$$

Noting $x = z$ and by (3.6), we have

$$d(g(x), g(y)) = d(F(x, y, z), F(y, x, y)) \leq (j + k + l)d(g(x), g(y)).$$

Since $j + k + l < 1$, we get $g(x) = g(y)$ which implies $x = y$ by the injective property of g. Therefore $x = y = z$ and x is a coincidence point of g and F.

Corollary 13. By adding to the hypothesis of Theorem 9 (resp. Theorem 10) the condition: X is a totally ordering set, we obtain that g and F have a unique coincidence point, that is, there exists a unique $x \in X$ such that $g(x) = F(x, x, x)$.

Proof. According to Theorem 12, it suffices to prove the uniqueness of the coincidence point of g and F. Suppose on the contrary that there exist two elements x and x_1 in X, such that $g(x) = F(x, x, x)$ and

$g(x_1) = F(x_1, x_1, x_1)$. Since X is a totally ordering set, $g(x)$ and $g(x_1)$ are comparable. By the mixed comparable property with respect to g of F, we have

$$d(g(x), g(x_1)) = d(F(x, x, x), F(x_1, x_1, x_1)) \leq (j + k + l)d(g(x), g(x_1)). \tag{3.7}$$

Noting $j + k + l < 1$, we get $g(x) = g(x_1)$ which implies $x = x_1$ as g is an injection.

Remark 3. The results of Theorem 11, Theorem 12 and Corollary 13 generalize the corresponding results of [3].

Example 3. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ and $F : X \times X \times X \to X$ and $g : X \to X$, as defined in Example 2. Then applying Corollary 13, we get that g and F have a unique coincidence point $(2, 2, 2)$.

References

[1] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7) (2006) 1379–1393.

[2] F. Sabetghadam, H. P. Masiha, A.H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory. Appl. 2009 (2009) 8. doi:10.1155/2009/125426. Art. ID 125426.

[3] V. Berinde, M. Borcut, Tripled coincidence point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897.

[4] A. Amini-Harandi, Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem, Math. Comput. Model., In Press, Corrected Proof, Available online 23 December 2011.

[5] M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput, 218 (2012) 5929–5936.

[6] M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput, 218 (2012) 7339–7346.

[7] N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983–992.

[8] S. Sedghi, I. Altun, N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Anal. 72 (3-4) (2010) 1298–1304.

[9] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–349.

[10] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72(12)(2010) 4508–4517.

[11] J. Harjani, B. Lopez, K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal. 74 (2011) 1749–1760.

[12] I. Altun, B. Damjanovic, D. Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (3) (2010) 310–316.

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Fixed Point Theorems for a Banach Type Contraction on Tvs-cone Metric Spaces Endowed with a Graph

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Abstract

We prove fixed point theorems for a Banach contractive type mapping on complete tvs-cone metric spaces associated with w-cone distance and endowed with a graph. The sufficient conditions for the existence are obtained. The main results extend some known others in the current literature.

Keywords: Banach contraction; connected digraph; orbitally continuous; tvs-cone metric space; w-cone distance

MSC: 47H09; 47H10.

1 Introduction and preliminaries

Let (X, d) be a metric space and $T : X \to X$. We say that T is a Picard operator (PO for short) if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$ (see Petruşel-Rus [1]).

It was proved in [2] that if (X, d) is complete and T is a contraction, *i.e.*,

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$ and for some $\alpha \in [0, 1)$, then T is a PO.

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Since then, there have been a number of articles concerning the fixed point theory of contractions/generalized contractions in a metric space.

Let (X, d) be a metric space endowed with a graph G. In 2007, Jachymski [3] first introduced the concept of G-contractions and proved some fixed point theorems in a metric space endowed with a graph. Recently, Bojor [4] defined the notion of G-Reich type mappings and obtained a fixed point theorem for such mappings in a metric space. Let us recall a few basic notions and concepts of graph theory.

Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph (or digraph) G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, *i.e.*, $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) : (y, x) \in G\}.$$

We denote $\tilde{G}$ by the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Recall that a graph G is *connected* if there is a path between any two vertices, that is, if x and y are vertices in a graph G, then there is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices from x to y of length N ($N \in \mathbb{N}$) such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$.

Using graph theory, we know the following definitions defined in [3]:

Definition 1.1. [3] Let (X, d) be a metric space endowed with a graph G. Then $T : X \to X$ is said to be a *G-contraction* if:

1. T preserves edges of G, *i.e.*,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \ \forall x, y \in X;$$

2. T decreases weights of edges of G in the following way: there exists $\alpha \in [0, 1)$, such that, for each $(x, y) \in E(G)$, we have:

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

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Definition 1.2. [3] A mapping $T : X \to X$ is called *orbitally G-continuous* if for all $x, y \in X$ and any sequence $\{k_n\}$ of positive integers,

$$T^{k_n}x \to y,\ (T^{k_n}x, T^{k_{n+1}}x) \in E(G) \Rightarrow T(T^{k_n}x) \to Ty.$$

Very recently, Ćirić et al. [5] introduced the concept of w-cone distance on a real Hausdorff topological vector space (tvs for short). This concept generalizes the w-distance on a metric space introduced and studied by Kada et al. [6].

Let E be a tvs with the zero vector θ. A proper, nonempty and closed subset P of E is called a *(convex) cone* if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{\theta\}$. We shall always assume that the cone P of E has a nonempty interior int P.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P on E by

$$x \preceq y \Leftrightarrow y - x \in P.$$

We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. $x \ll y$ will stand for $y - x \in \text{int } P$. The pair (E, P) is called an *ordered tvs*. The algebraic operations of the addition and the scalar multiplication in a tvs-cone metric are jointly continuous [5].

We next recall the concept of a tvs-cone metric.

Definition 1.3. [5] Let X be a nonempty set and (E, P) an ordered tvs. A function $d : X \times X \to E$ is called a *tvs-cone metric* and (X, d) is called a *tvs-cone metric space* if the following conditions hold:

(C1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 1.4. [5] Let (X, d) be a tvs-cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $\theta \ll c$, it is said that:

(a) $\{x_n\}$ is a *tvs-cone convergent sequence* if there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$;

(b) $\{x_n\}$ is a *tvs-cone Cauchy sequence* if there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$;

(c) (X, d) is *complete* if every tvs-cone Cauchy sequence is tvs-cone convergent in X.

Remark 1.5. [5] (1) If $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$, and (2) if $c \in \text{int } P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists n_0 such that $a_n \ll c$ for all $n > n_0$.

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Definition 1.6. [5] Let (X,d) be a tvs-cone metric space. Then $G : X \to P$ is *lower semi-continuous* at $x \in X$ if for any $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$,

$$G(x) \preceq G(x_n) + c,$$

whenever $\{x_n\}$ is a sequence in X and $x_n \to x$.

Definition 1.7. [5] Let (X,d) be a tvs-cone metric space. Then, a function $p : X \times X \to P$ is called a *w-cone distance* on X if the following are satisfied:

(W1) $p(x,z) \preceq p(x,y) + p(y,z)$ for all $x,y,z \in X$;

(W2) for any $x \in X$, $p(x,\cdot) : X \to P$ is lower semi-continuous;

(W3) for any $c \in E$ with $\theta \ll c$, there is δ in E with $\theta \ll \delta$ such that $p(z,x) \ll \delta$ and $p(z,y) \ll \delta$ imply $d(x,y) \ll c$.

Lemma 1.8. [5] *Let (X,d) be a tvs-cone metric space and let p be a w-cone distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in E converging to θ, and let $x,y,z \in X$. Then the following hold:*

(i) *If $p(x_n,y) \preceq \alpha_n$ and $p(x_n,z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = \theta$ and $p(x,z) = \theta$, then $y = z$.*

(ii) *If $p(x_n,y_n) \preceq \alpha_n$ and $p(x_n,z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*

(iii) *If $p(x_m,x_n) \preceq \alpha_n$ for any $m,n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a tvs-cone Cauchy sequence.*

(iv) *If $p(y,x_n) \preceq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a tvs-cone Cauchy sequence.*

We now define the concept of a (G,p)-contraction in a tvs-cone metric space.

Definition 1.9. Let (X,d) be a tvs-cone metric space associated with w-cone distance p and endowed with a graph G. Then $T : X \to X$ is said to be a *G-Banach contraction defined on p* or simply a *(G,p)-contraction* if:

1. $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G),\ \forall x,y \in X$;
2. there exists $\alpha \in [0,1)$ such that

$$(x,y) \in E(G) \Rightarrow p(Tx,Ty) \preceq \alpha p(x,y)$$

for each $(x,y) \in E(G)$.

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2 Main results

In this section, we prove fixed point theorems for a (G, p)-contraction in a tvs-cone metric space associated with a w-cone distance. To this end, we need the following propositions:

Proposition 2.1. *Let (X, d) be a tvs-cone metric space associated with w-cone distance p and endowed with a connected digraph G. Let $T : X \to X$ be a (G, p)-contraction. Then, for each $x \in X$, $\{T^n x\}$ is a tvs-cone Cauchy sequence in X.*

Proof. Let $x \in X$. We consider the following two cases:

Case 1. If $(x, Tx) \in E(G)$, then $(T^n x, T^{n+1}x) \in E(G)$. Since T is a (G, p)-contraction, there exists $\alpha \in [0, 1)$ such that

$$p(T^n x, T^{n+1}x) \preceq \alpha p(T^{n-1}x, T^n x)$$

for all $n \in \mathbb{N}$. It also follows that

$$p(T^n x, T^{n+1}x) \preceq \alpha^n p(x, Tx)$$

for all $n \in \mathbb{N}$. Hence, for $m > n$, we have from Definition 1.7 (W1)

$$\begin{aligned} p(T^n x, T^m x) &\preceq p(T^n x, T^{n+1}x) + p(T^{n+1}x, T^{n+2}x) + ... + p(T^{m-1}x, T^m x) \\ &\preceq \alpha^n p(x, Tx) + \alpha^{n+1} p(x, Tx) + ... + \alpha^{m-1} p(x, Tx) \\ &= (\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}) p(x, Tx) \\ &\preceq \frac{\alpha^n}{1-\alpha} p(x, Tx). \end{aligned}$$

From Lemma 1.8 (iii), $\{T^n x\}$ is a tvs-cone Cauchy sequence.

Case 2. If $(x, Tx) \notin E(G)$, since G is a connected digraph, then there exists a path $(x_i)_{i=0}^N$ from x to Tx such that $x_0 = x$, $x_N = Tx$ with $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, ..., N$. So we have $(T^n x_{i-1}, T^n x_i) \in E(G)$ for all $i = 1, ..., N$. Thus, for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} p(T^n x, T^{n+1}x) &= p(T^n x_0, T^n x_N) \\ &\preceq p(T^n x_0, T^n x_1) + p(T^n x_1, T^n x_2) + ... + p(T^n x_{N-1}, T^n x_N) \\ &\preceq \alpha^n p(x_0, x_1) + \alpha^n p(x_1, x_2) + ... + \alpha^n p(x_{N-1}, x_N) \\ &= \alpha^n r, \end{aligned}$$

where $r = p(x_0, x_1) + p(x_1, x_2) + ... + p(x_{N-1}, x_N)$. Hence, for $m > n$, we can show that

$$p(T^n x, T^m x) \preceq \frac{\alpha^n}{1-\alpha} r.$$

This implies that $\{T^n x\}$ is a tvs-cone Cauchy sequence. □

Proposition 2.2. *Let (X, d) be a complete tvs-cone metric space associated with w-cone distance p and endowed with a connected digraph G. Let $T : X \to X$ be a (G, p)-contraction. Then, for each $x \in X$, there exists a unique point x^* in X such that $T^n x \to x^*$.*

Proof. Let $x, y \in X$. Then $\{T^n x\}$ and $\{T^n y\}$ are Cauchy sequences in X by Proposition 2.2. Since X is complete, there exist $x^*, y^* \in X$ such that $T^n x \to x^*$ and $T^n y \to y^*$. Let $c \in E$ with $\theta \ll c$. Observe that if $(x, Tx) \in E(G)$, by Definitions 1.6 and 1.7 (W2), then there is $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} p(T^n x, x^*) &\preceq p(T^n x, T^m x) + c \\ &\preceq \frac{\alpha^n}{1-\alpha} p(x, Tx) + c \end{aligned}$$

for all $m > m_0$. Taking $c = \frac{c}{j}$, we have

$$p(T^n x, x^*) \preceq \frac{\alpha^n}{1-\alpha} p(x, Tx) + \frac{c}{j} \tag{2.1}$$

for each $j \geq 1$. From (2.1) we see that

$$\frac{\alpha^n}{1-\alpha} p(x, Tx) + \frac{c}{j} - p(T^n x, x^*) \in P.$$

By the continuity of algebraic operations and the closedness of P, we get

$$\lim_{j\to\infty} \frac{\alpha^n}{1-\alpha} p(x, Tx) + \frac{c}{j} - p(T^n x, x^*) = \frac{\alpha^n}{1-\alpha} p(x, Tx) - p(T^n x, x^*) \in P.$$

This shows that

$$p(T^n x, x^*) \preceq \frac{\alpha^n}{1-\alpha} p(x, Tx) \tag{2.2}$$

for all $n \in \mathbb{N}$. Also, if $(y, Ty) \in E(G)$ then $p(T^n y, y^*) \preceq \frac{\alpha^n}{1-\alpha} p(y, Ty)$ for all $n \in \mathbb{N}$.

On the other hand, if $(x, Tx) \notin E(G)$ then, for all $m > m_0$,

$$\begin{aligned} p(T^n x, x^*) &\preceq p(T^n x, T^m x) + c \\ &\preceq \frac{\alpha^n}{1-\alpha} r + c \end{aligned}$$

where $r = p(x, x_1) + p(x_1, x_2) + ... + p(x_{N-1}, Tx)$. In a similar way, we can show that

$$p(T^n x, x^*) \preceq \frac{\alpha^n}{1-\alpha} r \tag{2.3}$$

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for all $n \in \mathbb{N}$. Also, if $(y, Ty) \notin E(G)$, then $p(T^n y, y^*) \preceq \frac{\alpha^n}{1-\alpha} s$, where $s = p(y, y_1) + p(y_1, y_2) + ... + p(y_{M-1}, Ty)$ and $(y_i)_{i=0}^M$ is a path from y to Ty such that $y_0 = y$, $y_M = Ty$ with $(y_{i-1}, y_i) \in E(G)$ for all $i = 1, 2, ..., M$.

We next consider the following two cases:

Case 1. Suppose $(x, y) \in E(G)$. If $(y, Ty) \in E(G)$, then there exists $m_1 \in \mathbb{N}$ such that

$$\begin{aligned} p(T^n x, y^*) &\preceq p(T^n x, T^m y) + c \\ &\preceq p(T^n x, T^n y) + p(T^n y, T^m y) + c \\ &\preceq \alpha^n p(x, y) + \frac{\alpha^n}{1-\alpha} p(y, Ty) + c \end{aligned}$$

for $m > m_1$. So we have

$$p(T^n x, y^*) \preceq \alpha^n \big(p(x, y) + \frac{1}{1-\alpha} p(y, Ty)\big)$$

for all $n \in \mathbb{N}$. Using (2.2) and (2.3), we obtain $x^* = y^*$ by Lemma 1.8 (i). In case $(y, Ty) \notin E(G)$ we also obtain $x^* = y^*$.

Case 2. Suppose $(x, y) \notin E(G)$. Then there exists a path $(x_i)_{i=0}^L$ from x to y such that $x_0 = x$ and $x_L = y$ with $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., L$. So if $(y, Ty) \in E(G)$, then

$$\begin{aligned} p(T^n x, y^*) &\preceq p(T^n x, T^n x_1) + ... + p(T^n x_{L-1}, T^n y) + p(T^n y, y^*) \\ &\preceq \alpha^n p(x, x_1) + ... + \alpha^n p(x_{L-1}, y) + \frac{\alpha^n}{1-\alpha} p(y, Ty) \\ &= \alpha^n \big(p(x, x_1) + ... + p(x_{L-1}, y) + \frac{1}{1-\alpha} p(y, Ty)\big) \end{aligned}$$

This concludes that $x^* = y^*$. In case $(y, Ty) \notin E(G)$ we also obtain $x^* = y^*$. $\square$

Now we are ready to prove the main theorem.

Theorem 2.3. *Let (X, d) be a complete tvs-cone metric space associated with w-cone distance p and endowed with a connected digraph G. Let $T : X \to X$ be a (G, p)-contractive and G-orbitally continuous mapping. Assume that there exists $z \in X$ such that $(z, Tz) \in E(G)$. Then T is a PO. If $v = Tv$, then $p(v, v) = \theta$.*

Proof. Let $z \in X$ be such that $(z, Tz) \in E(G)$. Then $(T^n z, T^{n+1} z) \in E(G)$. From Proposition 2.2, there exists a unique point x^* in X such that $T^n z \to x^*$. Since T is G-orbitally continuous, $T^{n+1} z \to Tx^*$. Hence $x^* = Tx^*$ and $x^* \in F(T)$. If $y^* = Ty^*$, then $x^* = y^*$. Therefore T is a PO. Moreover, if $v = Tv$, then $p(v, v) = p(Tv, Tv) \preceq \alpha p(v, v)$. Thus $p(v, v) = \theta$ by Remark 1.5 (i). This completes the proof. $\square$

As a direct consequence of Theorem 2.3, we obtain the following result:

Theorem 2.4. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) a complete tvs-cone metric space associated with w-cone distance p. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following hold:*

(i) *there exists $\alpha \in [0, 1)$ such that*

$$p(Tx, Ty) \preceq \alpha p(x, y)$$

for each $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exists $z \in X$ such that $z \sqsubseteq Tz$.*

Then T is a PO. If $v = Tv$, then $p(v, v) = \theta$.

Proof. Set $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. □

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References

[1] A. Petruşel, I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134 (2006) 411-418.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133-181.

[3] J. Jachymski, The contracion principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2007) 1359-1373.

[4] F. Bojor, Fixed point theorems for Reich type contractions on metric spaces with a graph, Nonlinear Anal. TMA. 75 (2012) 3895-3901.

[5] L. Ċirić, H. Lakzian, V. Rakočević, Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces, Fixed Point Theory Appl. 2012, 2012:3.

[6] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996) 381-591.

APPROXIMATE (m,n)-CAUCHY–JENSEN MAPPINGS IN QUASI-β-NORMED SPACES

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Abstract. In 1940 S.M. Ulam proposed the famous Ulam stability problem. In this paper we introduce a general (m,n)-Cauchy–Jensen functional equation and establish new theorems about the Ulam stability of general (m,n)-Cauchy–Jensen additive mappings in quasi-β-normed spaces, which generalize results obtained for Cauchy–Jensen type additive mappings.

1. Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation? Thus we say a functional equation $E_1(f) = E_2(f)$ is stable if any mapping g subject to $d(E_1(g), E_2(g)) \le \varphi$ satisfying approximately the equation with some controlled function φ is near to an exact solution f such that $E_1(f) = E_2(f)$ and $d(f(x), g(x)) \le \Phi(x)$ for some function Φ depending on the given function φ. The stability problem was first raised by S.M. Ulam [24] during his talk at the University of Wisconsin in 1940. We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$? In 1941, D.H. Hyers [9] gave an affirmative answer to Ulam's problem for the case of approximate additive mappings under the assumption that G and G' are Banach spaces. And then T. Aoki [2], D.G. Bourgin [4] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences for approximate additive mappings. In 1978, Th.M. Rassias [21] provided a generalization of Hyers' theorem for approximate linear mappings by considering the case when the Cauchy difference is controlled by a sum of unbounded function $\varepsilon(\|x\|^p + \|y\|^p), 0 < p < 1$. J.M. Rassias [17] established another stability theorem for the unbounded Cauchy difference controlled by a product of unbounded function $\varepsilon(\|x\|^p \cdot \|y\|^q), p + q \ne 1$. Zhou [25] used a stability property of the functional equation $f(x-y) + f(x+y) = 2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of

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a mapping and the degree of its approximation by the associated Bernstein polynomials. The stability problems of several functional equations had been extensively investigated by a number of authors and there are many interesting results concerning these problem [7, 10, 11, 12, 22]. These stability results can be applied in stochastic analysis [13], probability theory [16], financial and actuarial mathematics [5], as well as in psychology and sociology [1].

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation, which is introduced by the first author,

$$\sum_{\substack{1 \le i_1 < \cdots < i_m \le n \\ 1 \le k_l (\ne i_j, \forall j \in \{1, \cdots, m\}) \le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^n f(x_i) \tag{1.1}$$

for all $x_1, \cdots, x_n \in X$, where $n, m \in \mathbb{N}$ are fixed integers with $n \ge 2$, $1 \le m \le n$. Specially, we observe that in case $m = 1$ the equation (1.1) yields Cauchy additive equation

$$f(\sum_{j=1}^n x_j) = \sum_{i=1}^n f(x_i).$$

In case $m = n$, the equation (1.1) yields Jensen equation

$$f\left(\frac{\sum_{j=1}^n x_j}{n}\right) = \frac{1}{n}\sum_{i=1}^n f(x_i),$$

of which the general solution is of the form $f(x) - f(0) := A(x)$ being additive. Therefore, the equation (1.1) is a generalized form of the Cauchy and Jensen additive equation and thus every solution of the equation (1.1) may be analogously called a *general* (m, n)*-Cauchy–Jensen additive mapping*. In particular, if $m = 2$ and $n = 3$, then the equation (1.1) yields Cauchy additive equation

$$f(\frac{x+y}{2} + z) + f(\frac{y+z}{2} + x) + f(\frac{x+z}{2} + y) = 2[f(x) + f(y) + f(z)],$$

of which stability results have been investigated in [14, 15]. For the case $m = 2$, the authors have already established the general solution of the equation (1.1) and new theorems about the Hyers–Ulam stability in quasi-β-normed spaces [19]. And then, the authors have investigated approximate homomorphisms and derivations on C^*-ternary algebras associated with the functional equation [18]. Recently, J.M. Rassias, K. Jun and H. Kim [20] have investigated approximate (m, n)-Cauchy–Jensen additive mappings in C^*-algebras associated with stability results for the equation (1.1).

In this paper, we are to establish the general solution of the equation (1.1), and we are going to investigate the generalized Hyers–Ulam stability problem for the equation (1.1) in quasi-β-normed linear spaces. As corollaries, we obtain the generalized

results of the Hyers–Ulam stability theorem for the equation (1.1) in normed linear spaces and thus we generalize results obtained for Cauchy–Jensen type additive mappings.

2. General (m,n)-Cauchy–Jensen additive mappings

First, we establish the general solution of the equation (1.1).

Theorem 2.1. *Let X and Y be linear spaces. For each m with $1 \le m \le n$, a mapping $f : X \to Y$ satisfies the equation (1.1) for all $n \ge 2$ if and only if $f - f(0)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f(\lambda x) = \lambda f(x)$ and $f(mx) = mf(x)$ for all $x \in X$, where $\lambda := n - m + 1$.*

Proof. Let m be fixed. If $n = 2$, then the result is correct. Assume by induction that f satisfies the equation (1.1) for $n = 3, \cdots, n$ and then f is Cauchy additive for $n = 3, \cdots, n$. Without loss of generality, we let $f(0) = 0$. Replacing $x_i := x$ for all $i \in \{1, \cdots, n\}$ in the equation (1.1), we get $f(\lambda x) = \lambda f(x)$. Substituting x_1 for x and x_i for 0 for all $i \in \{2, \cdots, n\}$ in the equation (1.1), we get $f(mx) = mf(x)$. Now, suppose f satisfies the equation (1.1) for $n+1$. Then by setting $x_3 = \cdots = x_{n+1} = 0$ in the equation (1.1), one has

$$\frac{1}{m}\binom{n-1}{m-2} f(x_1 + x_2) + \binom{n-1}{m-1}\left[f(\frac{x_1}{m} + x_2) + f(x_1 + \frac{x_2}{m})\right] + \binom{n-1}{m} f(x_1 + x_2) = \frac{n-m+2}{n+1}\binom{n+1}{m}[f(x_1) + f(x_2)],$$

which yields similarly

$$(m-1)f(x_1 + x_2) + m(n-m+1)\left[f(\frac{x_1}{m} + x_2) + f(x_1 + \frac{x_2}{m})\right] + (n-m+1)(n-m)f(x_1 + x_2) = n(n-m+2)[f(x_1) + f(x_2)]. \tag{2.1}$$

By inductive assumption on n, we have by setting $x_3 = \cdots = x_n = 0$ in the equation (1.1)

$$\frac{1}{m}\binom{n-2}{m-2} f(x_1 + x_2) + \binom{n-2}{m-1}\left[f(\frac{x_1}{m} + x_2) + f(x_1 + \frac{x_2}{m})\right] + \binom{n-2}{m} f(x_1 + x_2) = \frac{n-m+1}{n}\binom{n}{m}[f(x_1) + f(x_2)],$$

which yields according to Pascal's identity and other identities from combinatorial analysis

$$(m-1)f(x_1 + x_2) + m(n-m)\left[f(\frac{x_1}{m} + x_2) + f(x_1 + \frac{x_2}{m})\right] + (n-m)(n-m-1)f(x_1 + x_2) = (n-1)(n-m+1)[f(x_1) + f(x_2)]. \tag{2.2}$$

First, we note that if $m = n$ in the equation (1.1) then the equation (1.1) reduces Jensen equation

$$\sum_{1 \le i_1 < \cdots < i_n \le n} f\left(\frac{\sum_{j=1}^{n} x_{i_j}}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

and so f is Cauchy additive by assumption $f(0) = 0$. On the other case, if $m < n$ in the equation (1.1) then associating the equation (2.1) with the equation (2.2) to eliminate the common term $f(\frac{x_1}{m} + x_2) + f(x_1 + \frac{x_2}{m})$, we get

$$A_m(n)f(x_1 + x_2) = B_m(n)[f(x_1) + f(x_2)], \tag{2.3}$$

where

$$A_m(n) = m - 1 - \frac{m(n-m+1)(m-1) + m(n-m+1)(n-m)(n-m-1)}{m(n-m)} \\ +(n-m+1)(n-m),$$

$$B_m(n) = n(n-m+2) - \frac{m(n-m+1)(n-1)(n-m+1)}{m(n-m)}.$$

It is now easily observed that $A_m(n) = B_m(n) \neq 0$. Therefore the mapping f satisfying the equation (1.1) for $n+1$ is also Cauchy additive. Hence f is Cauchy additive for all (m, n).

Conversely, if a mapping f is Cauchy additive by assuming $f(0) = 0$ without loss of generality then it is easy to see that f satisfies the equation (1.1). $\square$

3. Stability of general (m, n)-Cauchy–Jensen additive mappings

We consider some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-β-Banach space* is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a *(β, p)-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a *(β, p)-Banach space*. We note that for the case $\beta = 1$ the quasi-1-normed space, $(1, p)$-Banach space are in fact quasi-normed spaces, p-Banach spaces, respectively, and one can refer to [3, 23] for the concept of quasi-normed spaces and p-Banach spaces. It is well known that given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. Thus, each quasi-norm is equivalent to some p-norm by the Aoki–Rolewicz theorem [3, 23].

We recall that a subadditive function is a function $\phi : A \to B$, having a domain A and a codomain $(B, \leq)$ that are both closed under addition, with the following property:

$$\phi(x + y) \leq \phi(x) + \phi(y),\ \forall x, y \in A.$$

Now, we say that a function $\phi : A \to B$ is *contractively subadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \leq L[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. It follows by the contractively subadditive condition of ϕ that

$$\phi(\lambda x) \leq \lambda L \phi(x), \quad \text{and so} \quad \phi(\lambda^i x) \leq (\lambda L)^i \phi(x), i \in \mathbb{N}$$

for all $x \in A$ and all positive integer $\lambda \geq 2$.

Similarly, it is said that a function $\phi : A \to B$ is *expansively superadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then it follows that ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$, $\phi(\frac{x}{2^n}) \leq (\frac{L}{2})^n \phi(x)$ and so $\phi(0) = 0$. Further, we observe that an expansively superadditive mapping ϕ satisfies the following properties $\phi(\lambda x) \geq \frac{\lambda}{L}\phi(x)$ and so $\phi(\frac{x}{\lambda^i}) \leq (\frac{L}{\lambda})^i \phi(x), i \in \mathbb{N}$ for all $x \in A$ and all positive integer $\lambda \geq 2$.

Throughout this section, we assume that X is a quasi-α-normed linear space with quasi-α-norm $\|\cdot\|_X$ and Y is a (β, p)-Banach space with p-norm $\|\cdot\|_Y$. Let $K \geq 1$ be the modulus of concavity of $\|\cdot\|_Y$ and let $1 \leq m < n, \lambda := n-m+1$ otherwise specific reference. Now we are going to investigate the modified Hyers–Ulam stability of the functional equation (1.1). For notational convenience, given a mapping $f : X \to Y$, we define the difference operator $Df : X^n \to Y$ of the equation (1.1) as

$$\begin{aligned} &Df(x_1, x_2, \cdots, x_n) \\ :=& \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l} \right) \\ &- \frac{n-m+1}{n} \binom{n}{m} \sum_{i=1}^n f(x_i) \end{aligned}$$

for all n-variables $x_1, \cdots, x_n \in X, (n \geq 2)$ which acts as a perturbation of the equation (1.1).

Theorem 3.1. *Assume that there exists a mapping $\varphi : X^n = \overbrace{X \times \cdots \times X}^{n-times} \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\|_Y \leq \varphi(x_1, \cdots, x_n) \tag{3.1}$$

for all n-variables $x_1, \cdots, x_n \in X$, and that the map φ is contractively subadditive with a constant $L, 0 < L < 1$, satisfying $\lambda^{1-\beta} L < 1$. Then there exists a unique (m,n)-Cauchy–Jensen additive mapping $T : X \to Y$ which satisfies the equation

(1.1) and the inequality

$$\|f(x) - T(x)\|_Y \le \frac{\varphi(\overbrace{x, \cdots, x}^{n-times})}{\binom{n}{m}^{\beta} \sqrt[p]{(\lambda^{\beta p} - \lambda^p L^p)}} \tag{3.2}$$

for all $x \in X$.

Proof. Substituting x for $x_1, \cdots, x_n$ in the functional inequality (3.1), we obtain

$$\left\| \binom{n}{m} f((n-m+1)x) - \binom{n}{m}(n-m+1) f(x) \right\|_Y \le \varphi(x, \cdots, x),$$

$$\left\| \frac{f(\lambda x)}{\lambda} - f(x) \right\|_Y \le \frac{1}{\binom{n}{m}^{\beta} \lambda^{\beta}} \varphi(x, \cdots, x) \tag{3.3}$$

for all $x \in X$. Therefore it follows from (3.3) with $\lambda^i x$ in place of x and iterative method that

$$\begin{aligned} \left\| \frac{f(\lambda^l x)}{\lambda^l} - \frac{f(\lambda^k x)}{\lambda^k} \right\|_Y^p &\le \frac{1}{\binom{n}{m}^{\beta p} \lambda^{\beta p}} \sum_{i=k}^{l-1} \frac{1}{\lambda^{\beta p i}} \varphi(\lambda^i x, \cdots, \lambda^i x)^p \\ &\le \frac{1}{\binom{n}{m}^{\beta p} \lambda^{\beta p}} \sum_{i=k}^{l-1} \frac{1}{\lambda^{\beta p i}} (\lambda L)^{pi} \varphi(x, \cdots, x)^p \\ &= \frac{\varphi(x, \cdots, x)^p}{\binom{n}{m}^{\beta p} \lambda^{\beta p}} \sum_{i=k}^{l-1} (\lambda^{1-\beta} L)^{pi} \end{aligned} \tag{3.4}$$

for all $x \in X$ and for any $l > k \ge 0$. Thus it follows by taking the limit $k \to \infty$ in (3.4) that a sequence $\left\{ \frac{f(\lambda^l x)}{\lambda^l} \right\}$ is Cauchy in the complete space Y and so it converges in Y. Therefore we see that a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{l \to \infty} \frac{f(\lambda^l x)}{\lambda^l} = \lim_{l \to \infty} \frac{f((n-m+1)^l x)}{(n-m+1)^l}$$

is well defined for all $x \in X$. In addition it is clear from (3.1) that the following inequality

$$\begin{aligned} \|DT(x_1, \cdots, x_n)\|_Y^p &= \lim_{l \to \infty} \frac{\|Df(\lambda^l x_1, \cdots, \lambda^l x_n)\|_Y^p}{\lambda^{\beta p l}} \\ &\le \lim_{l \to \infty} \frac{\varphi(\lambda^l x_1, \cdots, \lambda^l x_n)^p}{\lambda^{\beta p l}} \\ &\le \lim_{l \to \infty} (\lambda^{1-\beta} L)^{pl} \varphi(x_1, \cdots, x_n)^p = 0 \end{aligned}$$

holds for all $x_1, \cdots, x_n \in X$. Therefore the mapping T is (m, n)-Cauchy–Jensen additive and so it is Cauchy additive by Theorem 2.1. Taking the limit $l \to \infty$ in (3.4) with $k = 0$, we find that the mapping T is Cauchy additive mapping satisfying the inequality (3.2) near the approximate mapping $f : X \to Y$ of the equation (1.1).

To prove the afore-mentioned uniqueness, we assume now that there is another (m,n)-Cauchy–Jensen additive mapping $T' : X \to Y$ which satisfies the equation (1.1) and the inequality (3.2). Then one proves by the last equality and (3.2) that

$$\begin{aligned}\left\|\frac{f(\lambda^l x)}{\lambda^l} - T'(x)\right\|_Y &= \frac{1}{\lambda^{\beta l}}\|f(\lambda^l x) - T'(\lambda^l x)\|_Y \le \frac{1}{\lambda^{\beta l}}\frac{\varphi(\overbrace{\lambda^l x, \cdots, \lambda^l x}^{n-times})}{\binom{n}{m}^{\beta}\sqrt[p]{(\lambda^{\beta p} - \lambda^p L^p)}} \\ &\le \frac{\varphi(\overbrace{x, \cdots, x}^{n-times})}{\binom{n}{m}^{\beta}\sqrt[p]{(\lambda^{\beta p} - \lambda^p L^p)}}(\lambda^{1-\beta}L)^l\end{aligned}$$

for all $x \in X$ and all $l \in \mathbb{N}$. Therefore from $l \to \infty$, one establishes

$$T(x) - T'(x) = 0$$

for all $x \in X$, completing the proof of uniqueness. $\square$

Theorem 3.2. *Assume that there exists a mapping* $\varphi : \overbrace{X \times \cdots \times X}^{n-times} \to [0,\infty)$ *for which a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n) \tag{3.5}$$

for all n-variables $x_1, \cdots, x_n \in X$, *and that the map* φ *is expansively superadditive with a constant* $L, 0 < L < 1$, *satisfying* $\lambda^{\beta-1}L < 1$. *Then there exists a unique* (m,n)*-Cauchy–Jensen additive mapping* $T : X \to Y$ *which satisfies the equation (1.1) and the inequality*

$$\|f(x) - T(x)\|_Y \le \frac{L\varphi(\overbrace{x, \cdots, x}^{n-times})}{\binom{n}{m}^{\beta}\sqrt[p]{(\lambda^p - \lambda^{\beta p} L^p)}} \tag{3.6}$$

for all $x \in X$.

Proof. It follows from (3.3) with $\frac{x}{\lambda^i}$ in place of x and iterative method that

$$\begin{aligned}\left\|\lambda^l f(\frac{x}{\lambda^l}) - \lambda^k f(\frac{x}{\lambda^k})\right\|_Y^p &\le \sum_{i=k}^{l-1}\lambda^{\beta p i}\left\|f(\frac{x}{\lambda^i}) - \lambda f(\frac{x}{\lambda^{i+1}})\right\|_Y^p \\ &\le \frac{1}{\binom{n}{m}^{\beta p}\lambda^{\beta p}}\sum_{i=k}^{l-1}\lambda^{(i+1)\beta p}\varphi(\frac{x}{\lambda^{i+1}}, \cdots, \frac{x}{\lambda^{i+1}})^p \\ &\le \frac{\varphi(x, \cdots, x)^p}{\binom{n}{m}^{\beta p}\lambda^{\beta p}}\sum_{i=k}^{l-1}(\lambda^{\beta-1}L)^{(i+1)p}\end{aligned} \tag{3.7}$$

for all $x \in X$ and for any $l > k \ge 0$. Therefore we see that a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{l\to\infty}\lambda^l f(\frac{x}{\lambda^l}) = \lim_{l\to\infty}(n-m+1)^l f(\frac{x}{(n-m+1)^l})$$

is well defined for all $x \in X$. Taking the limit $l \to \infty$ in (3.7) with $k = 0$, we find that the mapping T is (m, n)-Cauchy–Jensen additive mapping satisfying the inequality (3.6) near the approximate mapping $f : X \to Y$ of the equation (1.1).

The remaining assertion goes through by the similar way to corresponding part of Theorem 3.1. □

Next, we are going to establish alternative theorems concerning the stability of the equation (1.1).

Theorem 3.3. *Assume that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. If the function $\varphi : X^n \to [0, \infty)$ satisfies

$$\Phi(x_1, \cdots, x_n) := \sum_{i=0}^{\infty} \frac{K^i \varphi(\lambda^i x_1, \cdots, \lambda^i x_n)}{\lambda^{\beta i}} < \infty$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique (m, n)-Cauchy–Jensen additive mapping $T : X \to Y$ such that T satisfies the equation (1.1) and the inequality

$$\|f(x) - T(x)\|_Y \le \frac{K}{\binom{n}{m}^\beta \lambda^\beta} \Phi(x, \cdots, x)$$

for all $x \in X$.

Proof. It follows from (3.3) with $\lambda^i x$ in place of x and iterative method that

$$\left\| f(x) - \frac{f(\lambda^l x)}{\lambda^l} \right\|_Y \le \frac{K}{\binom{n}{m}^\beta \lambda^\beta} \sum_{i=0}^{l-2} \frac{K^i \varphi(\lambda^i x, \cdots, \lambda^i x)}{\lambda^{\beta i}} + \frac{1}{\binom{n}{m}^\beta \lambda^\beta} \frac{K^{l-1} \varphi(\lambda^{l-1} x, \cdots, \lambda^{l-1} x)}{\lambda^{\beta(l-1)}} \tag{3.8}$$

for all $x \in X$ and for any $l > 1$, which is considered to be (3.3) for $l = 1$. In fact, we see by computation

$$\begin{aligned}
&\left\| f(x) - \frac{f(\lambda^{l+1} x)}{\lambda^{l+1}} \right\|_Y \\
&\le K \left\| f(x) - \frac{f(\lambda x)}{\lambda} \right\|_Y + \frac{K}{\lambda^\beta} \left\| f(\lambda x) - \frac{f(\lambda^{l+1} x)}{\lambda^l} \right\|_Y \\
&\le \frac{K}{\binom{n}{m}^\beta \lambda^\beta} \varphi(x, \cdots, x) + \frac{K^2}{\binom{n}{m}^\beta \lambda^{2\beta}} \sum_{i=0}^{l-2} \frac{K^i \varphi(\lambda^{i+1} x, \cdots, \lambda^{i+1} x)}{\lambda^{\beta i}} \\
&\quad + \frac{K}{\binom{n}{m}^\beta \lambda^{2\beta}} \frac{K^{l-1} \varphi(\lambda^l x, \cdots, \lambda^l x)}{\lambda^{\beta(l-1)}} \\
&= \frac{K}{\binom{n}{m}^\beta \lambda^\beta} \sum_{j=0}^{l-1} \frac{K^j \varphi(\lambda^j x, \cdots, \lambda^j x)}{\lambda^{\beta j}} + \frac{1}{\binom{n}{m}^\beta \lambda^\beta} \frac{K^l \varphi(\lambda^l x, \cdots, \lambda^l x)}{\lambda^{\beta l}}
\end{aligned}$$

for all $x \in X$, which proves the inequality (3.8) for $l + 1$ by induction.

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Thus it follows that a sequence $\left\{\frac{f(\lambda^l x)}{\lambda^l}\right\}$ is Cauchy in Y and it converges. Therefore we see that a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{l\to\infty} \frac{f(\lambda^l x)}{\lambda^l} = \lim_{l\to\infty} \frac{f((n-m+1)^l x)}{(n-m+1)^l}$$

is well defined for all $x \in X$.

Now, applying the similar way to corresponding part of Theorem 3.1, we get the desired results. □

Theorem 3.4. *Assume that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. If the function $\varphi : X^n \to [0,\infty)$ satisfies

$$\Psi(x_1, \cdots, x_n) := \sum_{i=1}^{\infty} (\lambda^\beta K)^i \varphi(\frac{x_1}{\lambda^i}, \cdots, \frac{x_n}{\lambda^i}) < \infty$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique (m,n)-Cauchy–Jensen additive mapping $T : X \to Y$ such that T satisfies the equation (1.1) and the inequality

$$\|f(x) - T(x)\|_Y \le \frac{1}{\binom{n}{m}^\beta \lambda^\beta} \Psi(x, \cdots, x)$$

for all $x \in X$.

Proof. It follows from (3.3) and the similar method to (3.8) that

$$\begin{aligned}\left\|f(x) - \lambda^l f(\frac{x}{\lambda^l})\right\|_Y \le{}& \frac{1}{\binom{n}{m}^\beta \lambda^\beta} \sum_{i=1}^{l-1} (\lambda^\beta K)^i \varphi(\frac{x}{\lambda^i}, \cdots, \frac{x}{\lambda^i}) \\ &+ \frac{1}{\binom{n}{m}^\beta \lambda^\beta K} (\lambda^\beta K)^l \varphi(\frac{x}{\lambda^l}, \cdots, \frac{x}{\lambda^l})\end{aligned}$$

for all $x \in X$ and for any $l > 1$. Therefore we see that a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{l\to\infty} \lambda^l f(\frac{x}{\lambda^l}) = \lim_{l\to\infty} (n-m+1)^l f(\frac{x}{(n-m+1)^l})$$

is well defined for all $x \in X$.

The remaining assertion goes through by the similar way to corresponding part of Theorem 3.3. □

We obtain the following corollaries concerning the stability for approximate (m,n)-Cauchy–Jensen additive mappings in terms of a product of powers of norms.

Corollary 3.5. *If there exist real numbers $r_i \in \mathbb{R}$ with $r := \sum_{i=1}^n r_i \ne 1$ such that a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x_1, \cdots, x_n)\|_Y \le \theta \prod_{i=1}^{n} \|x_i\|^{r_i}$$

for all $x_1, \cdots, x_n \in X$ $(X \setminus \{0\}$ *if* $r_i \le 0)$ *and some* $\theta \ge 0$*, then there exists a unique* (m,n)*-Cauchy–Jensen additive mapping* $T : X \to Y$ *which satisfies the equation (1.1) and the inequality*

$$\|f(x) - T(x)\|_Y \le \left\{ \begin{array}{ll} \frac{K\theta\|x\|^r}{\binom{n}{m}^\beta(\lambda^\beta - K\lambda^{\alpha r})}, & if \quad K\lambda^{\alpha r} < \lambda^\beta \\ \frac{K\theta\|x\|^r}{\binom{n}{m}^\beta(\lambda^{\alpha r} - K\lambda^{\beta})}, & if \quad K\lambda^{\beta} < \lambda^{\alpha r} \end{array} \right\}$$

for all $x \in X$ $(X \setminus \{0\}$ *if* $r \le 0)$.

Now, we state the following *counterexample* given by P. Gavruta [8] in the special $2D$ case: $n = 2; \lambda = 2; \alpha = \beta = K = r = 1$ for equation (1.1): Let $0 < r < 1$. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \left\{ \begin{array}{ll} xln|x|, & \text{if} \quad x \ne 0; \\ 0, & \text{if} \quad x = 0 \end{array} \right\}$$

and there is a constant $\theta > 0$ such that

$$|f(x+y) - f(x) - f(y)| \le \theta |x|^r |y|^{1-r}$$

for all $x, y \in \mathbb{R}$, but

$$\sup\left\{\frac{|f(x) - T(x)|}{|x|} : x \ne 0\right\} = \infty$$

for every additive mapping $T : \mathbb{R} \to \mathbb{R}$.

Corollary 3.6. *If there exists a fixed* $r \in \mathbb{R}$ *such that a mapping* $f : X \to Y$ *satisfies the functional inequality*

$$\|Df(x_1, \cdots, x_n)\|_Y \le \theta(\sum_{i=1}^n \|x_i\|^r)$$

for all $x_1, \cdots, x_n \in X$ $(X \setminus \{0\}$ *if* $r < 0)$*, then there exists a unique* (m,n)*-Cauchy–Jensen additive mapping* $T : X \to Y$ *which satisfies the equation (1.1) and the inequality*

$$\|f(x) - T(x)\|_Y \le \left\{ \begin{array}{ll} \frac{nK\theta\|x\|^r}{\binom{n}{m}^\beta(\lambda^\beta - K\lambda^{\alpha r})}, & if \quad K\lambda^{\alpha r} < \lambda^\beta \\ \frac{nK\theta\|x\|^r}{\binom{n}{m}^\beta(\lambda^{\alpha r} - K\lambda^{\beta})}, & if \quad K\lambda^{\beta} < \lambda^{\alpha r} \end{array} \right\}$$

for all $x \in X$ $(X \setminus \{0\}$ *if* $r < 0)$.

Now, in the last part, we are to consider a *singular case* $m = n$ of Theorem 3.3 and Theorem 3.4 concerning the stability of the equation (1.1).

Theorem 3.7. *Assume that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies*

$$\|Df(x_1, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$*. If the function* $\varphi : X^n \to [0, \infty)$ *satisfies*

$$\sum_{i=1}^\infty \frac{K^i \varphi(n^i x_1, \cdots, n^i x_n)}{n^{\beta i}} < \infty$$

for all $x_1,\cdots,x_n \in X$, then there exists a unique (m,n)-Cauchy–Jensen additive mapping $T: X \to Y$ such that T satisfies the equation (1.1) and the inequality

$$\|f(x)-T(x)\|_Y \le n^\beta \sum_{i=1}^{\infty} \frac{K^i\tilde{\varphi}(n^i x)}{n^{\beta i}}$$

for all $x \in X$, where $\tilde{\varphi}(x) := \min_{1\le i\le n} \varphi(0,\cdots,0,\overbrace{x}^{i-th},0,\cdots,0)$.

Proof. For each $i=1,\cdots,n$, substituting x for x_i and 0 for all x_j with $j \ne i$ in the functional inequality (3.9), we obtain

$$\left\|f(\frac{x}{n})-\frac{1}{n}f(x)\right\|_Y \le \varphi(0,\cdots,0,\overbrace{x}^{i-th},0,\cdots,0),$$

$$\left\|f(x)-\frac{f(nx)}{n}\right\|_Y \le \tilde{\varphi}(nx),\ \tilde{\varphi}(x) := \min_{1\le i\le n}\varphi(0,\cdots,0,\overbrace{x}^{i-th},0,\cdots,0) \tag{3.9}$$

for all $x \in X$. It follows from (3.9) with $n^i x$ in place of x and iterative method that

$$\left\|f(x)-\frac{f(n^l x)}{n^l}\right\|_Y \le n^\beta \sum_{i=1}^{l-1} \frac{K^i\tilde{\varphi}(n^i x)}{n^{\beta i}} + n^\beta \frac{K^{l-1}\tilde{\varphi}(n^l x)}{n^{\beta l}} \tag{3.10}$$

for all $x \in X$ and for any $l > 1$, which is considered to be (3.9) for $l = 1$.

Thus, it follows that a sequence $\left\{\frac{f(n^l x)}{n^l}\right\}$ is Cauchy in Y and it converges. Therefore we see that a mapping $T: X \to Y$ defined by

$$T(x) := \lim_{l\to\infty} \frac{f(n^l x)}{n^l}$$

is well defined for all $x \in X$.

Now, applying the similar way to corresponding part of Theorem 3.3, we get the desired results. □

Theorem 3.8. *Assume that a mapping $f: X \to Y$ satisfies*

$$\|Df(x_1,\cdots,x_n)\|_Y \le \varphi(x_1,\cdots,x_n)$$

for all $x_1,\cdots,x_n \in X$. If the function $\varphi: X^n \to [0,\infty)$ satisfies

$$\sum_{i=0}^{\infty}(n^\beta K)^i \varphi(\frac{x_1}{n^i},\cdots,\frac{x_n}{n^i}) < \infty$$

for all $x_1,\cdots,x_n \in X$, then there exists a unique (m,n)-Cauchy–Jensen additive mapping $T: X \to Y$ such that T satisfies the equation (1.1) and the inequality

$$\|f(x)-T(x)\|_Y \le n^\beta \sum_{i=0}^{\infty}(n^\beta K)^i \tilde{\varphi}(\frac{x}{n^i})$$

for all $x \in X$, where $\tilde{\varphi}$ is defined by the same as in Theorem 3.7.

Proof. It follows from (3.9) and the similar method to (3.10) that

$$\left\|f(x) - n^l f(\frac{x}{n^l})\right\|_Y \leq n^\beta \sum_{i=0}^{l-2} (n^\beta K)^i \tilde{\varphi}(\frac{x}{n^i}) + n^\beta (n^\beta K)^{l-1} \tilde{\varphi}(\frac{x}{n^{l-1}})$$

for all $x \in X$ and for any $l > 1$.

The remaining assertion goes through by the similar way to corresponding part of Theorem 3.7. □

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References

[1] J. Aczél, *A Short Course on Functional Equations Based Upon Recent Applications to the Social and Behavioral Sciences,* D. Reidel Publishing Company, 1987.
[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2(1950), 64-66.
[3] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Vol. 1*, Colloq. Publ. **48**, Amer. Math. Soc. Providence, 2000.
[4] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. 57(1951), 223-237.
[5] W. Eichhorn, *Functional Equations in Economic,* Addison-Wesley Publ. Co. 1978.
[6] I. Fenyö, *On an inequality of P.W. Cholewa,* in: General inequalities, vol. 5, Internat. Schriftenreiche Numer. Math. vol. 80, Birkhäuser, Basel-Boston, MA, 1987, 277-280.
[7] G.L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. 50(1995), 143-190.
[8] P. Găvruta, *An answer to a question of John M. Rassias concerning the stability of Cauchy equation*, Advances in Equations and Inequalities, Hadronic Math. Series, USA, 1999, 67-71.
[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. 27(1941), 222-224.
[10] D.H. Hyers, G. Isac and Th.M. Rassias, *"Stability of Functional Equations in Several Variables"*, Birkhäuser, Basel, 1998.
[11] D.H. Hyers and Th.M. Rassias, *Approximate homomorphisms*, Aequationes Math. 44(1992), 125-153.
[12] K. Jun, H Kim and J. Rassias, *Extended Hyers-Ulam stability for Cauchy-Jensen mappings*, J. Diff. Equ. Appl. 13(12)(2007), 1139-1153.
[13] P. Malliavin, *Stochastic Analysis*, Springer, Berlin, 1997.
[14] A. Najati, *Stability of homomorphisms on JB^*-triples associated to a Cauchy–Jensen type functional equation,* J. Math. Inequal, V.1 (1) (2007), 83–103;
[15] A. Najati and A. Ranjbari, *Stability of homomorphisms for a 3D Cauchy Jensen type functional equation on C^*-ternary algebras*, J. Math. Anal. Appl. 341(2008), 62–79.
[16] B. Ramachandran and Ka-sing Lau, *Functional Equations in Probability Theory*, Academic Press, Inc. 1991.
[17] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math. 108(1984), 445-446.

[18] J.M. Rassias, H. Kim, *Approximate homomorphisms and derivations between C^*-ternary algebras,* J. Math. Physics, 49 (ID. 063507), 2008. 10 pages.
[19] J.M. Rassias and H. Kim, *Generalized Hyers–Ulam stability for general additive functional equations in quasi-β-normed spaces*, J. Math. Anal. Appl. 356(2009), 302-309.
[20] J.M. Rassias, K. Jun and H. Kim, *Approximate (m, n)-Cauchy–Jensen additive mappings in C^*-algebras*, Acta Math. Sinica, English Series, 27(2011), 1907–1922.
[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72(1978), 297-300.
[22] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. 251(2000), 264-284.
[23] S. Rolewicz, *Metric Linear Spaces*, Reidel and Dordrecht, and PWN-Polish Sci. Publ. 1984.
[24] S.M. Ulam, *A collection of Mathematical Problems*, Interscience Publ., New York, 1960; *Problems in Modern Mathematics*, Wiley-Interscience, New York, 1964, Chap. VI.
[25] Ding-Xuan Zhou, *On a conjecture of Z. Ditzian*, J. Approx. Theory 69(1992), 167-172.

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FIXED POINT METHOD FOR INTUITIONISTIC FUZZY STABILITY OF MIXED TYPE CUBIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. Using the fixed point alternative, recently, Mohiuddine et al. [20] established the stability of Jensen functional equation in intuitionistic fuzzy normed spaces. The object of the present project is to determine the stability of the Hyers-Ulam-Rassias type results concerning the cubic-quartic functional equation in intuitionistic fuzzy normed spaces by using the fixed point method.

Keywords and phrases: t-norm; t-conorm; cubic-quartic functional equation; Cauchy functional equation; intuitionistic fuzzy normed space; fixed point.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th. M. Rassias [29] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [29] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability or as Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; see [1, 4, 7, 10, 11, 13, 15, 16, 27, 28, 30, 32] and references therein for more detailed information.

Jun and Kim [10] and Lee et al. [12] introduced the cubic and quartic functional equation, respectively, and established the Hyers-Ulam-Rassias type stability results of these functional equations. The functional equation

$$\begin{aligned} Df(x,y) := {} & f(2x+y) + f(2x-y) - 3f(x+y) - f(-x-y) - 3f(x-y) \\ & - f(y-x) - 18f(x) - 6f(-x) + 3f(y) + 3f(-y) \qquad (1.1) \end{aligned}$$

is called the *cubic-quartic functional equation* (see [9]). The functional equation (1.1) is a cubic-quartic functional equation because (1.1) is quartic when $f(x)$ is a even function and (1.1) is cubic when $f(x)$ is a odd function.

Recently, the stability problem for Jensen functional equation, Pexiderized quadratic functional equation and cubic functional equation is considered in [14, 19, 22] respectively in the intuitionistic fuzzy normed spaces; while the idea of intuitionistic fuzzy normed space was introduced in [31] and further studied in

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[17, 18, 21, 23, 24, 25, 26] to deal with some summability problems. Quite recently, Mohiuddine et al [20] established the stability of Jensen functional equation in intuitionistic fuzzy normed spaces through fixed point technique.

In [8], Isac and Rassias were the first to provide an applications of fixed point alternative for the stability of functional equations. Therefore, one can choose here this method to determine the stability of mixed type cubic-quartic functional equation in intuitionistic fuzzy normed spaces through fixed point alternative.

In this paper we establish the stability of mixed type cubic-quartic functional equation in intuitionistic fuzzy normed spaces with the fixed point alternative.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:
(a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0,1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2. A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:
(a') $\diamondsuit$ is associative and commutative, (b') $\diamondsuit$ is continuous, (c') $a \diamondsuit 0 = a$ for all $a \in [0,1]$, (d') $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [31] have recently introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$
(i) $\mu(x,t) + \nu(x,t) \leq 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$, (vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim\limits_{t\to\infty} \mu(x,t) = 1$ and $\lim\limits_{t\to 0} \mu(x,t) = 0$, $(viii)$ $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (xi) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y, t+s)$, (xii) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, $(xiii)$ $\lim\limits_{t\to\infty} \nu(x,t) = 0$ and $\lim\limits_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an *intuitionistic fuzzy norm*.

Example 1.1. Let $(X, \|.\|)$ be a normed space and let $a * b = ab$ and $a \diamondsuit b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} 0 \text{ if } t \leq \|x\|; \\ 1 \text{ if } t > \|x\|; \end{cases} \quad \text{and } \nu(x,t) = \begin{cases} 1 \text{ if } t \leq \|x\|; \\ 0 \text{ if } t > \|x\|. \end{cases}$$

Then $(X, \mu, \nu, *, \diamondsuit)$ is an intuitionistic fuzzy normed space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [31] and further studied in [23].

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then, a sequence $x = (x_k)$ is said to be *intuitionistic fuzzy convergent* to $L \in X$ if $\lim \mu(x_k - L, t) = 1$ and $\lim \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write (μ, ν)-$\lim x = L$ or $x_k \xrightarrow{IF} L$ as $k \to \infty$.

Let $(X,\mu,\nu,*,\diamondsuit)$ be an IFNS. Then, $x = (x_k)$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim \mu(x_{k+p} - x_k, t) = 1$ and $\lim \nu(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \cdots$.

Let $(X,\mu,\nu,*,\diamondsuit)$ be an IFNS. Then $(X,\mu,\nu,*,\diamondsuit)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X,\mu,\nu,*,\diamondsuit)$ is intuitionistic fuzzy convergent in $(X,\mu,\nu,*,\diamondsuit)$.

The following known theorems will be used to prove our anticipated results.

Theorem 1.1 (Banach's Contraction principle). Let (X, d) be a complete generalized metric space and consider a mapping $J : X \to X$ be a strictly contractive mapping, that is

$$d(Jx, Jy) \le Ld(x, y),\ \forall x, y \in X$$

for some (Lipschitz constant) $L < 1$. Then

(i)The mapping J has one and only one fixed point $x^* = J(x^*)$;

(ii) The fixed point x^* is globally attractive, that is

$$\lim_{n\to\infty} J^n x = x^*,$$

for any starting point $x \in X$;

(iii) One has the following estimation inequalities for all $x \in X$ and $n \ge 0$:

$$d(J^n x, x^*) \le L^n d(x, x^*)$$
$$d(J^n x, x^*) \le \frac{1}{1-L} d(J^n x, J^{n+1} x)$$
$$d(x, x^*) \le \frac{1}{1-L} d(x, Jx).$$

Theorem 1.2 (The alternative of fixed point) [3]. Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, with Lipschitz constant L. Then, for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = +\infty,\ \forall n \ge 0$$

or

$$d(J^n x, J^{n+1} x) < +\infty\ \forall n \ge n_\circ$$

for some natural number $n_\circ$. Moreover, if the second alternative holds then

(i) The sequence $(J^n x)$ is convergent to a fixed point y^* of J;

(ii) y^* is the unique fixed point of J in the set $Y = \{y \in X, d(J^{n_\circ} x, y) < +\infty\}$

(iii) $d(y, y*) \le \frac{1}{1-L} d(y, Jy)$, $y \in Y$.

2. Main Results

Throughout this paper, assume that X is a vector space and that (Y,μ,ν) is an intuitionistic fuzzy Banach space.

Theorem 2.1. Let $\varphi : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x, y) \le \frac{L}{8}\varphi(2x, 2y),$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu(Df(x, y), t) \ge \frac{t}{t + \varphi(x, y)} \text{ and } \nu(Df(x, y), t) \le \frac{\varphi(x, y)}{t + \varphi(x, y)}, \qquad (*)$$

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for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic mapping $T : X \to Y$ such that $T(x) = (\mu, \nu)$- $\lim_{n\to\infty} 8^n f(x/2^n)$,

$$\mu(f(x)-T(x), t) \geq \frac{(16-16L)t}{(16-16L)t + L\varphi(x,0)} \text{ and } \nu(f(x)-T(x), t) \leq \frac{L\varphi(x,0)}{(16-16L)t + L\varphi(x,0)}, \tag{**}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in $(*)$, we get

$$\mu(2f(2x) - 16f(x), t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(2f(2x) - 16f(x), t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}, \tag{2.1.1}$$

for all $x \in X$.

Consider the set $E = \{g : X \to Y, g(0) = 0\}$ together with the mapping d_M defined on $E \times E$ by

$$d_M(g, h) = \inf\{a \in \mathbb{R}^+ : \mu(g(x)-h(x), at) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(g(x)-h(x), at) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}\},$$

for all $x \in X$ and $t > 0$. It is known that $d_M(g, h)$ is a complete generalized metric space by Lemma 2.1[20]. Now, we define the linear mapping $J : E \to E$ such that

$$Jg(x) = 8g(\frac{x}{2}).$$

Let $g, h \in E$ be given such that $d_M(g, h) < \epsilon$. Then

$$\mu(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(g(x) - h(x), \epsilon t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}\},$$

for all $x \in X$ and $t > 0$. Thus

$$\mu\Big(Jg(x) - Jh(x), L\epsilon t\Big) = \mu\Big(8g(\frac{x}{2}) - 8h(\frac{x}{2}), L\epsilon t\Big) = \mu\Big(g(\frac{x}{2}) - h(\frac{x}{2}), \frac{L}{8}\epsilon t\Big)$$

$$\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi(\frac{x}{2}, 0)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}\varphi(x,0)} = \frac{t}{t + \varphi(x,0)}$$

and similarly,

$$\nu\Big(Jg(x) - Jh(x), L\epsilon t\Big) = \nu\Big(8g(\frac{x}{2}) - 8h(\frac{x}{2}), L\epsilon t\Big) = \nu\Big(g(\frac{x}{2}) - h(\frac{x}{2}), \frac{L}{8}\epsilon t\Big)$$

$$\leq \frac{\frac{L\varphi(x,0)}{8}}{\frac{Lt}{8} + \frac{L}{8}\varphi(x,0)} = \frac{\varphi(x,0)}{t + \varphi(x,0)}$$

for all $x \in X$ and $t > 0$.

It follow from (2.1.1) that

$$\mu(f(x) - 8f(\frac{x}{2}), \frac{L}{16}t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(f(x) - 8f(\frac{x}{2}), \frac{L}{16}t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}$$

for all $x \in X$ and $t > 0$. it follows that $d_M(f, Jf) \leq \frac{L}{16}$.

Using the fixed point alternative we deduce the existence of a fixed point of J , that is, the existence of a mapping $T : X \to Y$ such that $T(\frac{x}{2}) = \frac{1}{8}T(x)$, for all $x \in X$.

Moreover, We have $d_M(J^n f, T) \to 0$, which implies

$$(\mu,\nu)\text{-}\lim_n 8^n f(\frac{x}{2^n}) = T(x),$$

for all $x \in X$. Also

$$d_M(f,T) \le \frac{1}{1-L} d_M(f, Jf) \text{ implies } d_M(f,T) \le \frac{L}{16-16L}.$$

This implies that the inequality $(**)$ holds.

By $(*)$,

$$\mu(8^n Df(\frac{x}{2^n}, \frac{y}{2^n}), 8^n t) \ge \frac{t}{t+\varphi(\frac{x}{2^n}, \frac{y}{2^n})} \text{ and } \nu(8^n Df(\frac{x}{2^n}, \frac{y}{2^n}), 8^n t) \le \frac{\varphi(\frac{x}{2^n}, \frac{y}{2^n})}{t+\varphi(\frac{x}{2^n}, \frac{y}{2^n})},$$

for all $x, y \in X$, all $t > 0$ and all $n \in N$. Thus

$$\mu(8^n Df(\frac{x}{2^n}, \frac{y}{2^n}), t) \ge \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)} \text{ and } \nu(8^n Df(\frac{x}{2^n}, \frac{y}{2^n}), t) \le \frac{\frac{\varphi(x,y)}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)},$$

for all $x, y \in X$, all $t > 0$ and all $n \in N$. Letting $n \to \infty$ in the above two equations, we get

$$\mu(DT(x,y),t) = 1 \text{ and } \nu(DT(x,y),t) = 0$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $T : X \to Y$ is cubic, as desired.

Corollary 2.2. Let $\theta \ge 0$ and p be a real number with $p > 3$. Let X be a normed space. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu(Df(x,y),t) \ge \frac{t}{t+\theta(\|x\|^p + \|y\|^p)} \text{ and } \nu(Df(x,y),t) \le \frac{\theta(\|x\|^p + \|y\|^p)}{t+\theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then $T(x) = (\mu,\nu)\text{-}\lim_{n\to\infty} 8^n f(x/2^n)$ exists for each $x \in X$ and defines a cubic mapping $T : X \to Y$ such that

$$\mu(f(x)-T(x),t) \ge \frac{2(2^p-8)t}{2(2^p-8)t+\theta\|x\|^p} \text{ and } \nu(f(x)-T(x),t) \le \frac{\theta\|x\|^p}{2(2^p-8)t+\theta\|x\|^p},$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. By choosing $L = 2^{3-p}$, we get the desired result.

Theorem 2.3. Let $\varphi : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x,y) \le 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right),$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (*). Then $T(x) = (\mu,\nu)\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $T : X \to Y$ such that

$$\mu(f(x)-T(x),t) \ge \frac{(16-16L)t}{(16-16L)t+\varphi(x,0)} \text{ and } \nu(f(x)-T(x),t) \le \frac{\varphi(x,0)}{(16-16L)t+\varphi(x,0)},$$

for all $x \in X$ and $t > 0$.

Proof. Let (E, d_M) be the generalized metric space defined in the proof of Theorem 2.1 Consider the linear mapping $J : E \to E$ such that

$$Jg(x) = \frac{1}{8}g(2x)$$

for all $x \in X$. Let $g, h \in E$ be given such that $d_M(g, h) = \epsilon$. Then

$$\mu(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, 0)} \text{ and } \nu(g(x) - h(x), \epsilon t) \leq \frac{\varphi(x, 0)}{t + \varphi(x, 0)}\},$$

for all $x \in X$ and $t > 0$. Thus

$$\mu\bigg(Jg(x) - Jh(x), L\epsilon t\bigg) = \mu\bigg(\frac{1}{8}g(2x) - \frac{1}{8}h(2x), L\epsilon t\bigg) = \mu\bigg(g(2x) - h(2x), 8L\epsilon t\bigg)$$
$$\geq \frac{8Lt}{8Lt + \varphi(2x, 0)} \geq \frac{8Lt}{8Lt + 8L\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

and similarly,

$$\nu\bigg(Jg(x) - Jh(x), L\epsilon t\bigg) = \nu\bigg(\frac{1}{8}g(2x) - \frac{1}{8}h(2x), L\epsilon t\bigg) = \nu\bigg(g(2x) - h(2x), 8L\epsilon t\bigg)$$
$$\leq \frac{8L\varphi(x, 0)}{8Lt + 8L\varphi(x, 0)} = \frac{\varphi(x, 0)}{t + \varphi(x, 0)}$$

for all $x \in X$ and $t > 0$. So $d_M(g, h) = \epsilon$ implies that $d_M(Jg, Jh) \leq L\epsilon$. This means that

$$d_M(Jg, Jh) \leq L d_M(g, h)$$

for all $g, h \in E$. It follow from (2.1.1) that

$$\mu(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t) \geq \frac{t}{t + \varphi(x, 0)} \text{ and } \nu(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t) \leq \frac{\varphi(x, 0)}{t + \varphi(x, 0)}$$

for all $x \in X$ and $t > 0$. Hence that $d_M(f, Jf) \leq \frac{1}{16}$.

Using the fixed point alternative we deduce the existence of a fixed point of J , that is, the existence of a mapping $T : X \to Y$ such that $T(2x) = 8T(x)$, for all $x \in X$.

Moreover, We have $d_M(J^n f, T) \to 0$, which implies

$$(\mu, \nu)\text{-}\lim_n \frac{1}{8^n} f(2^n x) = T(x),$$

for all $x \in X$. Also

$$d_M(f, T) \leq \frac{1}{1 - L} d_M(f, Jf) \text{ implies } d_M(f, T) \leq \frac{1}{16 - 16L}.$$

This implies that the inequality $(**)$ holds.

The remaining of the proof is similar to the proof of Theorem 2.1

Corollary 2.4. Let $\theta \geq 0$ and p be a real number with $0 < p < 3$. Let X be a normed space. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu(Df(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \text{ and } \nu(Df(x, y), t) \leq \frac{\theta(\|x\|^p + \|y\|^p)}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then $T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $T : X \to Y$ such that

$$\mu(f(x) - T(x), t) \geq \frac{2(8 - 2^p)t}{2(8 - 2^p)t + \theta\|x\|^p} \text{ and } \nu(f(x) - T(x), t) \leq \frac{\theta\|x\|^p}{2(8 - 2^p)t + \theta\|x\|^p},$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. By choosing $L = 2^{3-p}$, we get the desired result.

Theorem 2.5. Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x, y) \leq \frac{L}{16}\varphi(2x, 2y),$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.1.1). Then $Q(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} 16^n f(x/2^n)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu(f(x)-Q(x), t) \geq \frac{(32-32L)t}{(32-32L)t + L\varphi(x,0)} \text{ and } \nu(f(x)-Q(x), t) \leq \frac{L\varphi(x,0)}{(32-32L)t + L\varphi(x,0)},$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in $(*)$, we get

$$\mu(2f(2x) - 32f(x), t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(2f(2x) - 32f(x), t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}, \tag{2.5.1}$$

for all $x \in X$.

Let (E, d_M) be the generalized metric space defined in the proof of Theorem 2.1.

We define the linear mapping $J : E \to E$ such that

$$Jg(x) = 16g(\frac{x}{2})$$

for all $x \in X$. Let $g, h \in E$ be given such that $d_M(g, h) = \epsilon$. Then

$$\mu(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(g(x) - h(x), \epsilon t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}\},$$

for all $x \in X$ and $t > 0$. Thus

$$\mu\left(Jg(x) - Jh(x), L\epsilon t\right) = \mu\left(16g(\frac{x}{2}) - 16h(\frac{x}{2}), L\epsilon t\right) = \mu\left(g(\frac{x}{2}) - h(\frac{x}{2}), \frac{L}{16}\epsilon t\right)$$

$$\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \varphi(\frac{x}{2}, 0)} \geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \frac{L}{16}\varphi(x,0)} = \frac{t}{t + \varphi(x,0)}$$

and similarly,

$$\nu\left(Jg(x) - Jh(x), L\epsilon t\right) = \nu\left(16g(\frac{x}{2}) - 16h(\frac{x}{2}), L\epsilon t\right) = \nu\left(g(\frac{x}{2}) - h(\frac{x}{2}), \frac{L}{16}\epsilon t\right)$$

$$\leq \frac{\frac{L\varphi(x,0)}{16}}{\frac{Lt}{16} + \frac{L}{16}\varphi(x,0)} = \frac{\varphi(x,0)}{t + \varphi(x,0)}$$

for all $x \in X$ and $t > 0$.

It follow from (2.1.1) that

$$\mu(f(x) - 16f(\frac{x}{2}), \frac{L}{32}t) \geq \frac{t}{t + \varphi(x,0)} \text{ and } \nu(f(x) - 16f(\frac{x}{2}), \frac{L}{132}t) \leq \frac{\varphi(x,0)}{t + \varphi(x,0)}$$

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for all $x \in X$ and $t > 0$. it follows that $d_M(f, Jf) \leq \frac{L}{32}$.

Using the fixed point alternative we deduce the existence of a fixed point of J , that is, the existence of a mapping $T : X \to Y$ such that $T(\frac{x}{2}) = \frac{1}{16}T(x)$, for all $x \in X$.

Moreover, We have $d_M(J^n f, T) \to 0$, which implies

$$(\mu, \nu)\text{-} \lim_n 16^n f(\frac{x}{2^n}) = T(x),$$

for all $x \in X$. Also

$$d_M(f, T) \leq \frac{1}{1-L} d_M(f, Jf) \text{ implies } d_M(f, T) \leq \frac{L}{32 - 32L}.$$

This implies that the inequality $(**)$ holds.

The remaining of the proof is similar to the proof of Theorem 2.1

Similarly, we can obtain the following. We will omit the proof.

Theorem 2.6. Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{x}{2}\right),$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.1.1). Then there exists a unique quartic mapping $Q : X \to Y$ such that $Q(x) = (\mu, \nu)\text{-} \lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$

$$\mu(f(x)-Q(x), t) \geq \frac{(32 - 32L)t}{(32 - 32L)t + \varphi(x, 0)} \text{ and } \nu(f(x)-Q(x), t) \leq \frac{\varphi(x, 0)}{(32 - 32L)t + \varphi(x, 0)},$$

for all $x \in X$ and $t > 0$.

Corollary 2.7. Let $\theta \geq 0$ and p be a real number with $0 < p < 4$. Let X be a normed space. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$,

$$\mu(Df(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \text{ and } \nu(Df(x, y), t) \leq \frac{\theta(\|x\|^p + \|y\|^p)}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then $T(x) = (\mu, \nu)\text{-} \lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $T : X \to Y$ such that

$$\mu(f(x)-T(x), t) \geq \frac{2(16 - 2^p)t}{2(16 - 2^p)t + \theta\|x\|^p} \text{ and } \nu(f(x)-T(x), t) \leq \frac{\theta\|x\|^p}{2(16 - 2^p)t + \theta\|x\|^p},$$

for all $x \in X$ and $t > 0$.

References

[1] A. Alotaibi, S.A. Mohiuddine, On the stability of a cubic functional equation in random 2-normed spaces, Adv. Difference Equ. 2012, 2012:39.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64-66.

[3] J.B. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on generalized complete metric space, Bull. Amer. Math. Soc. 126 (74) (1968) 305-309.

[4] Z. Gajda, On stability of additive mappings, Internat. J. Math. Math. Sci. 14 (1991) 431-434.

[5] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431-436.

[6] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222-224.

[7] D.H. Hyers, G. Isac, T.M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, Basel, 1998.

[8] G. Isac, T.M. Rassias, Stability of ψ-additive mappings: Appications to nonlinear analysis, Internat. J. Math. Math. Sci. 19 (1996) 219-228.
[9] S.-Y. Jang, C. Park, D.Y. Shin, Fuzzy stability of a cubic-quartic functional equation: A fixed point approach, Bull. Korean Math. Soc. 48 (2011) 491-503.
[10] K.W. Jun, H.M. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, J. Math. Anal. Appl. 274 (2002) 867-878.
[11] K.W. Jun, H.M. Kim, I.S. Chang, On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation, J. Comput. Anal. Appl. 7(2005)21-33.
[12] S. Lee, S. Im, I. Hwang, Quartic functional equations, J. Math. Anal. Appl. 307 (2005) 387-394.
[13] D. Miheţ, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. 343 (2008) 567-572.
[14] S.A. Mohiuddine, Stability of Jensen functional equation in intuitionistic fuzzy normed space, Chaos, Solitons Fract. 42 (2009) 2989-2996.
[15] S.A. Mohiuddine, M.A. Alghamdi, Stability of functional equation obtained through fixed point alternative in intuitionistic fuzzy normed spaces, Adv. Difference Equ. 2012, 2012:141.
[16] S.A. Mohiuddine, A. Alotaibi, Fuzzy stability of a cubic functional equation via fixed point technique, Adv. Difference Equ. 2012, 2012:48.
[17] S.A. Mohiuddine, A. Alotaibi, M. Obaid, Stability of various functional equations in non-Archimedean intuitionistic fuzzy normed spaces, Discrete Dyna. Nature Soc. Volume 2012, Article ID 234727, 16 pages.
[18] S.A. Mohiuddine, Q.M. Danish Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos Solitons Fract. 42 (2009) 1731-1737.
[19] S.A. Mohiuddine, H. Şevli, Stability of Pexiderized quadratic functional equation in intuitionistic fuzzy normed space, J. Comput. Appl. Math. 235 (2011) 2137-2146.
[20] S.A. Mohiuddine, M. Cancan, H. Şevli, Intuitionistic fuzzy stability of a Jensen functional equation via fixed point technique, Math. Comput. Model. 54 (2011) 2403-2409.
[21] M. Mursaleen, V. Karakaya, S.A. Mohiuddine, Schauder basis, separability, and approximation property in intuitionistic fuzzy normed space, Abstract Appl. Analy. Volume 2010, Article ID 131868, 14 pages.
[22] M. Mursaleen, S.A. Mohiuddine, On stability of a cubic functional equation in intuitionistic fuzzy normed spaces, Chaos, Solitons Fract. 42 (2009) 2997-3005.
[23] M. Mursaleen, S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos, Solitons Fract. 41 (2009) 2414-2421.
[24] M. Mursaleen, S.A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math. 233 (2009) 142-149.
[25] M. Mursaleen, S.A. Mohiuddine, Nonlinear operators between intuitionistic fuzzy normed spaces and Fréchet differentiation, Chaos, Solitons Fract. 42 (2009) 1010-1015.
[26] M. Mursaleen, S.A. Mohiuddine, Osama H.H. Edely, On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces, Comput. Math. Appl. 59 (2010) 603-611.
[27] V. Radu, The fixed point alternative and the stability of functional equations, Sem. Fixed Point Theory 4 (1) (2003) 91-96.
[28] J.M. Rassias, Solution of the Ulam stability problem for cubic mappings, Glas. Mat. Ser. III 36 (56) (2001) 63-72.
[29] T.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. 72 (1978) 297-300.
[30] T.M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, J. Math. Anal. Appl. 246 (2000) 352-378.
[31] R. Saadati, J.H. Park, On the intuitionistic fuzzy topological spaces. Chaos, Solitons Fract. 27 (2006) 331-344.
[32] F. Skof, Local properties and approximation of operators, Rend. Sem. Mat. Fis. Milano 53 (1983) 113-129.
[33] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publ., New York, 1960.

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ISOMETRIES OF P-NUCLEAR OPERATORS

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ABSTRACT. For Banach spaces X and Y, the operator ideal of p-nuclear operators from X into Y is denoted by $N_p(X,Y)$. In this paper, we study the onto isometries of $N_p(X,Y)$ for certain classes of Banach spaces. Full characterization is given for onto isometries of $N_p(X,\ell^p)$.

1. Introduction

Let X and Y be Banach spaces and X^* be the dual of X. We denote by $L(X,Y)$, the space of bounded linear operators from X to Y. For $x^* \in X^*$, $y \in Y$, we define the one rank operator $x^* \otimes y : X \to Y$, by $(x^* \otimes y)(z) = x^*(z)y$. The set of all such operators will be denoted by K.

Let $X^* \otimes Y$ denote the span(K) in $L(X,Y)$. Clearly, $X^* \otimes Y$ is just the space of all finite rank operators from X to Y. So, every $T \in X^* \otimes Y$ has a form:

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $x_i^* \in X^*$, $y_i \in Y$ and $y_1, y_2, \ldots, y_n$ are independent.

On $X^* \otimes Y$, one can define so many norms. In this paper, we are interested in one particular norm, namely, the p-nuclear norm which is defined as follows:

$$||T||_{n(p)} = \inf \left[\left(\sum_{i=1}^{n} ||x_i^*||^p \right)^{\frac{1}{p}} . \sup_{||y^*||=1} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \right], \ \frac{1}{p} + \frac{1}{p^*} = 1.$$

where the infimum is taken over all representations of $T = \sum_{i=1}^{n} x_i^* \otimes y_i$, (y_i) independent, in $X^* \otimes Y$. The space $(X^* \otimes Y, ||.||_{n(p)})$ is a normed space that need not be complete. Let $X^* \widehat{\otimes}_{n(p)} Y$ denote the completion of $X^* \otimes Y$ with respect to $||.||_{n(p)}$. The space $X^* \widehat{\otimes}_{n(p)} Y$ is called the Banach space of p-nuclear operators from X to Y and is denoted by $N_p(X,Y)$.

Now, let W be a Banach space. Then $T \in L(W,W)$ is called an isometry if $||Tx|| = ||x||$ for all $x \in W$.

Many studies about isometries have been done in the literature. For example; isometries of the operator ideal of compact operators were studied in [8]. Isometries

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of bounded linear operators between certain pairs of Banach spaces were studied in [1]. The isometries of the Schatten classes on Hilbert spaces were studied in [6].

In this paper, we study the onto isometries of $N_p(X,Y)$. For certain classes of Banach spaces Y, we give a full characterization of such isometries. In the next section, we present some known results in the literature.

2. Basic Results

Let X and Y be Banach spaces. For $1 < p, q < \infty$, we set

$$\ell^p(X) = \left\{ (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} ||x_n||^p < \infty,\ x_n \in X \right\}$$

and

$$\ell^q\langle Y\rangle = \left\{ (y_n)_{n=1}^{\infty} : \sup_{||y^*||=1} \sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^q < \infty,\ y_n \in Y, y^* \in Y \right\}$$

For $f \in \ell^p(X)$, let $||f||_{n(p)} = (\sum_{n=1}^{\infty} ||x_n||^p)^{\frac{1}{p}}$, where $f = (x_n)$. For $g \in \ell^q\langle Y\rangle$, let $||g||_{\epsilon(q)} = \sup_{||y^*||=1} (\sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^q)^{\frac{1}{q}}$, where $g = (y_n)$. It is known, see [7] , that both $\ell^p(X)$ and $\ell^q\langle Y\rangle$ are Banach spaces.

There are two main known characterizations for T to be an element in $N_p(X,Y)$, which are stated below.

Theorem 1. *Let X and Y be Banach spaces. The following are equivalent:*

(1) *$T \in N_p(X,Y)$, $1 < p < \infty$.*

(2) *The following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ \ell^\infty & \xrightarrow[D]{} & \ell^p \end{array}$$

where $D \in L(\ell^\infty, \ell^p)$ is defined as $D(a_i) = (a_i b_i)$ for some fixed $(b_i) \in \ell^p$, $A \in L(X, \ell^\infty)$ and $B \in L(\ell^p, Y)$. Furthermore,

$$||T||_{n(p)} = \inf\{||A||\ ||(b_i)||\ ||B||\}$$

where the infimum is taken over all possible factorizations of T.

Theorem 2. *Let X and Y be Banach spaces. The following are equivalent:*

(1) *$T \in N_p(X,Y)$*

(2) *There exists $(x_n^*) \in \ell^p(X^*)$, $(y_n) \in \ell^{p^*}(Y)$ such that $Tx = \sum_{n=1}^{\infty} \langle x_n^*, x^*\rangle y_n$, and $||T||_{n(p)} = \inf\{||(x_n^*)||_{n(p)}\ ||(y_n)||_{\epsilon(p^*)}\}$, where the infimum is taken over all representation of $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, (y_n) independent in Y.*

For more on $p-nuclear$ operators, we refer the reader to [3] and [7].

3. Main Results

Let X and Y be two Banach spaces, $N_p(X,Y)$ be the space of $p-$ *nuclear* operators from X to Y. The problem in studying isometries of $N_p(X,Y)$ is that extreme points need not be contained in rank-one operators. So the fact that isometries preserve extreme points is not useful. Now, we would like to start with the following result:

Theorem 3. *Let A be an isometry from X onto X, and B be an isometry from Y onto Y. Then $J(T) = B\ T\ A$ is an isometric onto operator of $N_p(X,Y)$.*

Proof. Let $T \in N_p(X,Y)$ be such that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, (y_n) independent. Now, $Tx = \sum_{n=1}^{\infty} \langle x_n^*, x\rangle y_n$. Thus,

$$\begin{aligned} J(T)x &= B\ T\ Ax \\ &= \sum_{n=1}^{\infty} B(\langle x_n^*, Ax\rangle y_n) \\ &= \sum_{n=1}^{\infty} \langle A^* x_n^*, x\rangle B y_n \\ &= \left(\sum_{n=1}^{\infty} A^* x_n^* \otimes B y_n\right) x \end{aligned}$$

Since A is an isometric onto operator of X, then A^* is an isometric onto operator of X^*. Now, we show that J preserves norm. Since A and B are isometric onto operators, we have:

$$\begin{aligned} \|J(T)\|_{n(p)} &= \inf \left(\sum_{n=1}^{\infty} \|A x_n^*\|^p\right)^{\frac{1}{p}} . \sup_{\|y^*\|=1} \left(\sum_{n=1}^{\infty} |\langle y_n, B^* y^*\rangle|^{p^*}\right)^{\frac{1}{p^*}} \\ &= \inf \left(\sum_{n=1}^{\infty} \|x_n^*\|^p\right)^{\frac{1}{p}} . \sup_{\|y^*\|=1} \left(\sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^{p^*}\right)^{\frac{1}{p^*}} \\ &= \|T\|_{n(p)} \end{aligned}$$

To prove J is onto, let $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$, (y_i) independent. Since A^* is an isometric onto operator, then $x_i^* = A^* z_i^*$. Similarly, since B is an isometric onto operator, then $y_i = B w_i$. Therefore, $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i = \sum_{i=1}^{\infty} A^* z_i^* \otimes B w_i$. Now, consider $\widehat{T} = \sum_{i=1}^{\infty} z_i^* \otimes w_i$. Since (y_i) is independent, and B is an isometry, then (w_i) is also independent. Further,

$$J(\widehat{T}) = \sum_{i=1}^{\infty} A^* z_i^* \otimes B w_i = \sum_{i=1}^{\infty} x_i^* \otimes y_i = T$$

Hence, J is an isometric onto operator. □

Now, we define the following sets: let $N_1(u) = [u] \otimes_{n(p)} Y = \{u \otimes y : y \in Y\}$ and $N_2(v) = X \otimes_{n(p)} [v] = \{x \otimes v : x \in X\}$

Theorem 4. *If $J \in N_p(X,Y)$ is an isometric onto operator that preserves rank, then $J(N_1(u)) = N_1(\hat{u})$ or $N_2(\hat{v})$ for some $\hat{u} \in X$ or $\hat{v} \in Y$.*

Proof. Let $\hat{x}_1 \otimes \hat{y}_1$ and $\hat{x}_2 \otimes \hat{y}_2$ be two elements in $J(N_1(u))$. Then $\hat{x}_1 \otimes \hat{y}_1 = J(u \otimes y_1)$ and $\hat{x}_2 \otimes \hat{y}_2 = J(u \otimes y_2)$. This means,

$$\begin{aligned} \hat{x}_1 \otimes \hat{y}_1 + \hat{x}_2 \otimes \hat{y}_2 &= J(u \otimes y_1) + J(u \otimes y_2) \\ &= J(u \otimes (y_1 + y_2)) \\ &= \hat{\theta} \otimes \hat{h} \end{aligned}$$

But this implies by [4, Lemma 2.1, p. 82] that either $\hat{x}_1,\ \hat{x}_2$ or $\hat{y}_1,\ \hat{y}_2$ are linearly dependent. Assume that $\hat{x}_1,\ \hat{x}_2$ are linearly independent. This implies that $\hat{y}_1,\ \hat{y}_2$ are linearly dependent, and therefore $\hat{x}_1 \otimes \hat{y}_1$ and $\hat{x}_2 \otimes \hat{y}_2$ are linearly independent.

Now, if $\hat{f} \otimes \hat{g} \in J(N_1(u))$, then $\hat{f} \otimes \hat{g} = J(u \otimes g)$. So,

$$\begin{aligned} \hat{f} \otimes \hat{g} + \hat{x}_1 \otimes \hat{y}_1 &= J(u \otimes g) + J(u \otimes y_1) \\ &= J(u \otimes (g + y_1)) \\ &= \hat{z}_1 \otimes \hat{w}_1 \end{aligned}$$

Since the above sum is one atom, we have two cases:

Case 1. If $\hat{f},\ \hat{x}_1$ are independent, then $\hat{g} = \lambda \hat{y}_1$ but $\hat{y}_1 = \zeta \hat{y}_2$. Hence, $J(N_1(u)) = N_2(\hat{v})$.

Case 2. If $\hat{f},\ \hat{x}_1$ are dependent, then $\hat{f} = \eta \hat{x}_1$ which implies that $\hat{f} \otimes \hat{g} + \hat{x}_2 \otimes \hat{y}_2 = \eta \hat{x}_1 \otimes \hat{g} + \hat{x}_1 \otimes \hat{y}_1$. But in this case $\hat{g}$ and $\hat{y}_1$ are independent, so it follows that $\hat{g}$ and $\hat{y}_2$ are also independent, this implies that $\hat{x}_2$ and $\eta \hat{x}_1$ are dependent which contradicts our assumption. Thus, $\hat{f}$ and $\hat{x}_1$ are independent.

Consequently, if $\hat{x}_1$ and $\hat{x}_2$ are independent then $J(N_1(u)) = N_2(\hat{v})$. A similar argument shows that if $\hat{y}_1$ and $\hat{y}_2$ are independent then $J(N_1(u)) = N_1(\hat{u})$. □

Theorem 5. *Let $J : N_p(X, Y) \to N_p(X, Y)$ be an isometric onto operator that preserves rank. Then there exist an isometric onto operators $A^* \in L(X^*, X^*)$ and $B \in L(Y, Y)$ such that $J(\sum_{n=1}^{\infty} x_n^* \otimes y_n) = \sum_{n=1}^{\infty} A^* x_n^* \otimes B y_n = BTA$, where $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$.*

Proof. Let $W_1 = \{x^* \in X^* : J([x^*] \otimes_{n(p)} Y) = [\hat{x^*}] \otimes_{n(p)} Y\}$, and let

$W_2 = \{x^* \in X^* : J([x^*] \otimes_{n(p)} Y) = X^* \otimes_{n(p)} [\hat{y}]\}$. One can easily prove that W_1 and W_2 are subspaces of X^*. Moreover, by Theorem 4 we get $W_1 \cup W_2 = X^*$. This implies that $W_1 = X^*$ or $W_2 = X^*$. Without loss of generality, we can assume that $W_1 = X^*$.

Similarly, let $J(X^* \otimes_{n(p)} [y]) = X^* \otimes_{n(p)} [\hat{y}]$ for all $y \in Y$. Fix $x_o^* \in X^*$ and $y_o \in Y$ such that $||x_o^*|| = 1$ and $||y_o|| = 1$. Then $J([x_o^*] \otimes_{n(p)} Y = [\hat{x_o^*}] \otimes_{n(p)} Y$ and $J(X^* \otimes_{n(p)} [y_o]) = X^* \otimes_{n(p)} [\hat{y_o}]$. Now, let $x^* \in X^*$ and $y \in Y$, then $J(x^* \otimes y_o) = \hat{x^*} \otimes \hat{y_o}$, $||\hat{y_o}|| = 1$ and $J(x_o^* \otimes y) = \hat{x_o^*} \otimes \hat{y}$, $||\hat{x_o^*}|| = 1$.

Define $A : X^* \to X^*$ by $Ax^* = \hat{x^*}$, and $B : Y \to Y$ by $By = \hat{y}$. Then A and B are well-defined. Further, since J is an isometry, then $A,\ B$ are isometries. It follows that, $J(x^* \otimes y) = Ax^* \otimes By$, which ends the proof of the theorem.

□

One can ask now: Does there exist examples of X and Y in which each isometric onto operator of $N_p(X, Y)$ preserves rank?. In the next section we present two examples.

4. Isometries of $N_p(X, \ell^p)$

Any operator $T : X \to \ell^p$ has the form $Tx = \sum_{n=1}^{\infty} a_n(x)\delta_n$, for $1 \leq p < \infty$. One can easily show that $a_n(x) = \langle z_n^*, x\rangle$, for some $z_n^* \in X^*$. So, $T = \sum_{n=1}^{\infty} z_n^* \otimes \delta_n$ and for $x_n^* = \dfrac{z_n^*}{||z_n^*||}$ and $\lambda_n = ||z_n^*||$, we can write T as $T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes \delta_n$. Therefore, if $(\lambda_n) \in \ell^p$, then $T \in N_p(X, \ell^p)$.

Theorem 6. $||T||_{n(p)} = \left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{\frac{1}{p}}$.

Proof. Let $T \in N_p(X, \ell^p)$, then $T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes \delta_n$. Let $T = \sum_{n=1}^{\infty} \eta_n y_n^* \otimes z_n$ be any other representation of T. Now, we claim that $||(\lambda_n)||_p \leq ||(\eta_n)||_p$, and we will prove our claim as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \eta_n y_n^* \otimes z_n &= \sum_{n=1}^{\infty} \eta_n y_n^* \otimes \sum_{j=1}^{\infty} a_{nj}\delta_j \\
&= \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} \eta_n y_n^* a_{nj}\right) \otimes \delta_j
\end{aligned}$$

This implies that $\lambda_j x_j^* = \sum_{n=1}^{\infty} \eta_n a_{nj} y_n^*$, and since $||x_j^*|| = 1$ we can see that $|\lambda_j| = |\sum_{n=1}^{\infty} \eta_n a_{nj}\langle y_n^*, x_j\rangle|$, where $x_j \in X^{**}$ and $||x_j|| = 1$. Therefore,

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^{\infty} \left|\sum_{n=1}^{\infty} \eta_n a_{nj}\langle y_n^*, x_j\rangle\right|^p\right)^{\frac{1}{p}} \\
&= \sum_{j=1}^{\infty} \alpha_j \sum_{n=1}^{\infty} \eta_n a_{nj}\langle y_n^*, x_j\rangle \qquad (1) \\
&= \sum_{j,n=1}^{\infty} \eta_n \alpha_j a_{nj}\langle y_n^*, x_j\rangle
\end{aligned}$$

where $(\alpha_j) \in \ell^{p^*}$ with $||(\alpha_j)||_{p^*} = 1$ and $(\eta_n) \in \ell^p$. Since $z_n = \sum_{j=1}^{\infty} a_{nj}\delta_j \in \ell^p$ and $\left(\sum_{n=1}^{\infty} |\langle z_n, w^*\rangle|^{p^*}\right)^{\frac{1}{p^*}} \leq 1$, it follows that

$$\left(\sum_{n=1}^{\infty} \left|\sum_{j=1}^{\infty} a_{nj}\langle \delta_j, w^*\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \leq 1, \ \forall w^* \in \ell^p \textit{ such that } ||w^*||_p \leq 1.$$

Therefore, $\left|\sum_{n=1}^{\infty} \beta_n \sum_{j=1}^{\infty} a_{nj}\langle \delta_j, w^*\rangle\right| \leq 1$ for $||(\beta_n)||_p \leq 1$. But $(\langle \delta_j, w^*\rangle) \in \ell^{p^*}$, since $w^* \in \ell^{p^*}$. Hence,

$$\left|\sum_{n,j=1}^{\infty} \zeta_n \beta_j a_{nj}\right| \leq 1, \ \forall (\zeta_n), (\beta_j) \in \ell^p \textit{ with } ||(\zeta_n)||_p = ||(\beta_j)||_p = 1. \qquad (2)$$

Now, since $(\alpha_j) \in \ell^{p^*}$ and $||(\alpha_j)||_{p^*} \leq 1$, then (1) can be written as:

$$\begin{aligned}\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} &= \left|\sum_{j,n=1}^{\infty} \frac{\eta_n}{||(\eta_n)||_p} \alpha_j a_{nj} \langle y_n^*, x_j\rangle\right| \\ &\leq ||(\eta_n)||_p \left|\sum_{n,j=1}^{\infty} \beta_n \alpha_j a_{nj} \langle y_n^*, x_j\rangle\right|\end{aligned}$$

But $|\langle y_n^*, x_j\rangle| \leq 1$, so according to inequality (2) we have $||(\lambda_n)||_p \leq ||(\eta_n)||_p$. Hence $||T||_{n(p)} = ||(\lambda_n)||_p$ which finishes the proof. □

As a consequence of the above result, we state the following theorem.

Theorem 7. *Every operator $T \in N_p(X, \ell^p)$ has a representation for which the infimum is attained.*

Proof. The proof follows directly from the previous Theorem. □

Now, one can ask: Do isometric onto operators of $N_p(X, \ell^p)$ preserve rank?. A positive answer is obtained in the next Theorem.

Theorem 8. *Let J be an isometric onto operator of $N_p(X, \ell^p)$. Then J preserves rank.*

Proof. Let $J : N_p(X, \ell^p) \to N_p(X, \ell^p)$ be an isometric onto operator. If $T \in N_p(X, \ell^{p^*})$, then $T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes \delta_n$, and by Theorem 6, $||T||_{n(p)} = (\sum_{n=1}^{\infty} |\lambda_n|^p)^{\frac{1}{p}}$. Since $J(T) \in N_p(X, \ell^{p^*})$, then $J(T) = \sum_{n=1}^{\infty} \zeta_n y_n^* \otimes \delta_n$, where $||(\zeta_n)||_p = ||(\lambda_n)||_p$ and $||y_n^*|| = ||x_n^*||$. This produces a linear isometry $\hat{J}$ of ℓ^p. But isometries of ℓ^p preserve the support. In other words, if (λ_n) and (η_n) are in ℓ^p and $supp(\lambda_n) \cap supp(\eta_n) = \phi$, then $supp(\hat{J}(\lambda_n)) \cap supp(\hat{J}(\eta_n)) = \phi$. This follows from the known identity:

$$\begin{aligned}||\hat{J}(f) + \hat{J}(g)||_p^p &= ||\hat{J}(f + g)||_p^p \\ &= ||f + g||_p^p \\ &= ||f||_p^p + ||g||_p^p \\ &= ||\hat{J}(f)||_p^p + ||\hat{J}(g)||_p^p\end{aligned}$$

Further, it is known that such identities hold only if f and g have disjoint support. Since $\hat{J}$ is an isometric onto, we must have $\hat{J}(\delta_k) = \delta_m$. Now, let $T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes \delta_n$. The rank of T is the same as the cardinality of the support of (λ_n). Hence, $rank(\hat{J}(T)) = rank(T)$ and now the proof is complete.

□

Corollary 1. *If $J : N_p(X, \ell^p) \to N_p(X, \ell^p)$ is an isometric onto operator, then J preserves atoms.*

The following is another example of Banach spaces where isometric onto operators preserve rank.

Theorem 9. *Let X^* and Y be finite dimensional Banach spaces. If $T \in N_p(X, Y)$, then T has a representation $T = \sum_{k=1}^{n} x_k^* \otimes y_k$, where $n = dim(Y)$, such that $||T||_{n(p)} = ||(x_k^*)||_{n(p)} ||(y_k)||_{\epsilon(p)}$.*

Proof. From the definition of the infimum, there is a sequence of representations $T_m = \sum_{k=1}^{n} x_k^{*m} \otimes y_k^m$ such that $||T||_{n(p)} = \lim_{m\to\infty} ||(x_k^m)||_{n(p)}||(y_k^m)||_{\epsilon(p)}$.

Since Y is finite dimensional, the sequence (y_1^m) has a subsequence $(y_1^{m_j})$ that converges to an element in Y call it y_1. Let $(y_2^{m_j})$ be the subsequence of (y_2^m) that corresponds to $(y_1^{m_j})$, then there exists a subsequence $(y_2^{m_{j_k}})$ that converges say to y_2. Clearly, the subsequence $(y_1^{m_{j_k}})$ of $(y_1^{m_j})$ corresponding to the subsequence $(y_2^{m_{j_k}})$, converges to y_1. We continue this process to get subsequences (y_1^j), (y_2^j), $\dots$, (y_n^j) that converge to $y_1,\ y_2, \dots,\ y_n$ respectively.

Similarly, let (x_i^{*j}) be the subsequence of (x_i^{*m}) corresponds to (y_i^j), for $1 \leq i \leq n$. It follows that, there exist $x_1^*,\ x_2^*, \dots, x_n^* \in X^*$ such that (x_i^{*j}) converges to x_i^* for all $i = 1, 2, \dots, n$.

Hence, for $T = \sum_{k=1}^{n} x_k^* \otimes y_k$, the proof is complete. □

We end our paper with the following:

Question: Must every isometric onto operator of an operator ideal preserve rank?.

References

[1] Arazy, J. The isometries of C_p. *Isr. J. Math.* **22**(1975), pp.247-256.

[2] Apiola, H. Duality between spaces of $p-$summable sequences, $(p,q)-$summing operators and charachterization of nuclearity. *Math. Ann.* **219**(1976), pp.53-64.

[3] Jarchaw, H. Locally convex spaces. *Teubner Stuttgart.*(1981).

[4] Khalil, R. Isometries of $L^p\hat{\otimes}L^p$, *Tamkang Journal of Mathematics* **16**, No.2, (1985), pp.77-85.

[5] Khalil, R. Extreme points and isometries of $\phi-$nuclear operators in Hilbert spaces. *J. Math. Analysis and App* **148**, (1990), pp.147-156.

[6] Khalil, R. and Saleh, A. Isometries of certain operator spaces. *Proc. Amer. Math. Soc.* **132**(2003), pp. 1473-1481.

[7] Pietsch, A. Operator ideals. *North-Holland publishing company, Amesterdam.* (1980).

[8] Sourous, A. Isometries of norm ideals of compact operators. *J. Func. Analysis.* **43**(1981), pp-69-77.

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Inequalities of Ostrowski's type for $m-$ and $(\alpha, m)-$ logarithmically convex functions via Riemann-Liouville Fractional integrals

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In this paper, we establish some new Ostrowski's type inequalities for $m-$ and $(\alpha, m)-$ logarithmically convex functions by using the Riemann-Liouville fractional integrals.

1 INTRODUCTION

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [7]):

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1) see ([7, 8, 9, 10, 11, 18]) and the references therein.

Let us recall some known definitions and results which we will use in this paper. The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We can define starshaped functions on $[0, b]$ which satisfy the condition

$$f(tx) \leq tf(x)$$

for $t \in [0, 1]$.

The concept of $m-$convexity has been introduced by Toader in [5], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1 *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

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for all $x, y \in [0, b]$ *and* $t \in [0, 1]$. *We say that* f *is* $m-$*concave if* $-f$ *is* $m-$*convex.*

In [4], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 2 *The function* $f : [0, b] \to \mathbb{R}$, $b > 0$ *is said to be* $(\alpha, m)-$*convex, where* $(\alpha, m) \in [0, 1]^2$, *if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. For the recent results based on the above definitions see the papers [2]-[9].

Definition 3 *([1]) A function* $f : [0, b] \to (0, \infty)$ *is said to be* $m-$*logarithmically convex if the inequality*

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)} \tag{2}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, *and* $t \in [0, 1]$.

Obviously, if putting $m = 1$ in Definition 3, then f is just the ordinary logarithmically convex function on $[0, b]$.

Definition 4 *([1]) A function* $f : [0, b] \to (0, \infty)$ *is said to be* $(\alpha, m)-$*logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t^{\alpha}} [f(y)]^{m(1-t^{\alpha})} \tag{3}$$

holds for all $x, y \in [0, b]$, $(\alpha, m) \in (0, 1] \times (0, 1]$, *and* $t \in [0, 1]$.

Clearly, when taking $\alpha = 1$ in Definition 4, then f becomes the standard m-logarithmically convex function on $[0, b]$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 5 *Let* $f \in L_1[a, b]$. *The Riemann-Liouville integrals* $J_{a^+}^{\mu} f$ *and* $J_{b^-}^{\mu} f$ *of order* $\mu > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a^+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a$$

and

$$J_{b^-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\mu) = \int_0^{\infty} e^{-t} u^{\mu-1} du$. *Here is* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see [11]-[18].

The aim of this study is to establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are $m-$ and $(\alpha, m)-$ logarithmically convex functions via Riemann-Liouville fractional integral.

2 THE NEW RESULTS

In order to prove our results, we need the following lemma that has been obtained in [11]:

Lemma 1 *([11]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then for all $x \in [a,b]$ and $\mu > 0$ we have:*

$$\begin{aligned} & \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu} f(a) + J_{x^+}^{\mu} f(b)\right] \\ = & \frac{(x-a)^{\mu+1}}{b-a}\int_0^1 t^{\mu} f'(tx + (1-t)a)\,dt + \frac{(b-x)^{\mu+1}}{b-a}\int_0^1 t^{\mu} f'(tx + (1-t)b)\,dt \end{aligned}$$

where $\Gamma(\mu) = \int\limits_0^{\infty} e^{-t} u^{\mu-1} du.$

Theorem 2 *Let $f : [0,\infty) \to (0,\infty)$ be differentiable mapping with $a, b \in [0,\infty)$ such that $a < b$. If $|f'(x)|$ is $(\alpha, m)-$logarithmically convex function with $|f'(x)| \leq M$, $f' \in L[a,b]$, $(\alpha, m) \in (0,1] \times (0,1]$ and $\mu > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|\frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu} f(a) + J_{x^+}^{\mu} f(b)\right]\right| \\ \leq & \left[\frac{1}{2\mu+1} + K_1(\alpha, m, t)\right]\left[\frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{2(b-a)}\right] \end{aligned} \tag{4}$$

where

$$K_1(\alpha, m, t) = \begin{cases} \frac{M^{2m}\left(M^{2\alpha - 2\alpha m} - 1\right)}{(2\alpha - 2\alpha m)\ln M} & , M < 1 \\ 1 & , M = 1 \end{cases}.$$

Proof. By Lemma 1 and since $|f'|$ is $(\alpha, m)-$logarithmically convex, we can write

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x-}^{\mu}f(a)+J_{x+}^{\mu}f(b)\right]\right| \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\int_0^1 t^{\mu}\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\mu+1}}{b-a}\int_0^1 t^{\mu}\left|f'(tx+(1-t)b)\right|dt \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\int_0^1 t^{\mu}\left|f'(x)\right|^{t^{\alpha}}\left|f'\left(\frac{a}{m}\right)\right|^{m(1-t^{\alpha})}dt+\frac{(b-x)^{\mu+1}}{b-a}\int_0^1 t^{\mu}\left|f'(x)\right|^{t^{\alpha}}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^{\alpha})}dt \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\int_0^1 t^{\mu}M^{m+t^{\alpha}(1-m)}dt+\frac{(b-x)^{\mu+1}}{b-a}\int_0^1 t^{\mu}M^{m+t^{\alpha}(1-m)}dt.
\end{aligned}$$

By using the elemantery inequality $cd\leq\frac{c^2+d^2}{2}$, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x-}^{\mu}f(a)+J_{x+}^{\mu}f(b)\right]\right| \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\int_0^1 \frac{t^{2\mu}+M^{2(m+t^{\alpha}(1-m))}}{2}dt+\frac{(b-x)^{\mu+1}}{b-a}\int_0^1 \frac{t^{2\mu}+M^{2(m+t^{\alpha}(1-m))}}{2}dt \\
=\ & \left[\frac{1}{2\mu+1}+\int_0^1 M^{2(m+t^{\alpha}(1-m))}dt\right]\left[\frac{(x-a)^{\mu+1}+(b-x)^{\mu+1}}{2(b-a)}\right].
\end{aligned}\tag{5}$$

If we choose $M=1$, then

$$\int_0^1 M^{2(m+t^{\alpha}(1-m))}dt=1.$$

If $M<1$, then $M^{2(m+t^{\alpha}(1-m))}\leq M^{2(m+\alpha t(1-m))}$, thus

$$\int_0^1 M^{2(m+\alpha t(1-m))}dt=\frac{M^{2m}\left(M^{2\alpha-2\alpha m}-1\right)}{(2\alpha-2\alpha m)\ln M}$$

which completes the proof. ■

Corollary 3 *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|$ is $m-$logarithmically convex function with $|f'(x)|\leq M, f'\in L[a,b]$, $m\in(0,1]$ and $\mu>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x-}^{\mu}f(a)+J_{x+}^{\mu}f(b)\right]\right| \\
\leq\ & \left[\frac{1}{2\mu+1}+\frac{M^2-M^{2m}}{2\ln M-2m\ln M}\right]\left[\frac{(x-a)^{\mu+1}+(b-x)^{\mu+1}}{2(b-a)}\right].
\end{aligned}\tag{6}$$

Proof. If we take $\alpha = 1$ in (4), we get the required result. ∎

Corollary 4 *Let $f : [0,\infty) \to (0,\infty)$ be differentiable mapping with $a,b \in [0,\infty)$ such that $a < b$. If $|f'(x)|$ is logarithmically convex function with $|f'(x)| \leq M, f' \in L[a,b]$ and $\mu > 0$, then the following inequality for fractional integrals holds:*

$$\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a} \left[J_{x^-}^{\mu} f(a) + J_{x^+}^{\mu} f(b) \right] \right| \leq \left[\frac{1}{2\mu+1} + M^2 \right] \left[\frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{2(b-a)} \right]. \tag{7}$$

Proof. If we take $\alpha = m = 1$ in (5), we get the required result. ∎

Corollary 5 *Let $f : [0,\infty) \to (0,\infty)$ be differentiable mapping with $a,b \in [0,\infty)$ such that $a < b$. If $|f'(x)|$ is logarithmically convex function with $|f'(x)| \leq M$ and $f' \in L[a,b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \left[\frac{1}{3} + M^2 \right] \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right].$$

Proof. If we choose $\mu = 1$ in (7), we get the required result. ∎

Theorem 6 *Let $f : [0,\infty) \to (0,\infty)$ be differentiable mapping with $a,b \in [0,\infty)$ such that $a < b$. If $|f'(x)|^q$ is $(\alpha,m)-$logarithmically convex function with $|f'(x)|^q \leq M, f' \in L[a,b]$, $(\alpha,m) \in (0,1] \times (0,1]$ and $\mu > 0$, then the following inequality for fractional integrals holds:*

$$\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a} \left[J_{x^-}^{\mu} f(a) + J_{x^+}^{\mu} f(b) \right] \right| \leq \left(\frac{q-1}{\mu(q-p)+q-1} \right)^{\frac{q-1}{q}} (K_2(\alpha,m,t))^{\frac{1}{q}} \left[\frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{(b-a)} \right] \tag{8}$$

where $q > 1$, $0 \leq p \leq q$ and

$$K_2(\alpha,m,t) = \begin{cases} \frac{M^{qm}\left(\Gamma(\mu p+1) - \Gamma\left(\mu p+1, \ln M^{q\alpha(m-1)}\right)\right)}{\left(\ln M^{q\alpha(m-1)}\right)^{\mu p+1}} & , M < 1 \\ \frac{1}{\mu p+1} & , M = 1 \end{cases}.$$

Proof. From Lemma 1 and by using the properties of modulus, we have

$$\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a} \left[J_{x^-}^{\mu} f(a) + J_{x^+}^{\mu} f(b) \right] \right| \leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^{\mu} |f'(tx+(1-t)a)| dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^{\mu} |f'(tx+(1-t)b)| dt.$$

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By applying the Hölder inequality for $q>1$, $0\leq p\leq q$, we get

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu}f(a)+J_{x^+}^{\mu}f(b)\right]\right| \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\left[\left(\int_0^1 t^{\mu\left(\frac{q-p}{q-1}\right)}dt\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^{\mu+1}}{b-a}\left(\int_0^1 t^{\mu\left(\frac{q-p}{q-1}\right)}dt\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is easy to see that

$$\int_0^1 t^{\mu\left(\frac{q-p}{q-1}\right)}dt=\frac{q-1}{\mu(q-p)+q-1}.$$

Hence, by (α,m) −logarithmically convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu}f(a)+J_{x^+}^{\mu}f(b)\right]\right| \\
\leq\ & \frac{(x-a)^{\mu+1}}{b-a}\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}\left|f'(x)\right|^{qt^{\alpha}}\left|f'\left(\frac{a}{m}\right)\right|^{qm(1-t^{\alpha})}dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\mu+1}}{b-a}\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}t^{\mu}\left|f'(x)\right|^{qt^{\alpha}}\left|f'\left(\frac{b}{m}\right)\right|^{qm(1-t^{\alpha})}dt\right)^{\frac{1}{q}} \\
=\ & \frac{(x-a)^{\mu+1}}{b-a}\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}M^{qm+qt^{\alpha}(1-m)}dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\mu+1}}{b-a}\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}\left(\int_0^1 t^{\mu p}M^{qm+qt^{\alpha}(1-m)}dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{9}$$

If we choose $M=1$, then

$$\int_0^1 t^{\mu p}dt=\frac{1}{\mu p+1}.$$

If $M<1$, then $M^{qm+qt^{\alpha}(1-m)}\leq M^{qm+q\alpha t(1-m)}$, thus

$$\int_0^1 t^{\mu p}M^{qm+q\alpha t(1-m)}dt=\frac{M^{qm}\left(\Gamma(\mu p+1)-\Gamma\left(\mu p+1,\ln M^{q\alpha(m-1)}\right)\right)}{\left(\ln M^{q\alpha(m-1)}\right)^{\mu p+1}}$$

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which completes the proof. ■

Corollary 7 *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is $m-$logarithmically convex function with $|f'(x)|^q\leq M, f'\in L[a,b]$, $m\in(0,1]$ and $\mu>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu}f(a)+J_{x^+}^{\mu}f(b)\right]\right| \\
\leq\quad &\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}(K_2(1,m,t))^{\frac{1}{q}}\left[\frac{(x-a)^{\mu+1}+(b-x)^{\mu+1}}{(b-a)}\right]
\end{aligned}$$

where $q>1$, $0\leq p\leq q$ and

$$K_2(1,m,t)=\begin{cases}\dfrac{M^{qm}\left(\Gamma(\mu p+1)-\Gamma\left(\mu p+1,\ln M^{q(m-1)}\right)\right)}{\left(\ln M^{q(m-1)}\right)^{\mu p+1}} & ,M<1\\[2ex] \dfrac{1}{\mu p+1} & ,M=1\end{cases}.$$

Proof. If we set $\alpha=1$ in 8, the proof is completed. ■

Corollary 8 *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is logarithmically convex function with $|f'(x)|^q\leq M, f'\in L[a,b]$ and $\mu>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{(x-a)^{\mu}+(b-x)^{\mu}}{b-a}f(x)-\frac{\Gamma(\mu+1)}{b-a}\left[J_{x^-}^{\mu}f(a)+J_{x^+}^{\mu}f(b)\right]\right| \\
\leq\quad & M^m\left(\frac{q-1}{\mu(q-p)+q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{\mu p+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\mu+1}+(b-x)^{\mu+1}}{(b-a)}\right]
\end{aligned}$$

where $q>1$, $0\leq p\leq q$.

Proof. If we set $\alpha=m=1$ in 9, the proof is completed. ■

Corollary 9 *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is $(\alpha,m)-$logarithmically convex function with $|f'(x)|^q\leq M, f'\in L[a,b]$ and $(\alpha,m)\in(0,1]\times(0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \qquad (10)\\
\leq\quad &\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}(K_1(\alpha,m,t))^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{(b-a)}\right]
\end{aligned}$$

where $q>1,\ 0\leq p\leq q$ *and*

$$K_3(\alpha,m,t)=\begin{cases}\frac{M^{qm}\left(\Gamma(p+1)-\Gamma\left(p+1,\ln M^{q\alpha(m-1)}\right)\right)}{\left(\ln M^{q\alpha(m-1)}\right)^{p+1}} & ,M<1\\ \frac{1}{p+1} & ,M=1\end{cases}.$$

Proof. If we set $\mu=1$ in 8, the proof is completed. ■

Corollary 10 *Let* $f:[0,\infty)\to(0,\infty)$ *be differentiable mapping with* $a,b\in[0,\infty)$ *such that* $a<b$. *If* $|f'(x)|^q$ *is* (α,m) *−logarithmically convex function with* $|f'(x)|^q\leq M, f'\in L[a,b]$ *and* $(\alpha,m)\in(0,1]\times(0,1]$, *then the following inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\\ \leq\ &\left(\frac{1}{2}\right)^{\frac{q-1}{q}}(K_1(\alpha,m,t))^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{(b-a)}\right]\end{aligned}$$

where $q>1,\ 0\leq p\leq q$ *and*

$$K_4(\alpha,m,t)=\begin{cases}\frac{M^{qm}\left(\Gamma(2)-\Gamma\left(2,\ln M^{q\alpha(m-1)}\right)\right)}{\left(\ln M^{q\alpha(m-1)}\right)^{2}} & ,M<1\\ \frac{1}{2} & ,M=1\end{cases}.$$

Proof. If we set $p=1$ in 10, the proof is completed. ■

References

[1] R.-F. Bai, F. Qi and B.-Y. Xi, *Hermite-Hadamard type inequalities for the* $m-$ *and* (α,m) *−logarithmically convex functions*, Filomat 27 (2013), 1-7.

[2] M.K. Bakula, J. Pečarić and M. Ribibić, *Companion inequalities to Jensen's inequality for* $m-$*convex and* $(\alpha,m)-$*convex functions*, J. Inequal. Pure and Appl. Math., 7 (5) (2006), Article 194.

[3] S.S. Dragomir and G. Toader, *Some inequalities for* $m-$*convex functions*, Studia University Babes Bolyai, Mathematica, 38 (1) (1993), 21-28.

[4] V.G. Miheşan, *A generalization of the convexity*, Seminar of Functional Equations, Approx. and Convex, Cluj-Napoca (Romania) (1993).

[5] G. Toader, *Some generalization of the convexity*, Proc. Colloq. Approx. *Opt.*, Cluj-Napoca, (1984), 329-338.

[6] G. Toader, *On a generalization of the convexity*, Mathematica, 30 (53) (1988), 83-87.

[7] A. Ostrowski, *Über die Absolutabweichung einer differentierbaren Funktion von ihren Integralmittelwert*, Comment. Math. Helv., 10, 226-227, (1938).

[8] M.E. Özdemir, H. Kavurmaci, E. Set, *Ostrowski's type inequalities for* $(\alpha, m)-$*convex functions*, KYUNGPOOK Math. J. 50 (2010) 371–378.

[9] H. Kavurmaci, M.E. Özdemir and M. Avci, *New Ostrowski type inequalities for* $m-$*convex functions and applications*, Hacettepe Journal of Mathematics and Statistics, Volume 40 (2) (2011), 135 – 145.

[10] M. Alomari and M. Darus, *Some Ostrowski type inequalities for convex functions with applications*, RGMIA Res. Rep. Coll., (2010) 13, 2, Article 3. [ONLINE: http://ajmaa.org/RGMIA/v13n2.php].

[11] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl., 63 (2012) 1147-1154.

[12] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Ineq. Pure and Appl. Math., 10(3), Art. 86 (2009).

[13] Z. Dahmani, *New inequalities in fractional integrals,* International Journal of Nonlinear Science, 9(4), 493-497 (2010).

[14] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(1), 51-58 (2010).

[15] Z. Dahmani, L. Tabharit and S. Taf, *Some fractional integral inequalities*, Nonl. Sci. Lett. A., 1(2), 155-160 (2010).

[16] M.Z. Sarıkaya, E. Set, H. Yaldiz and N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, In Press.

[17] Z. Dahmani, L. Tabharit and S. Taf, *New generalizations of Grüss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3), 93-99 (2010).

[18] M.E. Özdemir, H. Kavurmacı and M. Avcı, *New inequalities of Ostrowski type for mappings whose derivatives are* (α, m)*-convex via fractional integrals*, RGMIA Research Report Collection, 15, Article 10, 8 pp (2012).

ON A CLASS OF OPERATORS FROM BERGMAN-TYPE SPACES TO WEIGHTED-TYPE SPACES

LI KE AND JIANG ZHI-JIE

Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For φ an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$, we define two operators on $H(\mathbb{D})$ by $DW_{\varphi,g}f = (g \cdot f \circ \varphi)'$ and $W_{\varphi,g}Df = (g \cdot f' \circ \varphi)$. By using some growth properties of the inducing maps φ and g to the boundary of $\mathbb{D}$, we obtain an asymptotical expression of the essential norms for operators $DW_{\varphi,g}$ and $W_{\varphi,g}D$ from Bergman-type to weighted-type spaces in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. If u is a positive continuous function on $[0, 1)$ and there exist positive numbers $\delta \in [0, 1)$, s and t, $0 < s < t$, such that $u(r)/(1 - r)^s$ is decreasing on $[\delta, 1)$ and $\lim_{r\to 1^-} u(r)/(1 - r)^s = 0$; $u(r)/(1 - r)^t$ is increasing on $[\delta, 1)$ and $\lim_{r\to 1^-} u(r)/(1 - r)^t = \infty$, then u is called a normal weight function (see [1]). For such normal weights, one can consider the following examples:

$$u(r) = (1 - r^2)^\alpha, \ \ \alpha \in (0, \infty),$$

$$u(r) = (1 - r^2)^\alpha\{\log 2(1 - r^2)^{-1}\}^\beta, \ \ \alpha \in (0, 1), \ \ \beta \in \big[\frac{\alpha - 1}{2} \log 2, 0\big],$$

and

$$u(r) = (1 - r^2)^\alpha\{\log \log e^2(1 - r^2)^{-1}\}^\gamma, \ \ \alpha \in (0, 1), \ \ \gamma \in \big[\frac{\alpha - 1}{2} \log 2, 0\big].$$

For $0 < p < \infty$ and the normal weight function u, the Bergman-type space A^p_u on $\mathbb{D}$ is defined by

$$A^p_u = \Big\{f \in H(\mathbb{D}) : \|f\|^p = \int_{\mathbb{D}} |f(z)|^p \frac{u(|z|)^p}{1 - |z|} dA(z) < \infty\Big\}.$$

When $1 \le p < \infty$, A^p_u is a Banach space with the norm $\|\cdot\|$. When $0 < p < 1$, it is a Fréchet space with the translation invariant metric

$$d(f, g) = \|f - g\|^p.$$

For this space and some operators, see, e.g., [1] and [2].

Let $0 < \alpha < \infty$, and let H^∞_α be the weighted Banach space of analytical functions on $\mathbb{D}$ satisfying

$$\|f\|_{H^\infty_\alpha} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha|f(z)| < \infty.$$

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Let ν be a radial bounded continuous positive function $\mathbb{D}$. As the generalization of the weighted Banach space, the weighted-type space H_ν^∞ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\nu^\infty} = \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty.$$

It is known that H_ν^∞ is a Banach space under the norm $\|\cdot\|_{H_\nu^\infty}$.

Let φ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. For $f \in H(\mathbb{D})$, the weighted composition operator $W_{\varphi,g}$ is defined by

$$W_{\varphi,g} f(z) = g(z) \cdot f(\varphi(z)).$$

When $g \equiv 1$ on $\mathbb{D}$, $W_{\varphi,1} := C_\varphi$ is called the composition operator. When $\varphi(z) = z$, $W_{z,g} := M_g$ is the multiplication operator. During the past few decades, weighted composition operators have been studied extensively on spaces of analytic functions on $\mathbb{D}$. For some recent results, see, e.g., [3]-[10]. Using the weighted composition operator $W_{\varphi,g}$, we define the following two operators:

$$DW_{\varphi,g} f(z) = (g \cdot f \circ \varphi)'(z)$$

and

$$W_{\varphi,g} D f(z) = (g \cdot f' \circ \varphi)(z).$$

The present author introduced the definitions of $DW_{\varphi,g}$ and $W_{\varphi,g}D$ and studied them in [11]. If $g \equiv 1$ on $\mathbb{D}$, then $DW_{\varphi,1} = DC_\varphi$ and $W_{\varphi,1}D = C_\varphi D$ are called the products of differentiation and composition. Hibschweiler and Portnoy [12] considered the bounded and compact DC_φ and $C_\varphi D$ on Hardy and weighted Bergman spaces. Ohno [13] also studied these problems for them on Bloch and little Bloch spaces.

It is well-known that by calculating the essential norm one can also give a characterization of compactness of the linear operator. Recently, we have obtained some asymptotical expression of the essential norms of the weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit disk and unit ball in [14] and [15]. It is a natural problem how to express the essential norms of $DW_{\varphi,g}$ and $W_{\varphi,g}D$ from the Bergman-type spaces to the weighted-type spaces. In this paper, we shall consider this problem.

At the last of this section, we recall the definition of the essential norm of the bounded linear operators. Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of the operator $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\| : K \in \mathcal{K}\},$$

where $\mathcal{K}$ denotes the set of all compact linear operators from X to Y. By this definition, we have that the bounded linear operator $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. The main results

The following lemma is also right for the bounded operator $W_{\varphi,g}D : A^p_u \to H^\infty_\nu$. Since the proof is standard, it is omitted (see, e.g., Proposition 3.11 in [16]).

Lemma 2.1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and the operator $DW_{\varphi,g} : A^p_u \to H^\infty_\nu$ is bounded, then the operator $DW_{\varphi,g} : A^p_u \to H^\infty_\nu$ is compact if and only if for bounded sequence $(f_n)_{n\in\mathbb{N}}$ in A^p_u such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that*

$$\lim_{n\to\infty} \|DW_{\varphi,g} f_n\|_{H^\infty_\nu} = 0.$$

For the cases of $n = 0$ and $n = 1$, the following lemma had been obtained in [2]. However, here we shall give the proof for $n \in \mathbb{N}_+$.

Lemma 2.2. *For $n \in \mathbb{N}_+$, there is a positive constant C independent of $f \in A^p_u$ such that for every $z \in \mathbb{D}$ the following inequality holds*

$$|f^{(n)}(z)| \le C \frac{\|f\|}{u(|z|)(1-|z|^2)^{n+\frac{1}{p}}}.$$

Proof. For $f \in A^p_u$, by Lemma 2.1 in [2] we have that

$$|f(z)| \le C \frac{\|f\|}{u(|z|)(1-|z|^2)^{\frac{1}{p}}}. \tag{1}$$

Since u is normal, it is not difficult to see that for each $w \in \{z + \frac{1-|z|}{2}\xi : \xi \in \partial\mathbb{D}\} = D(z, \frac{1-|z|}{2})$, the following relationship holds

$$u(|z|) \asymp u(w), \tag{2}$$

where $D(z, \frac{1-|z|}{2})$ denotes the disk with center z and radius $\frac{1-|z|}{2}$.

Since $1 - |z + \frac{1-|z|}{2}\xi|^2 \ge \frac{1-|z|^2}{4}$ for all $\xi \in \partial\mathbb{D}$, by the Cauchy integral formula we get that

$$(1-|z|^2)^n |f^{(n)}(z)| \le \frac{n! 2^{2n+1}}{\pi} \int_{\partial\mathbb{D}} \big|f\big(z + \frac{1-|z|}{2}\xi\big)\big||d\xi|. \tag{3}$$

By (1), (2) and (3) we have that

$$\begin{aligned}(1-|z|^2)^n |f^{(n)}(z)| &\le \frac{n! 2^{2n+1}}{\pi} \int_{\partial\mathbb{D}} \big|f\big(z + \frac{1-|z|}{2}\xi\big)\big||d\xi| \\ &\le C \frac{\|f\|}{u(|z|)(1-|z|^2)^{\frac{1}{p}}},\end{aligned}$$

from which the desired result follows.

In fact, the next result was obtained in [2].

Lemma 2.3. *Suppose that φ is an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$, then the operator $DW_{\varphi,g} : A^p_u \to H^\infty_\nu$ is bounded if and only if the following conditions are satisfied:*

(i)

$$M_0 := \sup_{z\in\mathbb{D}} \frac{\nu(z)|g'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}}} < \infty,$$

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(ii)

$$M_1 := \sup_{z\in\mathbb{D}} \frac{\nu(z)|g(z)||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} < \infty.$$

Now we begin to formulate and proof the main result of this paper.

Theorem 2.4. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and $DW_{\varphi,g} : A^p_u \to H^\infty_\nu$ is bounded, then*

$$\|DW_{\varphi,g}\|_{e,A^p_u\to H^\infty_\nu} \asymp \limsup_{j\to\infty} \frac{\nu(z_j)|g'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{\frac{1}{p}}} + \limsup_{j\to\infty} \frac{\nu(z_j)|g(z_j)||\varphi'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{1+\frac{1}{p}}}.$$

Proof. Suppose that $\{\varphi(z_j)\}_{j\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1^-$ as $j \to \infty$. For this sequence $\{\varphi(z_j)\}_{j\in\mathbb{N}}$, by taking $\{f_{\varphi(z_j)}\}$ the functions in the proof of Lemma 2.3, we have seen that $\max_{j\in\mathbb{N}} \|f_{\varphi(z_j)}\| \le C$ and $f_{\varphi(z_j)} \to 0$ uniformly on compacta of $\mathbb{D}$ as $j \to \infty$. Hence for every compact operator $K : A^p_u \to H^\infty_\nu$, we have $\|Kf_{\varphi(z_j)}\|_{H^\infty_\nu} \to 0$ as $j \to \infty$. Thus it follows that

$$\begin{aligned}
\|DW_{\varphi,g} - K\| &= \sup_{\|f\|=1} \|(DW_{\varphi,g} - K)f\|_{H^\infty_\nu} \\
&\ge \limsup_{j\to\infty} \frac{\|(DW_{\varphi,g} - K)f_{\varphi(z_j)}\|_{H^\infty_\nu}}{\|f_{\varphi(z_j)}\|} \\
&\ge \limsup_{j\to\infty} \frac{\|DW_{\varphi,g}f_{\varphi(z_j)}\|_{H^\infty_\nu} - \|Kf_{\varphi(z_j)}\|_{H^\infty_\nu}}{\|f_{\varphi(z_j)}\|} \\
&\ge C^{-1} \limsup_{j\to\infty} \frac{\nu(z_j)|g(z_j)||\varphi'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{1+\frac{1}{p}}}. \qquad (4)
\end{aligned}$$

By taking the infimum in (4) over the set of all compact operators $K : A^p_u \to H^\infty_\nu$, we obtain that

$$\|DW_{\varphi,g}\|_{e,A^p_u\to H^\infty_\nu} \ge C^{-1} \limsup_{j\to\infty} \frac{\nu(z_j)|g(z_j)||\varphi'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{1+\frac{1}{p}}}. \qquad (5)$$

By using the similar method, we also can prove that

$$\|DW_{\varphi,g}\|_{e,A^p_u\to H^\infty_\nu} \ge C^{-1} \limsup_{j\to\infty} \frac{\nu(z_j)|g'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{\frac{1}{p}}}.$$

Consequently, we have obtained that

$$\begin{aligned}
\|DW_{\varphi,g}\|_{e,A^p_u\to H^\infty_\nu} \ge& C\Big(\limsup_{j\to\infty} \frac{\nu(z_j)|g'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{\frac{1}{p}}} \\
&+ \limsup_{j\to\infty} \frac{\nu(z_j)|g(z_j)||\varphi'(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{1+\frac{1}{p}}}\Big). \qquad (6)
\end{aligned}$$

Now suppose that $\{r_j\}_{j\in\mathbb{N}}$ is a positive sequence which increasingly converges to 1. For each r_j, consider the operator $DW_{r_j\varphi,g}$. By Lemma 2.3, the boundedness of $DW_{\varphi,g} : A^p_u \to H^\infty_\nu$ implies that the operator $DW_{r_j\varphi,g} : A^p_u \to H^\infty_\nu$ is bounded. Since $|r_j\varphi(z)| \le r_j < 1$, by Lemma 2.1 we have that the operator $DW_{r_j\varphi,g} : A^p_u \to H^\infty_\nu$ is also compact. Hence we have that

$$\|DW_{\varphi,g} - DW_{r_j\varphi,g}\| = \sup_{\|f\|=1} \|(DW_{\varphi,g} - DW_{r_j\varphi,g})f\|_{H^\infty_\nu}$$

$$
\begin{aligned}
&= \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)\big|\big(g(z)(f\circ\varphi(z))\big)' - \big(g(z)(f\circ r_j\varphi(z))\big)'\big| \\
&= \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)\Big|g'(z)f(\varphi(z)) + g(z)f'(\varphi(z))\varphi'(z) \\
&\quad - g'(z)f(r_j\varphi(z)) - r_jg(z)f'(r_j\varphi(z))\varphi'(z)\Big| \\
&\le \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)|g'(z)|\big|f(\varphi(z)) - f(r_j\varphi(z))\big| \\
&\quad + \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)|g(z)||\varphi'(z)|\big|f'(\varphi(z)) - r_jf'(r_j\varphi(z))\big| \\
&\le \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)|g'(z)|\big|f(\varphi(z)) - f(r_j\varphi(z))\big| \\
&\quad + (1-r_j)\sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)|g(z)||\varphi'(z)|\big|f'(r_j\varphi(z))\big| \\
&\quad + \sup_{\|f\|=1}\sup_{z\in\mathbb{D}}\nu(z)|g(z)||\varphi'(z)|\big|f'(\varphi(z)) - f'(r_j\varphi(z))\big| \\
&\le \sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\nu(z)|g'(z)|\big|f(\varphi(z)) - f(r_j\varphi(z))\big| \\
&\quad + \sup_{\|f\|=1}\sup_{|\varphi(z)|>\delta}\nu(z)|g'(z)|\big|f(\varphi(z)) - f(r_j\varphi(z))\big| \\
&\quad + C(1-r_j)\sup_{z\in\mathbb{D}}\frac{\nu(z)|g(z)||\varphi'(z)|}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{1+\frac{1}{p}}} \\
&\quad + \sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\nu(z)|g(z)||\varphi'(z)|\big|f'(\varphi(z)) - f'(r_j\varphi(z))\big| \\
&\quad + \sup_{\|f\|=1}\sup_{|\varphi(z)|>\delta}\nu(z)|g(z)||\varphi'(z)|\big|f'(\varphi(z)) - f'(r_j\varphi(z))\big| \\
&\le \|DW_{\varphi,g}1\|_{H_\nu^\infty}\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}\big|f(\varphi(z)) - f(r_j\varphi(z))\big| \\
&\quad + C\sup_{|\varphi(z)|>\delta}\frac{\nu(z)|g'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}}} \\
&\quad + C\sup_{|\varphi(z)|>\delta}\frac{\nu(z)|g'(z)|}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{\frac{1}{p}}} \\
&\quad + C(1-r_j)\sup_{z\in\mathbb{D}}\frac{\nu(z)|g(z)||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} \\
&\quad + \sup_{z\in\mathbb{D}}\nu(z)|g(z)||\varphi'(z)|\sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}|f'(\varphi(z)) - f'(r_j\varphi(z))\big| \\
&\quad + C\sup_{|\varphi(z)|>\delta}\frac{\nu(z)|g(z)||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} \\
&\quad + C\sup_{|\varphi(z)|>\delta}\frac{\nu(z)|g(z)||\varphi'(z)|}{u(r_j|\varphi(z)|)(1-r_j^2|\varphi(z)|^2)^{1+\frac{1}{p}}}.
\end{aligned}
\tag{7}
$$

We consider

$$
I_j^0 := \sup_{\|f\|=1}\sup_{|\varphi(z)|\le\delta}|f(\varphi(z)) - f(r_j\varphi(z))|.
$$

By using the mean value theorem and the subharmonicity of f and Lemma 2.2 we have

$$\begin{aligned} I_j^0 &\le \sup_{\|f\|=1} \sup_{|\varphi(z)|\le\delta} (1-r_j)|\varphi(z)| \sup_{|z|\le\delta} |f'(z)| \\ &\le \frac{C(1-r_j)}{\max\limits_{0\le r\le\delta} u(r)(1-\delta^2)^{1+\frac{1}{p}}}. \end{aligned} \tag{8}$$

By (8), we obtain that $I_j^0 \to 0$ as $j\to\infty$. Using the same method, we also have that $I_j^1 = \sup\limits_{\|f\|=1} \sup\limits_{|\varphi(z)|\le\delta} |f'(\varphi(z)) - f'(r_j\varphi(z))| \to 0$ as $j\to\infty$. Letting $j\to\infty$ in (7), from the above discussions and the boundedness of $DW_{\varphi,g} : A_u^p \to H_\nu^\infty$ we obtain that

$$\begin{aligned} \|DW_{\varphi,g} - DW_{r_j\varphi,g}\| \le 2C &\sup_{|\varphi(z)|>\delta} \frac{\nu(z)|g'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}}} \\ &+ 2C \sup_{|\varphi(z)|>\delta} \frac{\nu(z)|g(z)||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} \end{aligned}$$

as $j\to\infty$. Since $\|DW_{\varphi,g}\|_{e,A_u^p\to H_\nu^\infty} \le \|DW_{\varphi,g} - DW_{r_j\varphi,g}\|$, we end the proof.

Corollary 2.5. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$ and $DW_{\varphi,g} : A_u^p \to H_\nu^\infty$ is bounded, then $DW_{\varphi,g} : A_u^p \to H_\nu^\infty$ is compact if and only if the following conditions are satisfied:*

(i)

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|g'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}}} = 0,$$

(ii)

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|g(z)||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} = 0.$$

Similar to Theorem 2.4, we can prove the following result.

Theorem 2.6. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$ and $W_{\varphi,g}D : A_u^p \to H_\nu^\infty$ is bounded, then*

$$\|W_{\varphi,g}D\|_{e,A_u^p\to H_\nu^\infty} \asymp \limsup_{j\to\infty} \frac{\nu(z_j)|g(z_j)|}{u(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{1+\frac{1}{p}}}.$$

By Theorem 2.6, we have

Corollary 2.7. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$ and $W_{\varphi,g}D : A_u^p \to H_\nu^\infty$ is bounded, then $W_{\varphi,g}D : A_u^p \to H_\nu^\infty$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|g(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+\frac{1}{p}}} = 0.$$

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References

[1] Jiang, Z. J., On Volterra compositon operators from Bergman-type space to Bloch-type space, *Czechoslovak Mathematicial Journal,* 2011, 136:993-1005.

[2] Li, S. and Stević, S., Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci.,* 2007, 117:371-385.

[3] Jiang, Z. J. and Bai, H. B., Weighted composition operators on Hardy space $H^p(B_N)$(In Chinese), *Advances in Mathematics,* 2008, 37:749-754.

[4] Jiang, Z. J., Weighted composition operators from Bergman-type spaces to Bers-type spaces, *Acta Mathematics Sinica,* (Chinese Series), 2010, 53:67-74.

[5] Stević, S. and Jiang, Z. J., Differences of weighted composition operators on the unit polydisk, *Siberian Mathematics Journal,* 2011, 52:358-371.

[6] Stević, S., Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball, *Ars. Combin.,* 2008, 88:125-127.

[7] Li, S. and Stević, S., Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.,* 2008, 206:825-831.

[8] Ohno, S., Weighted composition operators between H^∞ and the Bloch space, *Taiwan. J. Math. Soc.,* 2001, 5:555-563.

[9] Sharma, A. K. and Abbas, Z., Weighted composition operators between weighted Bergman-Nevanlinna and Bloch-type spaces, *Applied Mathematical Sciences.,* 2010, 41:2039-2048.

[10] Zhou, Z. H. and Chen, R. Y., Weighted composition operators from $F(p, q, s)$ to Bloch space on the unit ball, *Internat. J. Math.,* 2008, 19:899-926.

[11] Jiang, Z. J., On a class of opertors from weighted Bergman spaces to some spaces of analytic functions, *Taiwanese Journal of Mathematics,* 2011, 15:2095-2121.

[12] Hibschweiler, R. A. and Portnoy, N., Composition followed by differentiation between Bergman and Hardy spaces, *Rocky Mountain J. Math.,* 2005, 35:843-855.

[13] Ohno, S., Products of composition and differentiation on Bloch spaces, *Bull. Korean Math. Soc.,* 2009, 46:1135-1140.

[14] Jiang, Z. J. and Stević, S., Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces, *Appl. Math. Comput.,* 2010, 217:3522-3530.

[15] Stević, S. and Jiang, Z. J., Compactness of differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, *Taiwanese Journal of Mathematics,* 2011, 15:2647-2665.

[16] Cowen, C. C. and MacCluer, B. D., Composition operators on spaces of analytic functions, CRC Press, 1995.

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GENERALIZED JORDAN HOMOMORPHISMS IN FRÉCHET ALGEBRAS

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ABSTRACT. A linear mapping $h : A \to B$ is called a generalized Jordan homomorphism if there exists a homomorphism $h' : A \to B$ such that $h(a^2) = h(a)h'(a)$ for all $a \in A$.

In this paper, we investigate generalized Jordan homomorphisms in Fréchet algebras, associated with the following functional equation

$$f\left(\frac{a+b}{2}\right) + f\left(\frac{a-b}{2}\right) = f(a).$$

Moreover, we prove the Hyers-Ulam stability of generalized Jordan homomorphisms in Fréchet algebras.

1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam [30] in 1940, concerning the stability of group homomorphisms. In 1941, Hyers [20] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Th. M. Rassias [26] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 4, 6, 7, 10, 11, 12, 15, 21, 24, 25, 28]).

Definition 1.1. A topological vector space X is a Fréchet space if it satisfies the following three properties:

(1) it is complete as a uniform space,

(2) it is locally convex,

(3) its topology can be induced by a translation invariant metric, i.e., a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x+a, y+a)$ for all $a, x, y \in X$.

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For more detailed definitions of such terminologies, we can refer to [8]. Note that a ternary algebra is called ternary Fréchet algebra if it is a Fréchet space with a metric d.

Fréchet algebras, named after Maurice Fréchet, are special topological algebras as follows.

Note that the topology on A can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

Trivially, every Banach algebra is a Fréchet algebra as the norm induces a translation invariant metric and the space is complete with respect to this metric.

Definition 1.2. Let A and B be two algebras. A linear mapping $h : A \to B$ is called a generalized Jordan homomorphism if there exists a homomorphism $h' : A \to B$ such that $h(a^2) = h(a)h'(a)$ for all $a, b \in A$.

For example, every Jordan homomorphism (resp., homomorphism) is a generalized Jordan homomorphism (resp., generalized homomorphism), but the converse is false, in general. For instance, let A be an algebra over $\mathbb{C}$ and let $h : A \to A$ be a non-zero Jordan homomorphism (resp., homomorphism) on A. Then we have $ih(a^2) = ih(a)^2 = ih(a)h(a)$, (resp., $ih(ab) = ih(a)h(b) = ih(a)h(b)$). This means that ih is a generalized Jordan homomorphism (resp., generalized homomorphism). It is easy to see that ih is not a Jordan homomorphism (resp., homomorphism).

Th.M. Rassias [27], Gajda [19] and Bourgin [5] proved the stability problem of ring homomorphisms between unital Banach algebras. Badora [1] proved the Hyers-Ulam stability of ring homomorphisms, which generalizes the result of Bourgin. Miura al et. [22] proved the Hyers-Ulam stability of Jordan homomorphisms. For more details about the results concerning stability of functional equations on Banach algebras, the reader refer to [2, 13, 14, 16, 17, 18].

Recently, Eshaghi Gordji and Bavand Savadkouhi [9] proved the Hyers-Ulam stability of generalized homomorphisms in quasi-Banach algebras.

In this paper, we prove the Hyers-Ulam stability of generalized Jordan homomorphisms in Fréchet algebras.

2. Hyers-Ulam stability of generalized Jordan homomorphisms in Fréchet algebras

In this section, we prove the Hyers-Ulam stability of generalized Jordan homomorphisms in Fréchet algebras.

Lemma 2.1. ([23]) *Let X and Y be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} \ : |\mu| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 2.2. *Let A be a Fréchet algebra with metric d and unit e, and B a Banach algebra with norm $\|\cdot\|$ and unit I. Let $f, g : A \to B$ be mappings with $f(0) = 0$, $g(0) = 0$ and $g(e) = I$ for which there exists a function $\phi : A^3 \to [0, \infty)$ such that*

$$\tilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \phi(2^j x, 2^j y, 2^j z) < \infty, \tag{2.1}$$

$$\left\| f\left(\frac{\mu x + \mu y}{2} + z^2\right) + f\left(\frac{\mu x - \mu y}{2}\right) - \mu f(x) + f(z)g(z) \right\| \le \phi(x, y, z), \tag{2.2}$$

$$\left\| 2g\left(\frac{\mu xy + \mu z}{2}\right) - \mu g(x)g(y) - \mu g(z) \right\| \le \phi(x, y, z) \tag{2.3}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan homomorphism $H : A \to B$ and a unique homomorphism $G : A \to B$ such that

$$H(x^2) = H(x)G(x),$$

$$\|f(x) - H(x)\| \le \tilde{\phi}(x, 0, 0), \tag{2.4}$$

$$\|g(x) - G(x)\| \le \tilde{\phi}(x, e, 0) \tag{2.5}$$

for all $x \in A$.

Proof. Putting $y = z = 0$ and $\mu = 1$ in (2.2), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \le \phi(x, 0, 0)$$

and so

$$\left\| \frac{f(2x)}{2} - f(x) \right\| \le \frac{\phi(2x, 0, 0)}{2} \tag{2.6}$$

for all $x \in A$. Using the Rassias' method on (2.6) ([19]), one can use induction on n to show that

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \le \sum_{j=1}^{n} \frac{\phi(2^j x, 0, 0)}{2^j} \tag{2.7}$$

for all $x \in A$ and all nonnegative integers n. Hence

$$\left\| \frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m} \right\| \leq \sum_{j=m+1}^{n+m} \frac{\phi(2^j x, 0, 0)}{2^j}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in A$. It follows from (2.1) that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2.8}$$

for all $x \in A$. Letting $z = 0$ and replacing x, y by $2^n x, 2^n y$, respectively, in (2.2) and multiplying both sides by $\frac{1}{2^n}$, we get

$$\begin{aligned}
&\left\| H\left(\frac{\mu x + \mu y}{2}\right) + H\left(\frac{\mu x - \mu y}{2}\right) - \mu H(x) \right\| \\
&= \lim_{n\to\infty} \frac{1}{2^n} \left\| f\left(\frac{2^n \mu(x+y)}{2}\right) + f\left(\frac{2^n \mu(x-y)}{2}\right) - \mu f(2^n x) \right\| \\
&\leq \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 0)}{2^n}
\end{aligned}$$

for all $x, y \in A$, $\mu \in \mathbb{T}^1$ and all nonnegative integers n. Taking the limit as $n \to \infty$, we obtain

$$H\left(\frac{\mu x + \mu y}{2}\right) + H\left(\frac{\mu x - \mu y}{2}\right) = \mu H(x) \tag{2.9}$$

for all $x, y \in A$ and all $\mu \in \mathbb{T}^1$. Letting $\mu = 1$ in (2.9), we can easily show that H is additive. Letting $y = 0$ in (2.9), we get

$$H(\mu x) = 2H\left(\frac{\mu x}{2}\right) = \mu H(x)$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.1, the mapping $H : A \to B$ is $\mathbb{C}$-linear. Moreover, it follows from (2.7) and (2.8) that

$$\|f(x) - H(x)\| \leq \tilde{\phi}(x, 0, 0)$$

for all $x \in A$.

Putting $y = e$, $z = 0$ and $\mu = 1$ in (2.3), we get

$$\left\| 2g\left(\frac{x}{2}\right) - g(x) \right\| \leq \phi(x, e, 0)$$

and so

$$\left\| \frac{g(2x)}{2} - g(x) \right\| \leq \frac{\phi(2x, e, 0)}{2}$$

for all $x \in A$.

By the same method as above, one can show that there is a $\mathbb{C}$-linear mapping $G : A \to B$ satisfying (2.5). The mapping $G : A \to B$ is given by

$$G(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$$

for all $x \in A$.

Putting $z = 0$ and $\mu = 1$ in (2.3), we get

$$\left\| 2g\left(\frac{xy}{2}\right) - g(x)g(y)\right\| \le \phi(x, y, 0)$$

and so

$$\frac{1}{4^n}\left\| 2g\left(\frac{4^n xy}{2}\right) - g(2^n x)g(2^n y)\right\| \le \frac{1}{4^n}\phi(2^n x, 2^n y, 0) \le \frac{1}{2^n}\phi(2^n x, 2^n y, 0),$$

which tends to zero as $n \to \infty$. Thus $G(xy) = 2G\left(\frac{xy}{2}\right) = G(x)G(y)$ for all $x, y \in A$. So $G : A \to B$ is a homomorphism.

Putting $x = y = 0$ and $\mu = 1$ in (2.2), we get

$$\left\| f\left(z^2\right) - f(z)g(z)\right\| \le \phi(0, 0, z)$$

and so

$$\frac{1}{4^n}\left\| f\left(4^n z^2\right) - f(2^n z)g(2^n z)\right\| \le \frac{1}{4^n}\phi(0, 0, 2^n z) \le \frac{1}{2^n}\phi(0, 0, 2^n z),$$

which tends to zero as $n \to \infty$. Thus $H(z^2) = H(z)G(z)$ for all $z \in A$. So $H : A \to B$ is a generalized Jordan homomorphism.

Now, let $H' : A \to B$ be another generalized Jordan homomorphism satisfying (2.4). Then we have

$$\begin{aligned}
\|H(x) - H'(x)\| &= \frac{1}{2^n}\|H(2^n x) - H'(2^n x)\| \\
&\le \frac{1}{2^n}(\|H(2^n x) - f(2^n x)\| + \|f(2^n x) - H'(2^n x)\|) \\
&\le \frac{2}{2^n}\tilde{\phi}(2^n x, 0, 0)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = H'(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique generalized Jordan homomorphism satisfying (2.4).

Similarly, one can prove the uniqueness of G. □

Corollary 2.3. *Let A be a Fréchet algebra with metric d and unit e, and B a Banach algebra with norm $\|\cdot\|$ and unit I. Let $0 < p < 1$. Let $f, g : A \to B$ be mappings with $f(0) = 0$, $g(0) = 0$ and $g(e) = I$ such that*

$$\left\|f\left(\frac{\mu x+\mu y}{2}+z^2\right)+f\left(\frac{\mu x-\mu y}{2}\right)-\mu f(x)+f(z)g(z)\right\| \le d(x,0)^p+d(y,0)^p+d(z,0)^p,$$

$$\left\|2g\left(\frac{\mu xy+\mu z}{2}\right)-\mu g(x)g(y)-\mu g(z)\right\| \le d(x,0)^p+d(y,0)^p+d(z,0)^p$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan homomorphism $H : A \to B$ and a unique homomorphism $G : A \to B$ such that

$$H(x^2) = H(x)G(x),$$

$$\|f(x) - H(x)\| \le \frac{2^p d(x,0)^p}{2-2^p},$$

$$\|g(x) - G(x)\| \le \frac{2^p (d(x,0)^p + d(e,0)^p)}{2-2^p}$$

for all $x \in A$.

Proof. Note that $d(2x, 0) \le 2d(x, 0)$. It follows from Theorem 2.2 by putting $\phi(x, y, z) = d(x,0)^p + d(y,0)^p + d(z,0)^p$ for all $x, y, z \in A$. □

Theorem 2.4. *Let A be a Banach algebra with norm $\|\cdot\|$ and unit I, and B a Fréchet algebra with metric d and unit e. Let $f, g : A \to B$ be mappings with $f(0) = 0$, $g(0) = 0$ and $g(I) = e$ for which there exists a function $\phi : A^3 \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} 4^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{2.10}$$

$$d\left(f\left(\frac{\mu x+\mu y}{2}+z^2\right)+f\left(\frac{\mu x-\mu y}{2}\right), \mu f(x) - f(z)g(z)\right) \le \phi(x,y,z), \tag{2.11}$$

$$d\left(2g\left(\frac{\mu xy+\mu z}{2}\right), \mu g(x)g(y)+\mu g(z)\right) \le \phi(x,y,z) \tag{2.12}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan homomorphism $H : A \to B$ and a unique homomorphism $G : A \to B$ such that

$$H(x^2) = H(x)G(x),$$

$$d(f(x), H(x)) \le \tilde{\phi}(x, 0, 0), \tag{2.13}$$

$$d(g(x), G(x)) \leq \tilde{\phi}(x, I, 0) \tag{2.14}$$

for all $x \in A$. *Here*

$$\tilde{\phi}(x,y,z) := \sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in A$.

Proof. Putting $y = z = 0$ and $\mu = 1$ in (2.11), we get

$$d\left(2f\left(\frac{x}{2}\right), f(x)\right) \leq \phi(x, 0, 0) \tag{2.15}$$

for all $x \in A$. Using the Rassias' method on (2.15) ([19]), one can use induction on n to show that

$$d\left(2^n f\left(\frac{x}{2^n}\right), f(x)\right) \leq \sum_{j=0}^{n-1} 2^j \phi\left(\frac{x}{2^j}, 0, 0\right) \tag{2.16}$$

for all $x \in A$ and all nonnegative integers n. Hence

$$d\left(2^{n+m} f\left(\frac{x}{2^{n+m}}\right), 2^m f\left(\frac{x}{2^m}\right)\right) \leq \sum_{j=m}^{n+m-1} 2^j \phi\left(\frac{x}{2^j}, 0, 0\right)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in A$. It follows from (2.10) that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.17}$$

for all $x \in A$.

Letting $z = 0$ and replacing x, y by $\frac{x}{2^n}, \frac{y}{2^n}$, respectively, in (2.11), we get

$$\begin{aligned}
& d\left(H\left(\frac{\mu x + \mu y}{2}\right) + H\left(\frac{\mu x - \mu y}{2}\right), \mu H(x)\right) \\
&= \lim_{n\to\infty} d\left(2^n f\left(\frac{\mu(x+y)}{2^{n+1}}\right) + 2^n f\left(\frac{\mu(x-y)}{2^{n+1}}\right), 2^n \mu f\left(\frac{x}{2^n}\right)\right) \\
&\leq \lim_{n\to\infty} 2^n d\left(f\left(\frac{\mu(x+y)}{2^{n+1}}\right) + f\left(\frac{\mu(x-y)}{2^{n+1}}\right), \mu f\left(\frac{x}{2^n}\right)\right) \\
&\leq \lim_{n\to\infty} 2^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right)
\end{aligned}$$

for all $x, y \in A$, $\mu \in \mathbb{T}^1$ and all nonnegative integers n. Taking the limit as $n \to \infty$, we obtain

$$H\left(\frac{\mu x + \mu y}{2}\right) + H\left(\frac{\mu x - \mu y}{2}\right) = \mu H(x) \tag{2.18}$$

for all $x, y \in A$ and all $\mu \in \mathbb{T}^1$. Letting $\mu = 1$ in (2.18), we can easily show that H is additive. Letting $y = 0$ in (2.18), we get

$$H(\mu x) = 2H\left(\frac{\mu x}{2}\right) = \mu H(x)$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.1, the mapping $H : A \to B$ is $\mathbb{C}$-linear. Moreover, it follows from (2.16) and (2.17) that

$$d(f(x), H(x)) \leq \tilde{\phi}(x, 0, 0)$$

for all $x \in A$.

Putting $y = I$, $z = 0$ and $\mu = 1$ in (2.12), we get

$$d\left(2g\left(\frac{x}{2}\right), g(x)\right) \leq \phi(x, I, 0)$$

for all $x \in A$.

By the same method as above, one can show that there is a $\mathbb{C}$-linear mapping $G : A \to B$ satisfying (2.14). The mapping $G : A \to B$ is given by

$$G(x) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

Putting $z = 0$ and $\mu = 1$ in (2.12), we get

$$d\left(2g\left(\frac{xy}{2}\right), g(x)g(y)\right) \leq \phi(x, y, 0)$$

and so

$$\begin{aligned} d\left(2 \cdot 4^n g\left(\frac{xy}{2 \cdot 4^n}\right), 2^n g\left(\frac{x}{2^n}\right) \cdot 2^n g\left(\frac{y}{2^n}\right)\right) &\leq 4^n d\left(2g\left(\frac{xy}{2 \cdot 4^n}\right), g\left(\frac{x}{2^n}\right) g\left(\frac{y}{2^n}\right)\right) \\ &\leq 4^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right), \end{aligned}$$

which tends to zero as $n \to \infty$. Thus $G(xy) = 2G\left(\frac{xy}{2}\right) = G(x)G(y)$ for all $x, y \in A$. So $G : A \to B$ is a homomorphism.

Putting $x = y = 0$ and $\mu = 1$ in (2.11), we get

$$d\left(f\left(z^2\right), f(z)g(z)\right) \leq \phi(0, 0, z)$$

and so

$$d\left(4^n f\left(\frac{z^2}{4^n}\right), 2^n f\left(\frac{z}{2^n}\right) \cdot 2^n g\left(\frac{z}{2^n}\right)\right) \leq 4^n d\left(f\left(\frac{z^2}{4^n}\right), f\left(\frac{z}{2^n}\right) g\left(\frac{z}{2^n}\right)\right) \leq 4^n \phi\left(0, 0, \frac{z}{2^n}\right),$$

which tends to zero as $n \to \infty$. Thus $H(z^2) = H(z)G(z)$ for all $z \in A$. So $H : A \to B$ is a generalized Jordan homomorphism.

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Now, let $H' : A \to B$ be another generalized Jordan homomorphism satisfying (2.13). Then we have

$$\begin{aligned} d(H(x), H'(x)) &= d\left(2^n H\left(\frac{x}{2^n}\right), 2^n H'\left(\frac{x}{2^n}\right)\right) \\ &\le 2^n d\left(H\left(\frac{x}{2^n}\right), H'\left(\frac{x}{2^n}\right)\right) \\ &\le 2^n \left(d\left(H\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right) + \left(f\left(\frac{x}{2^n}\right), H'\left(\frac{x}{2^n}\right)\right)\right) \\ &\le 2^{n+1}\tilde{\phi}\left(\frac{x}{2^n}, 0, 0\right) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = H'(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique generalized Jordan homomorphism satisfying (2.13).

Similarly, one can prove the uniqueness of G. □

Corollary 2.5. *Let A be a Banach algebra with norm $\|\cdot\|$ and unit I, and B a Fréchet algebra with metric d and unit e. Let θ, p be positive real numbers with $p > 2$. Let $f, g : A \to B$ be mappings with $f(0) = 0$, $g(0) = 0$ and $g(I) = e$ such that*

$$d\left(f\left(\frac{\mu x + \mu y}{2} + z^2\right) + f\left(\frac{\mu x - \mu y}{2}\right), \mu f(x) - f(z)g(z)\right) \le \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

$$d\left(2g\left(\frac{\mu xy + \mu z}{2}\right) - \mu g(x)g(y), \mu g(z)\right) \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan homomorphism $H : A \to B$ and a unique homomorphism $G : A \to B$ such that

$$H(x^2) = H(x)G(x),$$

$$d(f(x), H(x)) \le \frac{2^p\theta}{2^p - 2}\|x\|^p,$$

$$d(g(x), G(x)) \le \frac{2^p\theta(\|x\|^p + 1)}{2^p - 2}$$

for all $x \in A$.

Proof. Note that $d(2x, 0) \le 2d(x, 0)$. It follows from Theorem 2.4 by putting $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. □

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References

[1] R. Badora, *On approximate derivations*, Math. Inequal. Appl. **9** (2006), 167–173.

[2] M. Bavand Savadkouhi, M. Eshaghi Gordji, J.M. Rassias and N. Ghobadipour, *Approximate ternary Jordan derivations on Banach ternary algebras*, J. Math. Phys. **50** (2009), Article ID 042303, 9 pages.

[3] M. Bessenyei, G. Horváth and C.G. Kézi, *Functional equations on finite groups of substitutions*, Expo. Math. (to appear).

[4] M. Bessenyei and C.G. Kézi, *Functional equations and groups substitutions*, Linear Algebra and its Applications **434** (2011), 1525–1531.

[5] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuse function rings*, Duke Math, J. **16** (1949), 358–397.

[6] I. Chang, *Stability of higher ring derivations in fuzzy Banach algebras,* J. Computat. Anal. Appl. **14** (2012), 1059–1066.

[7] I. Cho, D. Kang and H. Koh, *Stability problems of cubic mappings with the fixed point alternative,* J. Computat. Anal. Appl. **14** (2012), 132–142.

[8] M. Eshaghi Gordji, *Stability of an additive-quadratic functional equation of two variables in F-spaces*, J. Nonlinear Sci. Appl. **2** (2009), 251–259.

[9] M. Eshaghi Gordji and M. Bavand Savadkouhi, *Approximation of generalized homomorphisms in quasi-Banach algebras*, Analele Univ. Ovidius Constata, Math. Series **17** (2009), 203–214.

[10] M. Eshaghi Gordji, M. Bavand Savadkouhi and M. Bidkham, *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **12** (2010), 454–462.

[11] M. Eshaghi Gordji and A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 724–729.

[12] M. Eshaghi Gordji, R. Farokhzad Rostami and S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras,* J. Computat. Anal. Appl. **13** (2011), 734–742.

[13] M. Eshaghi Gordji, M.B. Ghaemi, S. Kaboli Gharetapeh, S. Shams and A. Ebadian, *On the stability of J^*-derivations,* J. Geom. Phys. **60** (2010), 454–459.

[14] M. Eshaghi Gordji and N. Ghobadipour, *Stability of (α,β,γ)-derivations on Lie C^*-algebras,* Internat. J. Geom. Meth. Mod. Phys. **7** (2010), No. 7, 1–10.

[15] M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi and M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras,* J. Computat. Anal. Appl. **12** (2010), 463–470.

[16] M. Eshaghi Gordji and A. Najati, *Approximately J^*-homomorphisms: A fixed point approach,* J. Geom. Phys. **60** (2010), 809–814.

[17] M. Eshaghi Gordji, J.M. Rassias and N. Ghobadipour, *Generalized Hyers-Ulam stability of the generalized (n,k)-derivations*, Abs. Appl. Anal. **2009** (2009), Article ID 437931, 8 pages.

[18] R. Farokhzad and S. A. R. Hosseinioun, *Perturbations of Jordan higher derivations in Banach ternary algebras: An alternative fixed point approach,* Int. J. Nonlinear Anal. Appl. **1** (2010), 42–53.

[19] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[20] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[21] H.A. Kenary, J. Lee and C. Park, *Non-Archimedean stability of an AQQ-functional equation,* J. Computat. Anal. Appl. **14** (2012), 211–227.

[22] T. Miura, S.E. Takahashi and G. Hirasawa, *Hyers-Ulam-Rassias stability of Jordan homomorphisms on Banach algebras*, *J. Inequal. Appl* **4** (2004), 435–441.

[23] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[24] C. Park, Y. Cho and H.A. Kenary, *Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **14** (2012), 526–535.

[25] C. Park, S. Jang and R. Saadati, *Fuzzy approximate of homomorphisms,* J. Computat. Anal. Appl. **14** (2012), 833–841.

[26] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[27] Th.M. Rassias, *Problem 16; 2, Report of the 27th International Symp. on Functional Equations*, Aequationes Math. **39** (1990), 292–293; 309.

[28] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 1097–1105.

[29] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Computat. Anal. Appl. **13** (2011), 1097–1105.

[30] S.M. Ulam, *Problems in Modern Mathematics,* Chapter VI, Science ed., Wiley, New York, 1940.

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An note on S_r-covering approximation spaces *

Bin Qin† Xun Ge‡

February 21, 2013

Abstract: In this paper, we prove that a covering approximation space $(U, \mathcal{C})$ is an S_r-covering approximation space if and only if $\{N(x) : x \in U\}$ forms a partition of the universe of discourse U. Furthermore, we give some simple characterizations for S_r-space $(U, \mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering $\mathcal{C}$. Results of this paper answer affirmatively an open problem posed by Z.Yun et al. in [16].

Keywords: Universe of discourse; Covering approximation space; S_r-covering approximation space; Covering lower (upper) approximation operation; Neighborhood; Partition.

1 Introduction

Rough set theory, which was first proposed by Z.Pawlak in [4], is a useful tool in researches and applications of process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis and other fields [2, 3, 5, 6, 10, 14, 15, 18, 19]. In the classical rough set theory, Pawlak approximation spaces are based on partitions of the universe of discourse U, but this requirement is not satisfied in some situations [20]. In the past years, Pawlak approximation spaces have been extended to covering approximation spaces [1, 8, 12, 13, 16, 21].

Definition 1.1 ([21]). *Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of nonempty subsets of U.*

(1) $\mathcal{C}$ is called a covering of U if $\bigcup\{K : K \in \mathcal{C}\} = U$. Furthermore, $\mathcal{C}$ is called a partition of U if also $K \bigcap K' = \emptyset$ for all $K, K' \in \mathcal{C}$, where $K \neq K'$.

(2) The pair $(U, \mathcal{C})$ is called a covering approximation space (resp. a Pawlak approximation space) if $\mathcal{C}$ is a covering (resp. a partition) of U.

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(3) $\bigcap\{K : x \in K \in \mathcal{C}\}$ *is called the neighborhood of* x *and denoted as* $Neighbor_C(x)$. *When there is no confusion, we omit* C *at the lowercase and abbreviate* $Neighbor_C(x)$ *to* $N(x)$.

For a covering approximation spaces $(U, \mathcal{C})$, it is interesting to study the condition for $\{N(x) : x \in U\}$ to form a partition of universe U. In particular, it is an important issue in covering approximation spaces theory to characterize this condition by covering lower (upper) approximation operations [8, 16]. In order to give a more detailed description for this issue, we present some covering lower (upper) approximation operations as follows.

Definition 1.2 ([16])**.** *Let* (U, C) *be a covering approximation space and* $X \subseteq U$. *Put*

(1) $\underline{\mathcal{C}_2}(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \subseteq X\}$, $\overline{\mathcal{C}_2}(X) = U - \underline{\mathcal{C}_2}(U - X)$;

(2) $\underline{\mathcal{C}_3}(X) = \{x \in U : N(x) \subseteq X\}$, $\overline{\mathcal{C}_3}(X) = \{x \in U : N(x) \bigcap X \neq \emptyset\}$;

(3) $\underline{\mathcal{C}_4}(X) = \{x \in U : \exists u(u \in N(x) \wedge N(u) \subseteq X)\}$, $\overline{\mathcal{C}_4}(X) = \{x \in U : \forall u(u \in N(x) \rightarrow N(u) \cap X \neq \emptyset)\}$;

(4) $\underline{\mathcal{C}_5}(X) = \{x \in U : \forall u(x \in N(u) \rightarrow N(u) \subseteq X)\}$, $\overline{\mathcal{C}_5}(X) = \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\}$;

(5) $\underline{\mathcal{C}_6}(X) = \{x \in U : \forall u(x \in N(u) \rightarrow u \in X)\}$, $\overline{\mathcal{C}_6}(X) = \bigcup\{N(x) : x \in X\}$.

Then $\underline{\mathcal{C}_i}$ *(resp.* $\overline{\mathcal{C}_i}$*) is called a covering lower (resp. upper) approximation operation and* $\underline{\mathcal{C}_i}(X)$ *(resp.* $\overline{\mathcal{C}_i}(X)$*) is called covering lower (resp. upper) approximation of* X. *Here,* $i = 2, 3, 4, 5, 6$.

Remark 1.3. *In [8],* $\underline{\mathcal{C}_i}$ *and* $\overline{\mathcal{C}_i}$ *are denoted by* $\underline{\mathcal{C}_{i-1}}$ *and* $\overline{\mathcal{C}_{i-1}}$ *respectively. Here,* $i = 2, 3, 4, 5, 6$.

K.Qin et al. gave the following theorem.

Theorem 1.4 ([8])**.** *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ *forms a partition of* U.
(2) $\overline{C_5}(X) = \overline{C_6}(X)$ *for each* $X \subseteq U$.
(3) $\overline{C_5}(X) = \overline{C_4}(X)$ *for each* $X \subseteq U$.
(4) $\overline{C_3}(X) = \overline{C_4}(X)$ *for each* $X \subseteq U$.
(5) $\overline{C_6}(X) = \overline{C_4}(X)$ *for each* $X \subseteq U$.
(6) $\overline{C_3}(X) = \overline{C_6}(X)$ *for each* $X \subseteq U$.
(7) $\underline{C_5}(X) = \underline{C_3}(X)$ *for each* $X \subseteq U$.

Recently, taking Theorem 1.4 into account, Z. Yun et al. [16] investigated the following question.

Question 1.5 ([16])**.** *Can we characterize the conditions under which* $\{N(x) : x \in U\}$ *forms a partition of* U *by using only a single covering approximation operator among* C_2*-*C_6*?*

The following results were obtained.

Theorem 1.6 ([16]). *Let $(U,\mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ forms a partition of U.
(2) $\overline{C_3}(\underline{C_3}(X)) = \underline{C_3}(X)$ for each $X \subseteq U$.
(3) $\overline{C_6}(\underline{C_6}(X)) = \underline{C_6}(X)$ for each $X \subseteq U$.
(4) $\underline{C_4}(X) \subseteq X$ for each $X \subseteq U$.

Theorem 1.7 ([16]). *Let $(U,\mathcal{C})$ be a covering approximation space.*

(1) If $\overline{C_2}(\underline{C_2}(X)) = \underline{C_2}(X)$ for each $X \subseteq U$, then $\{N(x) : x \in U\}$ forms a partition of U, not vice versa.

(2) If $\{N(x) : x \in U\}$ forms a partition of U, then $\overline{C_5}(\underline{C_5}(X)) = \underline{C_5}(X)$ for each $X \subseteq U$, not vice versa.

As an open problem, the following question is raised in the end of [16].

Question 1.8 ([16]). *How to give sufficient and necessary conditions for $\{N(x) : x \in U\}$ to form a partition of U by using only a single covering approximation operator C_i $(i = 2, 5)$?*

In this paper, we investigate Question 1.5 and Question 1.8 by S_r-covering approximation spaces. Here, S_r-covering approximation spaces was introduced by X.Ge in [1].

Definition 1.9 ([1]). *A covering approximation space $(U,\mathcal{C})$ is called an S_r-space (S_r-space is the abbreviation of S_r-covering approximation space) if $x \in K \in \mathcal{C}$ implies $D(x) \subset K$, where $D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$.*

In this paper, we gives a "nice" characterization for S_r-space. By this result, we translate the condition for $\{N(x) : x \in U\}$ to form a partition of the universe of discourse U into S_r-space $(U,\mathcal{C})$ in Question 1.5 and Question 1.8. Furthermore, we obtain some simple characterizations for S_r-space $(U,\mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering $\mathcal{C}$, which answer Question 1.5 and Question 1.8 and improve some results obtained in [16].

2 Preliminaries

For a covering approximation space $(U,\mathcal{C})$, we say that $\{N(x) : x \in U\}$ forms a partition of U if for every pair $x, y \in U$, $N(x) = N(y)$ or $N(x) \bigcap N(y) = \emptyset$. Before our discussion, we give some notations.

Note 2.1. *Let $(U,\mathcal{C})$ be a covering approximation space. Throughout this paper, we use the following notations, where $x \in U$, $X \subseteq U$ and $\mathcal{F} \subseteq 2^U$.*

(1) $\bigcap \mathcal{F} = \bigcap\{F : F \in \mathcal{F}\}$.
(2) $\bigcup \mathcal{F} = \bigcup\{F : F \in \mathcal{F}\}$.

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(3) $\mathcal{C}_x = \{K : x \in K \in \mathcal{C}\}$.
(4) $N(x) = \bigcap \mathcal{C}_x$.
(5) $D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$.

Remark 2.2. *It is clear that $x \in N(x)$ and $x \in D(x)$. Note that $x \in K \in \mathcal{C}$ if and only if $K \in \mathcal{C}_x$, we also replace $x \in K \in \mathcal{C}$ by $K \in \mathcal{C}_x$ in this paper.*

The following three lemmas are known.

Lemma 2.3 ([8, 9])**.** *Let $(U, \mathcal{C})$ be a covering approximation space and $X, Y \subseteq U$. Then the following hold.*

(1) $\underline{\mathcal{C}_i}(U) = U = \overline{\mathcal{C}_i}(U)$, $\underline{\mathcal{C}_i}(\emptyset) = \emptyset = \overline{\mathcal{C}_i}(\emptyset)$ *for* $i = 2, 3, 4, 5, 6$.
(2) $\underline{\mathcal{C}_i}(X) \subseteq X \subseteq \overline{\mathcal{C}_i}(X)$ *for* $i = 2, 3, 5, 6$.
(3) $X \subseteq Y \subseteq U \Longrightarrow \underline{\mathcal{C}_i}(X) \subseteq \underline{\mathcal{C}_i}(Y)$, $\overline{\mathcal{C}_i}(X) \subseteq \overline{\mathcal{C}_i}(Y)$ *for* $i = 2, 3, 4, 5, 6$.
(4) $\underline{\mathcal{C}_i}(X \bigcap Y) = \underline{\mathcal{C}_i}(X) \bigcap \underline{\mathcal{C}_i}(Y)$, $\overline{\mathcal{C}_i}(X \bigcup Y) = \overline{\mathcal{C}_i}(X) \bigcup \overline{\mathcal{C}_i}(Y)$ *for* $i = 3, 5, 6$.
(5) $\underline{\mathcal{C}_i}(X) = U - \overline{\mathcal{C}_i}(U - X)$, $\overline{\mathcal{C}_i}(X) = U - \underline{\mathcal{C}_i}(U - X)$ *for* $i = 2, 3, 4, 5, 6$.

Lemma 2.4 ([8])**.** *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $\{N(x) : x \in U\}$ *forms a partition of* U.
(2) For every pair $x, y \in U$, $x \in N(y) \Longrightarrow y \in N(x)$.

Lemma 2.5 ([1])**.** *Let $(U, \mathcal{C})$ be a covering approximation space and $x, y \in U$. Then the following are equivalent.*

(1) $x \in N(y)$.
(2) $\mathcal{C}_y \subseteq \mathcal{C}_x$.
(3) $N(x) \subseteq N(y)$.
(4) $D(y) \subseteq D(x)$.
(5) $y \in D(x)$.

Proposition 2.6. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *is an* S_r*-spaces.*
(2) $\{N(x) : x \in U\}$ *forms a partition of* U.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-spaces. Let $x, y \in U$ and $x \in N(y)$. Then $y \in D(x)$ by Lemma 2.5. For each $K \in \mathcal{C}_x$, $D(x) \subseteq K$, we have $y \in K$. This proves that $y \in N(x)$. By Lemma 2.4, $\{N(x) : x \in U\}$ forms a partition of U.

(2) $\Longrightarrow$ (1): Suppose that $\{N(x) : x \in U\}$ forms a partition of U. Let $K \in \mathcal{C}$ and $x \in K$. Then $N(x) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ by Lemma 2.5. By Lemma 2.4, $y \in N(x) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Proposition 2.6 gives a "nice" characterization for S_r-space, which is help for us to further comprehend [1, Remark 1.2]).

3 The main results

Theorem 3.1. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $D(x) \subseteq K$. If $y \in \overline{C_2}(\{x\}) = U - \underline{C_2}(U - \{x\})$, then $y \notin \underline{C_2}(U - \{x\})$. So, for each $K' \in \mathcal{C}$, if $K' \subseteq U - \{x\}$ then $y \notin K'$. That is, for each $K' \in \mathcal{C}$, if $x \notin K'$ then $y \notin K'$, and hence $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. It follows that $y \in U - \bigcup(\mathcal{C} - \mathcal{C}_x) = D(x) \subseteq K$. This proves that $\overline{C_2}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_2}(\{x\}) \subseteq K$. If $y \in D(x) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$, then $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. So $y \notin K$ for each $K \in \mathcal{C} - \mathcal{C}_x$. That is, for each $K \in \mathcal{C}$, if $x \notin K$ then $y \notin K$. Note that $x \notin K$ if and only if $K \subseteq U - \{x\}$. Thus, $y \notin \underline{C_2}(U - \{x\})$. It follows that $y \in U - \underline{C_2}(U - \{x\}) = \overline{C_2}(\{x\}) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Let $(U, \mathcal{C})$ be a covering approximation space. It is clear that if $\overline{C_2}(\underline{C_2}(X)) = \underline{C_2}(X)$ for each $X \subseteq U$. Then $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. So the following corollary improves Theorem 1.7(1), and the proof is quite simple.

Corollary 3.2. *Let $(U, \mathcal{C})$ be a covering approximation space. If $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$, then $(U, \mathcal{C})$ is an S_r-space.*

Proof. Let $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. If $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_2}(\{x\}) \subseteq \overline{C_2}(K) = K$ from Lemma 2.3(3). By Theorem 3.1, $(U, \mathcal{C})$ is an S_r-space. □

Remark 3.3. *[16, Example 3.9] and Proposition 2.6 show that Corollary 3.2 can not be reversed.*

What are sufficient and necessary conditions such that $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$? The following proposition gives an answer.

Proposition 3.4. *Let $(U, \mathcal{C})$ be a covering approximation space. Then $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$ if and only if the following hold.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$ for each $K \in \mathcal{C}$.

Proof. Necessity: Let $\overline{C_2}(K) = K$ for each $K \in \mathcal{C}$. By Corollary 3.2, $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 2.3(3), $\overline{C_2}(\{x\}) \subseteq \overline{C_2}(K)$ for each $x \in K$. Thus $\bigcup\{\overline{C_2}(\{x\}) : x \in K\} \subseteq \overline{C_2}(K)$. On the other hand, by Lemma 2.3(2), $x \in \overline{C_2}(\{x\})$ for each $x \in K$, so $\overline{C_2}(K) = K \subseteq \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$. Consequently, $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\}$.

Sufficiency: Suppose that (1) and (2) hold. Let $K \in \mathcal{C}$. By Theorem 3.1, $\overline{C_2}(\{x\}) \subseteq K$ for each $x \in K$. Thus, $\overline{C_2}(K) = \bigcup\{\overline{C_2}(\{x\}) : x \in K\} \subseteq K$. On the other hand, $K \subseteq \overline{C_2}(K)$ by Lemma 2.3(2). So $\overline{C_2}(K) = K$. □

Similarly, the following proposition is obtained, which gives sufficient and necessary conditions such that $\overline{C_2}(\underline{C_2}(X)) = \underline{C_2}(X)$ for each $X \subseteq U$. We omit its proof.

Proposition 3.5. *Let $(U, \mathcal{C})$ be a covering approximation space. Then $\overline{C_2}(\underline{C_2}(X)) = \underline{C_2}(X)$ for each $X \subseteq U$ if and only if the following hold.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_2}(X) = \bigcup\{\overline{C_2}(\{x\}) : x \in X\}$ for each union X of elements of $\mathcal{C}$.

Lemma 3.6. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_3}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $D(x) \subseteq K$. If $y \in \overline{C_3}(\{x\}) = \{z \in U : N(z) \bigcap \{x\} \neq \emptyset\}$, then $N(y) \bigcap \{x\} \neq \emptyset$, so $x \in N(y)$. By Lemma 2.5, $y \in D(x) \subseteq K$. This proves that $\overline{C_3}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_3}(\{x\}) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5, i.e., $N(y) \bigcap \{x\} \neq \emptyset$. It follows that $y \in \{z \in U : N(z) \bigcap \{x\} \neq \emptyset\} = \overline{C_3}(\{x\}) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Theorem 3.7. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_3}(K) = K$ for each $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 3.6, $\overline{C_3}(\{x\}) \subseteq K$ for each $x \in K$. By Lemma 2.3(4), $\overline{C_3}(K) = \bigcup\{\overline{C_3}(\{x\}) : x \in K\} \subseteq K$. On the other hand, by Lemma 2.3(3), $K \subseteq \overline{C_3}(K)$. Consequently, $\overline{C_3}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$. Then $\overline{C_3}(K) = K$. By Lemma 2.3(3), $\overline{C_3}(\{x\}) \subseteq \overline{C_3}(K) = K$. By Lemma 3.6, $(U, \mathcal{C})$ is an S_r-space. □

The following shows that "$\underline{C_4}(X) \subseteq X$" in Theorem 1.6(4) can not be replaced by "$\underline{C_4}(X) = X$"

Example 3.8. *There exists a covering approximation space $(U, \mathcal{C})$ such that $(U, \mathcal{C})$ is an S_r-space and $\underline{C_4}(X) \neq X$ for some $X \subseteq U$.*

Proof. Let $U = \{a, b.c\}$ and $\mathcal{C} = \{\{a, b\}, \{c\}\}$. Then $(U, \mathcal{C})$ is a Pawlak approximation space. It is known that each Pawlak approximation space is an S_r-space (see [1, Remark 3.4]). Put $X = \{a, c\}$. It is not difficult to check that $\underline{C_4}(X) = \{c\}$. So $\underline{C_4}(X) \neq X$. □

However, we have the following.

Theorem 3.9. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\underline{C_4}(K) = K$ for each $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Theorem 1.6 and Proposition 2.6, $\underline{C_4}(K) \subseteq K$. On the other hand, Let $x \in K$. Then $x \in N(x)$ and $N(x) \subseteq K$. By the definition of $\underline{C_4}(K)$, $x \in \underline{C_4}(K)$. This proves that $K \subseteq \underline{C_4}(K)$. Consequently, $\underline{C_4}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_4}(K) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. Note that $N(x) \subseteq K$. So $y \in \{z \in U : \exists u(u \in N(z) \wedge N(u) \subseteq K)\} = \overline{C_4}(K) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Lemma 3.10. *Let $(U, \mathcal{C})$ be a covering approximation space and $X \subset U$. Then $\underline{C_5}(X) = X$ if and only if $\overline{C_5}(X) = X$.*

Proof. Necessity: Suppose that $\underline{C_5}(X) = X$. Let $y \in \overline{C_5}(X) = \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\}$. Then there is $z \in U$ such that $y \in N(z)$ and $N(z) \bigcap X \neq \emptyset$. Pick $v \in N(z) \bigcap X$, then $v \in X = \underline{\mathcal{C}_5}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subseteq X)\}$. It follows that $N(z) \subseteq X$ since $v \in N(z)$. So $y \in N(z) \subseteq X$. This proves that $\overline{C_5}(X) \subseteq X$. By Lemma 2.3(2), $X \subseteq \overline{C_5}(X)$. Consequently, $\overline{C_5}(X) = X$.

Sufficiency: Suppose that $\overline{C_5}(X) = X$. By Lemma 2.3(2), $\underline{C_5}(X) \subseteq X$. It suffices to prove that $X \subseteq \underline{C_5}(X)$. If $X \not\subseteq \underline{C_5}(X)$, then there is $y \in X$ such that $y \notin \underline{C_5}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subseteq X)\}$. So there is $v \in U$ such that $y \in N(v) \not\subseteq X$. Pick $z \in N(v)$ such that $z \notin X$. Note that $y \in N(v) \bigcap X$. So $N(v) \bigcap X \neq \emptyset$. Thus $z \in \bigcup\{N(x) : x \in U \wedge N(x) \bigcap X \neq \emptyset\} = \overline{C_5}(X) = X$. This contradicts that $z \notin X$. □

Lemma 3.11. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.
(2) $\overline{C_5}(\{x\}) \subseteq K$ for each $x \in U$ and each $K \in \mathcal{C}_x$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $x \in U$ and $K \in \mathcal{C}_x$, then $D(x) \subseteq K$. If $y \in \overline{C_5}(\{x\}) = \bigcup\{N(z) : z \in U \wedge x \in N(z)\}$, then there exists $z \in U$ such that $x \in N(z)$ and $y \in N(z)$. By Lemma 2.5, $z \in D(x) \subseteq K$, hence $N(z) \subseteq K$. It follows that $y \in N(z) \subseteq K$. This proves that $\overline{C_5}(\{x\}) \subseteq K$.

(2) $\Longrightarrow$ (1). Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_5}(\{x\}) \subseteq K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. So $N(y) \subseteq \bigcup\{N(z) : z \in U \wedge x \in N(z)\} = \overline{C_5}(\{x\}) \subseteq K$. It follows that $y \in N(y) \subseteq K$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

Theorem 3.12. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ is an S_r-space.

(2) $\overline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(3) $\underline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. By Lemma 3.11, $\overline{C_5}(\{x\}) \subseteq K$ for each $x \in K$. By Lemma 2.3(4), $\overline{C_5}(K) = \bigcup\{\overline{C_5}(\{x\}) : x \in K\} \subseteq K$. On the other hand, by Lemma 2.3(2), $K \subseteq \overline{C_5}(K)$. Consequently, $\overline{C_5}(K) = K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\overline{C_5}(K) = K$. By Lemma 2.3(3), $\overline{C_3}(\{x\}) \subseteq \overline{C_5}(K) = K$. By Lemma 3.11, $(U, \mathcal{C})$ is an S_r-space.

(2) $\Longleftrightarrow$ (3): It holds by Lemma 3.10. □

Theorem 3.13. *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *is an* S_r*-space.*
(2) $\underline{C_6}(K) = K$ *for each* $K \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U, \mathcal{C})$ is an S_r-space. Let $K \in \mathcal{C}$. Then $\underline{C_6}(K) \subseteq K$ by Lemma 2.3(2). It suffices to prove that $K \subseteq \underline{C_6}(K)$. Let $x \in K$, then $D(x) \subseteq K$ since $(U, \mathcal{C})$ is an S_r-space. For each $u \in U$, if $x \in N(u)$, then $u \in D(x)$ by Lemma 2.5. It follows that $u \in K$. So $x \in \{z \in U : \forall u(z \in N(u) \to u \in K)\} = \underline{C_6}(K)$. This proves that $K \subseteq \underline{C_6}(K)$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$ and $K \in \mathcal{C}_x$, then $\underline{C_6}(K) = K$. If $y \in D(x)$, then $x \in N(y)$ from Lemma 2.5. $x \in K = \underline{C_6}(K) = \{x \in U : \forall u(x \in N(u) \to u \in K)\}$, so $x \in N(u)$ implies $u \in K$ for each $u \in U$. It follows that $y \in K$ since $x \in N(y)$. This proves that $D(x) \subseteq K$. So $(U, \mathcal{C})$ is an S_r-space. □

4 Conclusions

This paper answers an open problem posed by Z.Yun et al. in [16]. We give some simple characterizations for S_r-space $(U, \mathcal{C})$ by using only a single covering approximation operator and by using only elements of covering. The main results are summarized as follows.

Theorem 4.1. *Let* $(U, \mathcal{C})$ *be a covering approximation space. Then the following are equivalent.*

(1) $(U, \mathcal{C})$ *ia an* S_r*-space.*
(2) $\{N(x) : x \in U\}$ *forms a partition of* U.
(3) $\overline{C_2}(\{x\}) \subseteq K$ *for each* $x \in U$ *and each* $K \in \mathcal{C}_x$.
(4) $\overline{C_3}(K) = K$ *for each* $K \in \mathcal{C}$.
(5) $\underline{C_4}(K) = K$ *for each* $K \in \mathcal{C}$.
(6) $\overline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(7) $\underline{C_5}(K) = K$ *for each* $K \in \mathcal{C}$.
(8) $\underline{C_6}(K) = K$ *for each* $K \in \mathcal{C}$.

8

In the previous sections, covering approximation operators $\mathcal{C}_2$-$\mathcal{C}_6$ are used for our discussion. However, there are also other useful covering approximation operators, which play an important role in research of covering approximation spaces [7, 11, 17, 20, 21].

Definition 4.2 ([20]). *Let (U, C) be a covering approximation space and $x \in U$.*
$Md(x) = \{K : K \in \mathcal{C}_x \wedge (\forall S \in \mathcal{C}_x \wedge S \subseteq K \rightarrow K = S)\}$
is called the minimal description of x.

Definition 4.3 ([20]). *Let (U, C) be a covering approximation space and $X \subseteq U$. Put*

(1) $CL(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \subseteq X\}$;
(2) $FH(X) = CL(X) \bigcup\{Md(x) : x \in X - CL(X)\}$;
(3) $SH(X) = \bigcup\{K : K \in \mathcal{C} \wedge K \bigcap X \neq \emptyset\}$;
(4) $TH(X) = \bigcup\{Md(x) : x \in X\}$;
(5) $RH(X) = CL(X) \bigcup\{K : K \in \mathcal{C} \wedge K \bigcap(X - CL(X)) \neq \emptyset\}$;
(6) $IH(X) = CL(X) \bigcup\{N(x) : x \in X - CL(X)\}$.

CL is called covering lower approximation operation. FH, SH, TH, RH and IH are called the first, the second, the third, the fourth, and the fifth covering upper approximation operations, respectively.

Can we characterize the conditions under which $(U, \mathcal{C})$ is an S_r-space by using only a single covering approximation operator in Definition 4.3? It is an interesting question and is still worthy to be considered in research of covering approximation spaces.

References

[1] X.Ge, An application of covering approximation spaces on network security, Computers and Mathematics with Applications, 60(2010), 1191-1199.

[2] W.Li, N.Zhong, Y.Y.Yao, J.Liu, C.Liu, Developing intelligent applications in social e-mail networks, Lecture Notes in Artificial Intelligence, 4259(2006), 776-785.

[3] S.K.Pal, B.U.Shankar, P.Mitra, Granular computing, rough entropy and object extraction, Pattern Recognition Letters, 26(2005), 2509-2517.

[4] Z.Pawlak, Rough sets, International Journal of Computer and Information Science, 11(1982), 341-356.

[5] Z.Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Boston, 1991.

[6] L.Polkowski, A.Skowron (Eds.), Rough sets and current trends in computing, Vol. 1424, Springer, Berlin, 1998.

[7] J.A.Pomykala, Approximation operations in approximation space, Bulletin of the Polish Academy of Sciences, 35(9-10)(1987), 653-662.

[8] K.Qin, Y.Gao, Z.Pei, On Covering rough sets, Lecture Notes in Artificial Intelligence, 4481(2007), 34-41.

[9] P.Samanta, M.Chakraborty, Covering based approaches to rough sets and implication lattices, Lecture Notes in Artificial Intelligence, 5908(2009), 127-134.

[10] A.Skowron, J.F.Peters, Rough-Granular Computing, in: Handbook of Granular Computing, Edited by W. Pedrycz, A. Skowron, V. Kreinovich, John Wiley & Sons, Ltd., 2008, 285-328.

[11] E.Tsang, D.Cheng, J.Lee, D.Yeung, On the upper approximations of covering generalized rough sets, in: Proceedings of the 3rd International Conference Machine Learning and Cybernetics, 2004, pp. 4200-4203.

[12] E.Tsang, D.Chen, D.Yeung, Approximations and reducts with covering generalized rough sets, Computers and Mathematics with Applications, 56(2008), 279-289.

[13] Y.Y.Yao, On generalizing rough set theory, Lecture Notes in Artificial Intelligence, 2639(2003), 44-51.

[14] Y.Y.Yao, N.Zhong, Potential applications of granular computing in knowledge discovery and data mining, in: Proceedings of World Multiconference on Systemics, Cybernetics and Informatics, 1999, pp. 573-580.

[15] D.Yeung, D.Chen, E.Tsang, J.Lee, W.Xizhao, On the generalization of fuzzy rough sets, IEEE Transactions on Fuzzy Systems, 13(3)(2005), 343-361.

[16] Z.Yun, X.Ge, X.Bai, Axiomatization and conditions for neighborhoods in a covering to form a partition, Information Sciences, 181(2011), 1735-1740.

[17] W.Zakowski, Approximations in the space (u, Π), Demonstratio Mathematica, 16(1983), 761-769.

[18] N.Zhong, J.Dong, S.Ohsuga, Using rough sets with heuristics to feature selection, J. Intell. Inform. Syst., 16(3)(2001), 199-214.

[19] N.Zhong, Y.Y.Yao, M.Ohshima, Peculiarity oriented multidatabase mining, IEEE Transactions on Knowledge and Data Engineering, 15(4)(2003), 952-960.

[20] W.Zhu, Relationship between generalized rough sets based on binary relation and covering, Information Sciences, 179(2009), 210-225.

[21] W.Zhu, F.Wang, On three types of covering rough sets, IEEE Transactions on Knowledge and Data Engineering, 19(2007), 1131-1144.

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Difference of Generalized Composition Operators from H^∞ to the Bloch Space

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1

Abstract

We characterized the difference of generalized composition operator on the bounded analytic function space to the Bloch space in the disk. The boundedness and compactness of it were investigated.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

The Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$, which is Möbius invariant.

The space $\mathcal{B}$ becomes a Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

Denote $H^\infty(\mathbb{D})$ by H^∞ ,the space of all bounded analytic functions in the unit disk with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|.$$

Let φ be an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$, the generalized composition operator C_φ^g induced by φ and g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The definition of the generalized composition was first introduced by S. Li, S. Stević in [9], and in the paper, the boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces were investigated by them.

In the past few decades, boundedness, compactness, isometries and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors, see, e.g., [1, 2, 6, 14, 18, 19, 21, 22]. Recently, many papers focused on studying the mapping properties of the difference of two composition operators, i.e.,

$$(C_\varphi - C_\psi)(f) = f\circ\varphi - f\circ\psi.$$

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Differences of composition operators were studies first on hardy space $H^2(\mathbb{D})$ (see,e.g[3]). In [13], MacCluer, Ohno and Zhao, characterized the compactness of the difference of two composition operators on $H^\infty(\mathbb{D})$ in terms of the Poincaré distance. A fewer years later, these results were extended to the setting of $H^\infty(B_n)$ by Toews [20] . In [23], Z. H. Zhou and L. Zhang discussed the differences of the products of integral type and composition operators from H^∞ to the Bloch space, more results ,for example, can be seen in [4, 5, 8, 15, 16, 17].

Building on those foundation, this paper continues the research of this part, and discusses the difference of two generalized composition operators from the bounded analytic function space to the Bloch space in the disk.

2 Notation and Lemmas

First, we will introduce some notation and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a, is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}.$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\bar{z}w}\right|,$$

and the hyperbolic metric is given by

$$\beta(z,w) = \inf_\gamma \int_\gamma \frac{|d\xi|}{1-|\xi|^2} = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.

The following lemma is well known in [24].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1-\rho^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}.$$

A little modification of Lemma 1 in [7] shows the following lemma.

Lemma 2. *There exists a constant $C > 0$ such that*

$$\left|\left(1-|z|^2\right)f'(z) - \left(1-|w|^2\right)f'(w)\right| \le C\,\|f\|_{\mathcal{B}} \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}$.

Lemma 3. *Assume that $f \in H^\infty(\mathbb{D})$, then for each $n \in N$, there is a positive constant C independent of f such that*

$$\sup_{z\in D}(|1-|z|)^n \left|f^{(n)}(z)\right| < C||f||_\infty.$$

Remark The Lemma 3 can be concluded from [11].

Throughout the remainder of this paper, we will denote $\frac{1-|z|^2}{1-|\varphi(z)|^2}$ by the φ^* and constants are denoted by C , they are positive and not necessarily the same in each appearance.

3 Main theorems

Theorem 1. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is bounded;*

(ii)

$$\sup_{z\in D} |\varphi_1^*(z)|\, |g_1(z)|\, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{1}$$

$$\sup_{z\in D} |\varphi_2^*(z)|\, |g_2(z)|\, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{2}$$

and

$$\sup_{z\in D} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| < \infty. \tag{3}$$

Proof. We first prove $(ii) \Rightarrow (i)$. Assume that $(1), (2), (3)$ hold.

As the definition of C_φ^g, obviously, $\left|(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})f(0) = 0\right|$

By Lemma 2 and Lemma3, for every $f \in H^\infty$, we have

$$\begin{aligned}
& ||C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}||_{\mathcal{B}} \\
= \;& \sup_{z\in\mathbb{D}} (1-|z|^2)\, |f'(\varphi_1(z))g_1(z) - f'(\varphi_2(z))g_2(z)| \\
= \;& \sup_{z\in\mathbb{D}} \left|(1-|\varphi_1(z)|^2)\varphi_1^*(z)f'(\varphi_1(z))g_1(z) - (1-|\varphi_2(z)|^2)\varphi_2^*(z)f'(\varphi_2(z))g_2(z)\right| \\
\le \;& \sup_{z\in\mathbb{D}} |\varphi_1^*(z)g_1(z)| \left|(1-|\varphi_1(z)|^2)f'(\varphi_1(z)) - (1-|\varphi_2(z)|^2)f'(\varphi_2(z))\right| \\
+ \;& \sup_{z\in\mathbb{D}} (1-|\varphi_2(z)|^2)\, |f'(\varphi_2(z))|\, |g_1(z)\varphi_1^*(z)) - g_2(z)\varphi_2^*(z))| \\
\le \;& C \sup_{z\in\mathbb{D}} |\varphi_1^*(z)g_1(z)|\, \rho(\varphi_1(z), \varphi_2(z))\, \|f\|_{\mathcal{B}} \\
+ \;& \sup_{z\in\mathbb{D}} |g_1(z)\varphi_1^*(z)) - g_2(z)\varphi_2^*(z))|\, \|f\|_{\mathcal{B}} \\
\le \;& C\, \|f\|_\infty .
\end{aligned}$$

That is $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}$ is bounded.

Next we show that (i) implies (ii). We assume $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ is bounded.

For every $\omega \in \mathbb{D}$, we take the test function

$$f_{\varphi_1,\omega}(z) = \frac{\varphi_1(\omega) - z}{1 - \overline{\varphi_1(\omega)}z}.$$

We can obtain easily that $f_{\varphi_1,\omega} \in H^\infty$ and $||f_{\varphi_1,\omega}||_\infty \le 1$.
Therefore, we have

$$\begin{aligned}
& C \ge ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})f_{\varphi_1,\omega}||_{\mathcal{B}} \\
= \;& \sup_{z\in\mathbb{D}} (1-|z|^2) \left|f'_{\varphi_1,\omega}(\varphi_1(z))g_1(z) - f'_{\varphi_1,\omega}(\varphi_2(z))g_2(z)\right| \\
\ge \;& (1-|\omega|^2) \left|f'_{\varphi_1,\omega}(\varphi_1(\omega))g_1(\omega) - f'_{\varphi_1,\omega}(\varphi_2(\omega))g_2(\omega)\right| \\
= \;& \left|\varphi_1^*(\omega)g_1(\omega) - \frac{(1-|\varphi_1(\omega)|^2)(1-|\varphi_2(\omega)|^2)}{(1-\overline{\varphi_1(\omega)}\varphi_2(\omega))^2}\varphi_2^*(\omega)g_2(\omega)\right| \\
\ge \;& \left||\varphi_1^*(\omega)g_1(\omega)| - (1-\rho(\varphi_1(\omega), \varphi_2(\omega))^2\, |\varphi_2^*(\omega)g_2(\omega)|\right| .
\end{aligned}$$

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This is

$$\left||\varphi_1^*(\omega)g_1(\omega)| - (1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2\,|\varphi_2^*(\omega)g_2(\omega)|\right| \le C. \tag{4}$$

Similarly,letting the test function $f_{\varphi_2,\omega}(z) = \frac{\varphi_2(\omega)-z}{1-\overline{\varphi_2(\omega)}z}$, we can obtain

$$\left||\varphi_2^*(\omega)g_2(\omega)| - (1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2\,|\varphi_1^*(\omega)g_1(\omega)|\right| \le C. \tag{5}$$

We take the test functions as follow:

$$f(z) = f^2_{\varphi_1,\omega}(z) = (\frac{\varphi_1(\omega)-z}{1-\overline{\varphi_1(\omega)}z})^2, g(z) = f^2_{\varphi_2,\omega}(z) = (\frac{\varphi_2(\omega)-z}{1-\overline{\varphi_2(\omega)}z})^2. \tag{6}$$

The following conclusions can be easily concluded

$$2(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\rho(\varphi_1(\omega),\varphi_2(\omega))\,|\varphi_2^*(\omega)g_2(\omega)| \le C, \tag{7}$$

$$2(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\rho(\varphi_1(\omega),\varphi_2(\omega))\,|\varphi_1^*(\omega)g_1(\omega)| \le C. \tag{8}$$

If $\rho(\varphi_1(\omega),\varphi_2(\omega)) \le \frac{1}{2}$,then by (8), we have

$$|\varphi_1^*(z)g_1(z)|\,\rho(\varphi_1(z),\varphi_2(z)) < C.$$

If $\rho(\varphi_1(\omega),\varphi_2(\omega)) > \frac{1}{2}$,then by (7), we have

$$(1-\rho(\varphi_1(\omega),\varphi_2(\omega))^2)\,|\varphi_2^*(\omega)g_2(\omega)| \le C,$$

then, $|\varphi_1^*(\omega))g_1(\omega)| \le C$ is followed by (4), so

$$|\varphi_1^*(\omega)|\,|g_1(\omega)|\,\rho(\varphi_1(\omega),\varphi_2(\omega)) < C.$$

We can get (1) by use of the arbitrary of ω. Analogously, (2) was also can be obtained.

Finally, in order to prove the condition (3), using Lemma 2 and Lemma 3, we have

$$\begin{aligned}
& C \ge ||(C^{g_1}_{\varphi_1} - C^{g_2}_{\varphi_2})f_{\varphi_1,\omega}||_{\mathcal{B}} \\
\ge\ & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| \\
-\ & |g_2(\omega)\varphi_2^*(\omega)| \left| 1 - \frac{(1-|\varphi_1(\omega)|^2)(1-|\varphi_2(\omega)|^2)}{(1-\overline{\varphi_1(\omega)}\varphi_2(\omega))^2} \right| \\
\ge\ & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| \\
-\ & |g_2(\omega)\varphi_2^*(\omega)| \left|(1-|\varphi_1(\omega)|^2)f'_{\varphi_1,\omega}(\varphi_1(\omega)) - (1-|\varphi_2(\omega)|^2)f'_{\varphi_1,\omega}(\varphi_2(\omega))\right| \\
\ge\ & |g_1(\omega)\varphi_1^*(\omega) - g_2(\omega)\varphi_2^*(\omega)| - C\,|g_2(\omega)\varphi_2^*(\omega)|\,\rho(\varphi_1(\omega),\varphi_2(\omega)).
\end{aligned}$$

Then,

$$\sup_{z\in D} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| < \infty.$$

This is completes the proof of this theorem. □

By the studying similarly to the proof of Theorem 3.2 in the paper [7], the following theorem can be obtained.

Theorem 2. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ are bounded but not compact, Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is compact;*

(ii) Both (a) *and* (b) *hold:*

$$(a)\Gamma^*(\varphi_1) = \Gamma^*(\varphi_2) \neq \emptyset, then \Gamma^*(\varphi_1) \subset \Gamma(\varphi_1) \cap \Gamma(\varphi_2)$$

$$(b) For \{z_n\} \in \Gamma(\varphi_1) \cap \Gamma(\varphi_2),$$

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\,|g_1(z_n)|\,\rho(\varphi_1(z_n), \varphi_2(z_n)) = 0$$

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\,|g_2(z_n)|\,\rho(\varphi_1(z_n), \varphi_2(z_n)) = 0$$

and

$$\lim_{n\to\infty} |\varphi_1^*(z_n) g_1(z_n) - \varphi_2^*(z_n) g_2(z_n)| = 0$$

(iii)

$$\lim_{|\lambda|\to 1} ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})\varphi_\lambda||_{\mathcal{B}} = 0$$

and

$$\lim_{|\lambda|\to 1} ||(C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2})(\varphi_\lambda)^2||_{\mathcal{B}} = 0.$$

Here, $\Gamma(\varphi_1)$ is the set of sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi_1(z_n)| \to 1$. $\Gamma^*(\varphi_1)$ is the set of sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi_1(z_n)| \to 1$ and $\varphi_1^*(z_n) g_1(z_n)$ does not approach the 0.

Next, the other major theorem will be given

Theorem 3. *Let φ_1, φ_2 be analytic self-maps of the unit disk and $g_1, g_2 \in H(\mathbb{D})$, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ are bounded, Then the following statements are equivalent.*

(i) $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ *is compact;*

(ii)

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0$$

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0$$

and

$$\lim_{|\varphi_1(z)|, |\varphi_2(z)|\to 1} |\varphi_1^*(z) g_1(z) - \varphi_2^*(z) g_2(z)| = 0.$$

Proof. We first prove $(i) \Rightarrow (ii)$.We assume that $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}$ is compact, then, $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ are compact or noncompact.

If they are compact, the following conclusions are obtained obviously by the Theorem 2 in [12],

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)| = 0, \lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)| = 0,$$

then, the (ii) holds by them.

If they are all noncompact, for a sequence $\{z_n\}$, such that $|\varphi_1(z_n)| \to 1$, if

$$|\varphi_1^*(z_n)|\,|g_1(z_n)| \to 0,$$

then,

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\,|g_1(z_n)|\,\rho(\varphi_1(z_n), \varphi_2(z_n)) = 0;$$

if

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\,|g_1(z_n)| \neq 0,$$

then $\{z_n\} \in \Gamma^*(\varphi_1)$. By Theorem 2,

$$\{z_n\} \in \Gamma^*(\varphi_1) \subset \Gamma(\varphi_1) \cap \Gamma(\varphi_2),$$

and

$$\lim_{n\to\infty} |\varphi_1^*(z_n)|\,|g_1(z_n)|\,\rho(\varphi_1(z_n), \varphi_2(z_n)) = 0.$$

Hence,

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

According to similarly proof, we can get

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

For $\{z_n\}$ such that $|\varphi_1(z_n)|, |\varphi_2(z_n)| \to 1$, using Theorem 2, we have

$$\lim_{n\to\infty} |\varphi_1^*(z_n)g_1(z_n) - \varphi_2^*(z_n)g_2(z_n)| = 0.$$

Due to the arbitrary of $\{z_n\}$, we have

$$\lim_{|\varphi_1(z)|,|\varphi_2(z)|\to 1} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| = 0.$$

This is completes the proof of $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ If the operators $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ are all noncompact, (i) holds obviously by Theorem 2. If one of the operators $C_{\varphi_1}^{g_1}, C_{\varphi_2}^{g_2}$ is compact, we may also assume that $C_{\varphi_1}^{g_1}$ is compact, then by the Theorem 2 in [10], we have

$$\lim_{|\varphi_1(z)|\to 1} |\varphi_1^*(z)|\,|g_1(z)| = 0.$$

Let $\{z_n\}$ be an arbitrary sequence in $\mathbb{D}$, such that $|\varphi_2(z_n)| \to 1$ as $n \to \infty$.

If $|\varphi_1(z_n)|$ approach 1, since

$$\lim_{|\varphi_1(z)|,|\varphi_2(z)|\to 1} |\varphi_1^*(z)g_1(z) - \varphi_2^*(z)g_2(z)| = 0,$$

We obtain

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\,|g_2(z_n)| = 0.$$

If $|\varphi_1(z_n)|$ does not approach 1, then $\rho(\varphi_1(z), \varphi_2(z))$ does not approach 0, since,

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)|\,\rho(\varphi_1(z), \varphi_2(z)) = 0.$$

We also obtain

$$\lim_{n\to\infty} |\varphi_2^*(z_n)|\,|g_2(z_n)| = 0.$$

Due to the arbitrary of $\{z_n\}$, we have

$$\lim_{|\varphi_2(z)|\to 1} |\varphi_2^*(z)|\,|g_2(z)| = 0.$$

Therefore, $C_{\varphi_2}^{g_2}$ is a compact operator, therefore, $C_{\varphi_1}^{g_1} - C_{\varphi_2}^{g_2}$ is compact.

This is completes the proof of this theorem. $\square$

References

[1] E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math Soc.*, 81, 230-232 (1981).

[2] J. Bonet, M.Lindström and E. Wolf, Isometric weighted composition operators on weighted Banach spaces of type H^∞, *Proc. Amer. Math. Soc.*, 136(12), 4267-4273 (2008).

[3] C.C.Cowen, B.D. Macclur, Composition Operators on Spces of Analytic Functions, CRC PRESS, 1995.

[4] Z.S. Fang and Z.H. Zhou,Differences of composition operators on the space of bounded analytic functions in the polydisc, *Abstract and Applied Analysis*, 2008, Article ID 983132, 10 pages, (2008).

[5] Z.S. Fang and Z.H. Zhou,Differences of composition operators on the space of bounded analytic functions in the polydisc, *Bull. Aust. Math. Soc.*, 79, 465-471 (2009).

[6] T. Hosokawa, K. Izuchi and D. Zheng, Isolated points and essential components of composition operators on H^∞, *Proc. Amer. Math. Soc.*, 130, 1765-1773 (2002).

[7] T. Hosokawa and S. Ohno, Differences of composition operators on the Bloch space, *J. Operator Theory*, 57,229-242 (2007).

[8] M. Lindström and E. Wolf. Essential norm of the difference of weighted composition operators , *Monatsh. Math.*, 153, 133-143 (2008).

[9] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, 338,1282-1295 (2008).

[10] S. Li and S. Stević, Product of integral-type operators and composition operators between Bloch type spaces, *J. Math. Anal. Appl.*, 349, 596-610 (2009).

[11] S. Li and S. Stević, Products of Volterra type operators from H^∞ and the Bloch type spaces to Zygmund spaces, *J. Math. Anal. Appl.*, 345, 40-52 (2008).

[12] S. Li and S. Stević, Product of composition and integral type operators from H^∞ to the Bloch spaces, *Complex variables and Elliptic Equations*, 53, 463-474 (2008).

[13] B. MacCluer, S. Ohno and R. Zhao, Topological structure of the space of composition operators on H^∞, *Integr. Equ. Oper. Theory*, 40(4), 481-494 (2001).

[14] A. Montes-Rodríguez, Weighted composition operators on weighted Banach spaces of analytic functions, *J. London Math. Soc.*, 61(3), 872-884 (2000).

[15] J. Moorhouse, Compact difference of composition operators, *J. Funct. Anal.*, 219, 70-92 (2005).

[16] Pekka J. Nieminen. compact differences of composition operators on Bloch and Lipschitz spaces, *CMFT*, 7 (2), 325-344 (2007).

[17] E. Wolf. Differences of composition operators between weighted Banach spaces of holomorphic functions on the unit polydisk, *Result. Math.*, 51, 361-372 (2008).

[18] J. H. Shapiro, Composition operators and classical function theory, Spriger-Verlag, 1993.

[19] J. H. Shi and L. Luo, Composition operators on the Bloch space, *Acta Math. Sinica*, 16, 85-98 (2000).

[20] C. Toews, Topological components of the set of composition operators on $H^\infty(B_N)$, *Integr. Equ. Oper. Theory*, 48, 265-280 (2004).

[21] Z. H. Zhou and R. Y. Chen, Weighted composition operators fom $F(p,q,s)$ to Bloch type spaces, *International Jounal of Mathematics*, 19(8), 899-926 (2008).

[22] Z. H. Zhou and J. H. Shi, Compactness of composition operators on the Bloch space in classical bounded symmetric domains, *Michigan Math. J.*, 50, 381-405 (2002).

[23] Z. H. Zhou and L. Zhang, Differences of the products of integral type and composition operators from H^∞ to the Bloch space, *Complex Variables and Elliptic Equations*, DOI: 10.1080/17476933.2011.636150

[24] K. H. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag (GTM 226), 2004.

Isometries among the generalized composition operators on Bloch type spaces

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[1]

Abstract

In this paper, we characterize the isometries among the generalized composition operators on Bloch type spaces in the disk.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

We recall that the Bloch type space $\mathcal{B}^\alpha$ $(\alpha > 0)$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\alpha}$ is a complete semi-norm on $\mathcal{B}^\alpha$, which is Möbius invariant.

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}.$$

Let φ be an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$, the generalized composition operator C_φ^g induced by φ and g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The definition of generalized composition operator was first introduced by S. Li, S. Stević in [20], and in the paper, the boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces were investigated by them.

If we use the derivative of some function g to instead of g in operator C_φ^g, we can get a new integral operator L_g^φ, which is also called generalized composition operator. Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, the operator L_g^φ induced by φ and g is defined by

$$(L_g^\varphi f)(z) = \int_0^z f'(\varphi(\xi))g'(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

More results about boundedness, compactness, differences and essential norms of composition and closely related operators between various spaces of holomorphic functions have

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been studied by many authors, see, e.g., [12, 18, 19, 21, 25, 27]. Recently, many papers focused on studying isometries of the composition operators on various spaces of holomorphic functions.

Let X and Y be two Banach spaces, and recall that a linear isometry is a linear operator T from X to Y such that $\|Tf\|_Y = \|f\|_X$ for all $f \in X$.

In [3], Banach showed great interet in the form of an isometry on a specific Banach space. In most cases the isometries of a space of analytic functions on the disk or the ball have the canonical form of weighted composition operators, which is also true for most symmetric function spaces. For example, the surjective isometries of Hardy and Bergman spaces are certain weighted composition operators. See [13, 14, 15].

The description of all isometric composition operators is known for the Hardy space H^2 (see [8]). An analogous statement for the Bergman space A^2_α with standard radial weights has recently been obtained in [7], and there is a unified proof for all Hardy spaces and also for arbitrary Bergman spaces with reasonable radial weights [24]. In [9], Colonna gave a characterization of the isometric composition operators on the Bloch space in terms of the factorization of the symbol in H^∞, which shows that there is a very large class of isometries besides the rotations. By contrast, in [26], Zorboska showed that in the case $\alpha \neq 1$, the isometries of the composition operators on $\mathcal{B}^\alpha$ are the operators whose symbol is a rotation.

Continued the work of isometry, in 2008, Bonet, Lindström and Wolf [4] studied isometric weighted composition operators on weighted Banach spaces of type H^∞. Cohen and Colonna [6] discussed the spectrum of an isometric composition operators on the Bloch space of the polydisk. In 2009, Allen and Colonna [1] investigated the isometric composition operators on the Bloch space in $\mathcal{C}^n$. They [2] also discussed the isometries and spectra of multiplication operators on the Bloch space in the disk. Isometries of weighted spaces of holomorphic functions on unbounded domains were discussed by Boyd and Rueda in [5]. In 2010, Li and Zhou discussed the isometries on products of composition and integral operators on Bloch type space in [10].more results ,for example, can be seen in [11, 16, 17, 22, 23].

The paper continues the research of it, and discusses the isometries among the generalized composition operators on Bloch type space in the disk.

2 Notation and Lemmas

First, we will introduce some notations and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a, is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

For z, w in $\mathbb{D}$, the pseudohyperbolic distance between z and w is given by

$$\rho(z, w) = |\varphi_z(w)| = \left|\frac{z - w}{1 - \bar{z}w}\right|.$$

The following lemma is well known [25].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

For $\varphi \in S(D)$, the Schwarz-Pick lemma shows that $\rho(\varphi(z), \varphi(w)) \leq \rho(z, w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on $\mathcal{B}$.

A little modification of Lemma 1 in [4] shows the following lemma.

Lemma 2. *There exists a constant $C > 0$ such that*

$$\left|\left(1-|z|^2\right)^{\alpha} f'(z) - \left(1-|w|^2\right)^{\alpha} f'(w)\right| \leq C\,\|f\|_{\mathcal{B}^\alpha} \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}^\alpha$.

Throughout the rest of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

3 Main theorems

Theorem 1. *Let φ be analytic self maps of the unit disk and $g \in H(\mathbb{D})$. Then the operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is an isometry in the semi-norm if and only if the following conditions hold:*

(A) $\sup_{z\in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| \leq 1$;

(B) *For every $a \in \mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that* $\lim_{n\to\infty} \rho(\varphi(z_n), a) = 0$ *and* $\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |g(z_n)| = 1.$

Proof. We prove the sufficiency first.

By condition (A), for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
||C_\varphi^g f||_{\mathcal{B}^\beta} &= \sup_{z\in D}(1-|z|^2)^\beta\,|f'(\varphi(z))|\,|g(z)| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f'(\varphi(z))| \\
&\leq \|f\|_{\mathcal{B}^\alpha}.
\end{aligned}$$

Next we show that the property (B) implies $||C_\varphi^g f||_{\mathcal{B}^\beta} \geq ||f||_{\mathcal{B}^\alpha}$

Given any $f \in \mathcal{B}^\alpha$, then $||f||_{\mathcal{B}^\alpha} = \lim_{m\to\infty}(1-|a_m|^2)^\alpha|f'(a_m)|$ for some sequence $\{a_m\} \subset D$. For any fixed m, by property (B), there is a sequence $\{z_k^m\} \subset \mathbb{D}$ such that

$$\rho(\varphi(z_k^m), a_m) \to 0 \text{ and } \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha}\,|g(z_k^m)| \to 1$$

as $k \to \infty$. By Lemma 2, for all m and k,

$$\left|(1-|\varphi(z_k^m)|^2)^\alpha f'(\varphi(z_k^m)) - (1-|a_m|^2)^\alpha f'(a_m)\right| \leq C||f||_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Hence

$$(1-|\varphi(z_k^m)|^2)^\alpha\,|f'(\varphi(z_k^m))| \geq (1-|a_m|^2)^\alpha|f'(a_m)| - C||f||_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m)$$

Therefore,

$$\begin{aligned}
||C_\varphi^g f||_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f'(\varphi(z))| \\
&\geq \limsup_{k\to\infty} \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha}\,|g(z_k^m)|\,(1-|\varphi(z_k^m)|^2)^\alpha\,|f'(\varphi(z_k^m))| \\
&= (1-|a_m|^2)^\alpha|f'(a_m)|.
\end{aligned}$$

The inequality $||C_\varphi^g f||_{\mathcal{B}^\beta} \geq ||f||_{\mathcal{B}^\alpha}$follows by letting $m \to \infty$.

From the above discussions, we have $||C_\varphi^g f||_{\mathcal{B}^\beta} = ||f||_{\mathcal{B}^\alpha}$, which means that C_φ^g is an isometry operator in the semi-norm from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$.

Now we turn to the necessity.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}} dt. \tag{1}$$

It is clear that $f_a'(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1-|z|^2)^\alpha |f_a'(z)| = \frac{(1-|z|^2)^\alpha (1-|a|^2)^\alpha}{|1-\bar{a}z|^{2\alpha}} = (1-\rho^2(a,z))^\alpha. \tag{2}$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f_a'(z)| \le 1. \tag{3}$$

On the other hand, since $(1-|a|^2)^\alpha |f_a'(a)| = \frac{(1-|a|^2)^{2\alpha}}{(1-|a|^2)^{2\alpha}} = 1$, we have $\|f_a\|_{\mathcal{B}^\alpha} = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned} 1 &= ||f_{\varphi(a)}||_{\mathcal{B}^\alpha} = ||C_\varphi^g f_{\varphi(a)}||_{\mathcal{B}^\beta} \\ &= \sup_{z\in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |g(z)| (1-|\varphi(z)|^2)^\alpha \left| f_{\varphi(a)}'(\varphi(z)) \right| \\ &\ge \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha} |g(a)| . \end{aligned}$$

So (A) follows by the arbitrariness of a.

Since $||f_a||_{\mathcal{B}^\alpha} = ||C_\varphi^g f_a||_{\mathcal{B}^\beta} = 1$, there exists a sequence $\{z_m\} \subset \mathbb{D}$ such that

$$((1-|z_m|^2)^\beta \left| \frac{d(C_\varphi^g f_a)}{dz}(z_m) \right| = (1-|z_m|^2)^\beta |f_a'(\varphi(z_m))||g(z_m)| \to 1 \tag{4}$$

as $m \to \infty$.

It follows from (A) that

$$\begin{aligned} & (1-|z_m|^2)^\beta |f_a'(\varphi(z_m))||g(z_m)| \\ = & \frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^\alpha} |g(z_m)| (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| \qquad (5) \\ \le & (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| . \qquad (6) \end{aligned}$$

Combining (4) and (6), it follows that

$$\begin{aligned} 1 &\le \liminf_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| \\ &\le \limsup_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| \le 1. \end{aligned}$$

The last inequality follows by (2) since $\varphi(z_m) \in \mathbb{D}$.

Consequently,

$$\lim_{m\to\infty} (1-|\varphi(z_m)|^2)^\alpha |f_a'(\varphi(z_m))| = \lim_{m\to\infty} (1-\rho^2(\varphi(z_m),a))^\alpha = 1. \tag{7}$$

That is, $\lim_{m\to\infty} \rho(\varphi(z_m), a) = 0$.

Combining (4), (5) and (7), we know

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^\alpha}|g(z_m)|=1.$$

This completes the proof of this theorem. □

Corollary 1. *Let U denote unitary transformation in the unit disk, then $C_U^1:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is an isometry in the semi-norm.*

□

If we use the derivative of some function g to instead of g in operator C_φ^g, by the above theorem. we can easily get the following result about the operator L_g^φ.

Theorem 2. *Let φ be analytic self maps of the unit disk and $g\in H(\mathbb{D})$. Then the operator $C_g^\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is an isometry in the semi-norm if and only if the following conditions hold:*

(C) $\sup\limits_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|g'(z)|\le 1;$

(D) *For every $a\in\mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that* $\lim\limits_{n\to\infty}\rho(\varphi(z_n),a)=0$ *and* $\lim\limits_{n\to\infty}\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha}|g'(z_n)|=1.$

Remark If $\alpha=1$, $\beta=1$, then $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ will be Bloch space $\mathcal{B}$. There are similar results on the Bloch space $\mathcal{B}$ corresponding to Theorems 1 and 2.

References

[1] R. F. Allen and F. Colonna, Isometric composition operators on the Bloch space in $\mathcal{C}^n$, *J. Math. Anal. Appl.*, 355 (2009), 675-688.

[2] R. F. Allen and F. Colonna, Isometries and spectra of multiplication operators on the Bloch space, *Bull. Aust. Math. Soc.*, 79 (2009),147-160.

[3] S. Banach, Theorie des Operations Lineares, Chelsea, Warzaw, 1932.

[4] J.Bonet, M. Lindström and E. Wolf, Isometric weighted composition operators on weighted Banach spaces of type H^∞, *Proc. Amer. Math. Soc.*, 136(12)(2008), 4267-4273.

[5] C. Boyd and P. Rueda, Isometries of weighted spaces of holomorphic functions on unbounded domains, *Proceedings of the Royal Society of Edinburgh Section A-mathematics*, 139A (2009), 253-271.

[6] J.M.Cohen and F. Colonna, Isometric Composition Operators on the Bloch Space in the Polydisk, Contemporary Mathematics, *Banach Spaces of Analytic Functions*, 454 (2008), 9-21.

[7] B. J. Carswell and C. Hammond, Composition operators with maximal norm on weighted Bergman spaces, *Proc. Amer. Math. Soc.*, 134 (2006), 2599-2605.

[8] B. A. Cload, Composition operators: hyperinvariant subspaces, quasi-normals and isometries, *Proc. Amer. Math. Soc.*, 127 (1999),1697-1703.

[9] F. Colonna, Characterisation of the isometric composition operators on the Bloch space, *Bull. Austral. Math. Soc.* 72 (2005), 283-290.

[10] Geng-Lei Li and Ze-Hua Zhou,Isometries on Products of composition and integral operators on Bloch type space, *Journal of Inequalities and Applications,* 2010 (2010), Article ID 184957: 9 pages. (SCI No. 664HU)

[11] W. Hornor and J. E. Jamison, Isometries of some Banach spaces of analytic functions, *Integral Equations Operator Theory* ,41(2001), 410-425.

[12] T. Hosokawa, S. Ohno, Differences of composition operators on the Bloch space, *J. Operator Theory,* 57(2007), 229-242.

[13] C.J. Kolaski, Isometries of weighted Bergman spaces, *Can.J. Math.*, 34 (1982), 910-915.

[14] C. J. Kolaski, Isometries of some smooth spaces of analytic functions, *Complex Variables,* 10 (1988), 115-122.

[15] A. Koranyi and S. Vagi, On isometries of H^p of bounded symmetruc domains, *Cann. J. Math.*, 28(1976), 334-340.

[16] S. Y. Li, Composition operators and isometries on holomorphic function spaces over domains in C^n. *AMS/IP Stud Adv Math.* 39 (2007), 161-174.

[17] S. Y. Li and Y. B. Ruan, On characterizations of isometries on function spaces, *Science in China Series A: Mathematics*, 51(4)(2008),620-631.

[18] S. Li, S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, *Complex variables and EllipticEquations*, 53 (2008), 463-474.

[19] S. Li, S. Stević, Products of integral-type operators and composition operators between Bloch-type spaces, *J. Math. Anal. Appl.*, 349(2009),596-610.

[20] S. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, 338(2008),1282-1295.

[21] M. Lindström and E. Wolf. Essential norm of the difference of weighted composition operators, *Monatsh .Math.*, 153(2008), 133-143.

[22] A. Matheson, Isometries into function algebras, *Houston J. Math.*, 30 (2004), 219-230 .

[23] M. J. Martt́n and D. Vukotić, Isometries of the Bloch space among the composition operators, *Bull. London Math. Soc.*, 39 (2007), 151-155.

[24] M. J. Martt́n and D. Vukotić, Isometries of some classical function spaces among the composition operators, in *Recent advances in Operator-related function theory,* A. L. Matheson, M. I. Stessin, and R. M. Timoney,Eds., vol. 393 of *Contemporary Mathematics,*133-138, American Mathematical Society, Providence, RI,USA, 2006.

[25] K. H. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag (GTM 226), 2004.

[26] N. Zorboska, Isometric composition operators on the Bloch-type spaces, *C. R. Math. Acad. Sci. Soc.* 29(3)(2007), 91-96.

[27] Z. H. Zhou and J. H. Shi, Compactness of composition operators on the Bloch space in classical bounded symmetric domains, *Michigan Math. J.*, 50(2002),381-405.

Coupled Fixed Point Theorems for Generalized Symmetric Contractions in Partially Ordered Metric Spaces and applications

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Abstract

In the setting of partially ordered metric spaces, we introduce the notion of generalized symmetric g-Meir-Keeler type contractions and use the notion to establish the existence and uniqueness of coupled common fixed points. Our notion extends the notion of generalized symmetric Meir-Keeler contractions given by Berinde et. al. [V. Berinde, and M. Pacurar, Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces, Fixed Point Theory and Appl., 2012, 2012:115, doi:10.1186/1687-1812-2012-115] to a pair of mappings. We also give some applications of our main results.

AMS Subject Classification: 47H10, 46T99, 54H25 .

Key Words : partially ordered metric space, fixed point, generalized symmetric contractions, coupled fixed point.

1 Introduction

Banach [1] in his classical work gave the following contractive theorem:

Theorem 1.1. *Let (X, d) be a metric space and $T : X \to X$ be a self mapping. If (X, d) is complete and T is a contraction, that is, there exists a constant $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y), \forall x, y \in X \tag{1.1}$$

then, T has a unique fixed point $u \in X$ and for any $x_0 \in X$, the Picard iteration $\{T^n(x_0)\}$ converges to u.

This contraction principle proved to be a very powerful tool in nonlinear analysis, and different authors have generalized it in many ways. One can refer to the works noted in references [2]- [17]. Meir and Keeler [9] generalized the contraction principle due to Banach by considering a more general contractive condition in their work as follows:

Theorem 1.2. *[9] Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \Rightarrow d(T(x), T(y)) < \epsilon \tag{1.2}$$

*Corresponding author

for all $x, y \in X$. Then T admits a unique fixed point $x_0 \in X$ and for all $x \in X$, the sequence $\{T^n(x)\}$ converges to x_0.

By extending the Banach contraction principle to partially ordered sets, Turinici [16] laid the foundation for a new trend in fixed point theory. Ran and Reurings [17] developed some applications of Turinici's theorem to matrix equations. The work of Bhaskar and Lakshmikantham [18] is worth mentioning, as they introduced the new notion of fixed points for mappings having domain the product space $X \times X$, which they called coupled fixed points, and thereby proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ciric [19] extended the notion of the mixed monotone property to the mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [18] by establishing the existence of coupled coincidence points, using a pair of commutative maps.This proved to be a milestone in the development of fixed point theory with applications to partially ordered sets. Since then much work has been done in this direction by different authors. For more details the reader may consult [20]-[31].
Gordji et. al. [32], extended the results of Bhaskar and Lakshmikantham [18], and Samet [33] by introducing the concept of generalized g-Meir-Keeler type contractions. Abdeljawad et. al. [34] and Jain et. al. [36] proved some interesting results in partially ordered partial metric spaces and remarked that the metric space case of their results, proved recently in Gordji et. al. [32] has gaps. They claimed that some of the results proved by Gordji et. al.[32] cannot be true if obtained via nonstrongly minihedral cones. On the other hand, Berinde et. al. [35] with their outstanding new approach introduced the notion of generalized symmetric Meir-Keeler contractions and complemented the results due to Samet [33]. In this paper, we introduce the notion of generalized symmetric g-Meir-Keeler type contractions that extends the concept of generalized symmetric Meir-Keeler contractions given by Berinde et. al. [35] to a pair of mappings. Following Abdeljawad et. al. [34], we establish the existence and uniqueness of coupled common fixed points for mixed g-monotone mappings satisfying generalized symmetric conditions in partially ordered metric spaces. To validate our results we also give some applications. Before we proceed, we first summarize some basic results and definitions useful in our study.

Definition 1.3. [18] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2)$$

Definition 1.4. [18] An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.5. [19] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument; that is, for any $x, y \in X$,
$x_1, x_2 \in X, g(x_1) \leq g(x_2)$ implies $F(x_1, y) \leq F(x_2, y)$
and
$y_1, y_2 \in X, g(y_1) \leq g(y_2)$ implies $F(x, y_1) \geq F(x, y_2)$

Definition 1.6. [19] An element $(x, y) \in X \in X$, is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.7. [19] An element $(x, y) \in X \in X$, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.8. [19] Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say that F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Later, Choudhury and Kundu[20] introduced the notion of compatibility in the context of coupled coincidence point problems and used this notion to improve the results noted in [19].

Definition 1.9. [20] The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(g(F(x_n, y_n), F(gx_n, gy_n)) = 0$ and $\lim_{n\to\infty} d(g(F(y_n, x_n), F(gy_n, gx_n)) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$.

Recently, Gordji et. al. [32] replaced the mixed g-monotone property with the mixed strict g-monotone property and extended the results of Bhaskar and Lakshmikantham [18].

Definition 1.10. [32] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed strict g-monotone property if for any $x, y \in X$,
$x_1, x_2 \in X, g(x_1) < g(x_2)$ implies $F(x_1, y) < F(x_2, y)$
and
$y_1, y_2 \in X, g(y_1) < g(y_2)$ implies $F(x, y_1) > F(x, y_2)$

Here if we replace g with identity mapping in Definition 1.10, we get the definition of mixed strict monotone property of F.

Theorem 1.11. *[36] Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that there exists a real number $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)] \tag{1.3}$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$. Suppose that either
(a) F is continuous
or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n > 0$;
(ii) if a non-decreasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n > 0$;
Then F has a coupled fixed point in X.

We now introduce our notion.

Definition 1.12. Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say that F is a generalized symmetric g-Meir-Keeler type contraction if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that , for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$ (or $g(x) \geq g(u)$ and $g(y) \leq g(v)$),

$$\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))] < \epsilon \tag{1.4}$$

If, in Definition 1.12, we replace g by the identity mapping, we obtain the definition of a generalized symmetric Meir-Keeler type contraction due to Berinde et. al. [35].

Definition 1.13. [35] Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ be the given mapping. We say that F is a generalized symmetric Meir-Keeler type contraction if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that , for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$ (or $x \geq u$ and $y \leq v$),

$$\epsilon \leq \frac{1}{2}[d(x,u)+d(y,v)] < \epsilon+\delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))] < \epsilon \tag{1.5}$$

Proposition 1.14. *Let $(X, d, \leq)$ be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. If contractive condition* (1.3) *is satisfied for $0 < k < 1$, then F is a generalized symmetric Meir-Keeler type contraction.*

Proof. Assume that (1.3) is satisfied for $0 < k < 1$. For all $\epsilon > 0$, it is easy to check that (1.5) is satisfied with $\delta(\epsilon) = \left(\frac{1}{k} - 1\right)\epsilon$. □

Lemma 1.15. *Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. If F is a generalized symmetric g-Meir-Keeler type contraction, then we have*

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) < d(g(x),g(u))+d(g(y),g(v)) \tag{1.6}$$

for all $x, y, u, v \in X$ with $g(x) < g(u), g(y) \geq g(v)$ (or $g(x) \leq g(u), g(y) > g(v)$).

Proof. Without loss of generality, we may assume that $g(x) < g(u), g(y) \geq g(v)$ where $x, y, u, v \in X$. Then $d(g(x), g(u)) + d(g(y), g(v)) > 0$. Since F is a generalized symmetric g-Meir- Keeler type contraction, for $\epsilon = (\frac{1}{2})[d(g(x), g(u)) + d(g(y), g(v))]$, there exists a $\delta(\epsilon) > 0$ such that , for all $x_0, y_0, u_0, v_0 \in X$ with $g(x_0) < g(u_0)$ and $g(y_0) \geq g(v_0)$,

$$\epsilon \leq \frac{1}{2}[d(g(x_0),g(u_0))+d(g(y_0),g(v_0))] < \epsilon+\delta(\epsilon)$$

implies

$$\frac{1}{2}[d(F(x_0,y_0),F(u_0,v_0))+d(F(y_0,x_0),F(v_0,u_0))] < \epsilon$$

Then the result follows by choosing $x = x_0, y = y_0, u = u_0, v = v_0$; that is,

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) < d(g(x),g(u))+d(g(y),g(v))$$

□

2 Existence of Coupled Coincidence Points

We now establish our first main result.

Theorem 2.1. *Let $(X, \leq, d)$ be a partially ordered metric space. Suppose that X has the following properties:*
(i) if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1, 2, \ldots$ and $x_n \to x$, then $x_n < x$ for each $n = 1, 2, \ldots$.
(ii) if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1, 2, \ldots$ and $y_n \to y$, then $y_n > y$ for each $n = 1, 2, \ldots$.
Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, d). Also, suppose that
(a) F has the mixed strict g-monotone property;
(b) F is a generalized symmetric g-Meir-Keeler type contraction;
(c) there exists $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$(or $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$).
Then, there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. Without loss of generality, we may assume that there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0), g(y_1) = F(y_0, x_0)$. Again we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1), g(y_2) = F(y_1, x_1)$. Continuing this process, we construct sequences $\{gx_n\}$ and $\{gy_n\}$ such that

$$g(x_{n+1}) = F(x_n, y_n), g(y_{n+1}) = F(y_n, x_n), \forall n \geq 0 \tag{2.1}$$

Using conditions (a), (c) and mathematical induction, it is easy to see that

$$g(x_0) < g(x_1) < g(x_2) < \ldots < g(x_n) < g(x_{n+1}) < \ldots \tag{2.2}$$

and

$$g(y_{n+1}) < g(y_n) < \ldots < g(y_2) < g(y_1) < g(y_0). \tag{2.3}$$

Denote by

$$\delta_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \tag{2.4}$$

Using (2.1) of Lemma 1.15, and condition (b), we have

$$\begin{aligned} \delta_n &:= d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \\ &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &< d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)) = \delta_{n-1} \end{aligned} \tag{2.5}$$

Thus, the sequence $\{\delta_n\}$ is a decreasing sequence. Therefore there exists some $\delta^* \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta^*$.

We claim that $\delta^* = 0$. Suppose, to the contrary, that $\delta^* \neq 0$. Then there exists a positive integer m such that, for any $n \geq m$, we have

$$\epsilon \leq \frac{\delta_n}{2} = \frac{1}{2}[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] < \epsilon + \delta(\epsilon) \tag{2.6}$$

where $\epsilon = \delta^*/2$ and $\delta(\epsilon)$ is chosen by condition (b).

In particular, for n = m, we have

$$\epsilon \leq \frac{\delta_m}{2} = \frac{1}{2}[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon + \delta(\epsilon) \tag{2.7}$$

Then, by condition (b), it follows that

$$\frac{1}{2}[d(F(x_m, y_m), F(x_{m+1}, y_{m+1})) + d(F(y_m, x_m), F(y_{m+1}, x_{m+1}))] < \epsilon \tag{2.8}$$

and hence, from (2.1), we have

$$\frac{1}{2}[d(g(x_{m+1}), g(x_{m+2})) + d(g(y_{m+1}), g(y_{m+2}))] < \epsilon \tag{2.9}$$

a contradiction to (2.6) for $n = m + 1$. Thus we must have $\delta^* = 0$ and hence

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = 0 \tag{2.10}$$

We now prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Take an arbitrary $\epsilon > 0$. Then, by (2.10), it follows that there exists some $k \in \mathbb{N}$ such that

$$\frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(y_k), g(y_{k+1}))] < \delta(\epsilon) \tag{2.11}$$

Without loss of generality, assume that k has been chosen so large that $\delta(\epsilon) \leq \epsilon$ and define the set

$$\begin{aligned} \wedge := \{(g(x), g(y)) : (x, y) \in X^2, d(g(x), g(x_k)) + d(g(y), g(y_k)) < 2(\epsilon + \delta(\epsilon)), \text{ and} \\ g(x) > g(x_k), g(y) \leq g(y_k)\} \end{aligned} \tag{2.12}$$

We claim that $(g(x), g(y)) \in \wedge$ implies that

$$(F(x, y), F(y, x)) \in \wedge \tag{2.13}$$

where $x, y \in X$.
Take $(g(x), g(y)) \in \wedge$. Then, using the triangle inequality and (2.11), we have

$$\begin{aligned}
\frac{1}{2}[d(g(x_k), F(x, y)) &+ d(g(y_k), F(y, x))] \leq \frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(x_{k+1}), F(x, y))] \\
&+ \frac{1}{2}[d(g(y_k), g(y_{k+1})) + d(g(y_{k+1}), F(y, x))] \\
&= \frac{1}{2}[d(g(x_k), g(x_{k+1})) + d(g(y_k), g(y_{k+1}))] \\
&+ \frac{1}{2}[d(g(x_{k+1}), F(x, y)) + d(g(y_{k+1}), F(y, x))] \\
&< \delta(\epsilon) + \frac{1}{2}[d(F(x, y), F(x_k, y_k)) + d(F(y, x), F(y_k, x_k))] \qquad (2.14)
\end{aligned}$$

We distinguish two cases.
First Case: $\frac{1}{2}[d(g(x_k), F(x, y)) + d(g(y_k), F(y, x))] \leq \epsilon$. By Lemma 1.15 and Definition of $\wedge$, the inequality (2.14) becomes

$$\begin{aligned}
\frac{1}{2}[d(g(x_k), F(x, y)) &+ d(g(y_k), F(y, x))] \leq \delta(\epsilon) \\
&+ \frac{1}{2}[d(F(x, y), F(x_k, y_k)) + d(F(y, x), F(y_k, x_k))] \\
&< \delta(\epsilon) + \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] \\
&\leq \delta(\epsilon) + \epsilon \qquad (2.15)
\end{aligned}$$

Second Case: $\epsilon < \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] < \delta(\epsilon) + \epsilon$. In this case, we have

$$\epsilon < \frac{1}{2}[d(g(x), g(x_k)) + d(g(y), g(y_k))] < \delta(\epsilon) + \epsilon \tag{2.16}$$

Then, since $g(x) > g(x_k)$ and $g(y) \leq g(y_k)$, by condition (b), we have

$$\frac{1}{2}[d(F(x, y), F(x_k, y_k)) + d(F(y, x), F(y_k, x_k))] < \epsilon \tag{2.17}$$

Using (2.17) in (2.14), we get

$$\frac{1}{2}[d(g(x_k), F(x, y)) + d(g(y_k), F(y, x))] < \delta(\epsilon) + \epsilon \tag{2.18}$$

Since F satisfies the mixed strict g -monotone property and $(g(x), g(y)) \in \wedge$, it follows that

$$F(x, y) > g(x_k), F(y, x) > g(y_k) \tag{2.19}$$

Also, $F(X \times X) \subseteq g(X)$. Consequently, we have $(F(x, y), F(y, x)) \in \wedge$; that is (2.13) holds. By (2.11), we have $(g(x_{k+1}), g(y_{k+1})) \in \wedge$.Then, using (2.13), we have

$$\begin{aligned}
(g(x_{k+1}), g(y_{k+1})) \in \wedge &\Rightarrow d(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1})) = (g(x_{k+2}), g(y_{k+2}) \in \wedge \\
&\Rightarrow d(F(x_{k+2}, y_{k+2}), F(y_{k+2}, x_{k+2})) = (g(x_{k+3}), g(y_{k+3}) \in \wedge \\
&\Rightarrow \ldots \Rightarrow (g(x_n), g(y_n)) \in \wedge \Rightarrow \ldots \qquad (2.20)
\end{aligned}$$

Then, for all $n > k$, we have $(g(x_n), g(y_n)) \in \wedge$. This implies that, for all $n, m > k$, we have

$$\begin{aligned} d(g(x_n), g(x_m)) &+ d(g(y_n), g(y_m)) \le d(g(x_n), g(x_k)) + d(g(x_k), g(x_m)) \\ &+ d(g(y_n), g(y_k)) + d(g(y_k), g(y_m)) \\ &= [d(g(x_n), g(x_k)) + d(g(y_n), g(y_k))] + [d(g(x_k), g(x_m)) + d(g(y_k), g(y_m))] \\ &\le 4(\epsilon + \delta(\epsilon)) \le 8\epsilon \end{aligned}$$

Therefore, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy. Since $(g(X), d)$ is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x)) = 0, \lim_{n\to\infty} d(g(y_n), g(y)) = 0 \tag{2.21}$$

Since the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are monotone increasing and monotone decreasing, respectively, by conditions (i) and (ii), we have

$$g(x_n) < g(x), \quad g(y_n) > g(y) \tag{2.22}$$

for each $n \ge 0$. Therefore, by (2.22) and Lemma 1.15, along with condition (b), we obtain

$$\begin{aligned} d(g(x_{n+1}), F(x, y)) &+ d(g(y_{n+1}), F(y, x)) \\ &= d(F(x_n, y_n), F(x, y)) + d(F(y_n, x_n), F(y, x)) \end{aligned}$$

$$< d(g(x_n), g(x)) + d(g(y_n), g(y)) \tag{2.23}$$

Letting $n \to \infty$ in (2.23) and using (2.21), we get

$$d(g(x), F(x, y)) + d(g(y), F(y, x)) \le \lim_{n\to\infty}[d(g(x_n), g(x)) + d(g(y_n), g(y))] \tag{2.24}$$

which yields $F(x, y) = g(x)$, $F(y, x) = g(y)$. This completes the proof. □

Corollary 2.2. *Let $(X, \le, d)$ be a partially ordered metric space. Suppose that (X, d) is complete and has the following properties:*
(i) if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1, 2, \dots$ and $x_n \to x$, then $x_n < x$ for each $n = 1, 2, \dots$.
(ii) if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1, 2, \dots$ and $y_n \to y$, then $y_n > y$ for each $n = 1, 2, \dots$.
Let $F : X \times X \to X$ be a mapping. Also, suppose that
(d) F has the mixed strict monotone property;
(e) F is a generalized symmetric Meir-Keeler type contraction;
(f) there exists $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$(or $x_0 \le F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$).
Then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.3. If, in Theorem 2.1 condition (c) is replaced by the following condition:
(g) there exist $x_0, y_0 \in X$ such that $g(x_0) > F(x_0, y_0)$ and $g(y_0) \le F(y_0, x_0)$ (or $g(x_0) \ge F(x_0, y_0)$ and $g(y_0) < F(y_0, x_0)$),
then we also get the existence of some $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.
And, if in Corollary 2.2, condition (f) is replaced by the following condition:
(h) there exist $x_0, y_0 \in X$ such that $x_0 > F(x_0, y_0)$ and $y_0 \le F(y_0, x_0)$
(or $x_0 \ge F(x_0, y_0)$ and $y_0 < F(y_0, x_0)$),
then we also get the existence of some $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.4. Corollary 2.2, along with Remark 2.3, improves on the result of Berinde et. al. ([35], Theorem 2) by removing the continuity assumption on the mixed monotone operator F.

3 Existence and Uniqueness of Coupled Fixed Points

In this section we prove the existence and uniqueness of coupled fixed points. Before we proceed, we need to consider the following.
For a partially ordered set $(X, \leq)$, we endow $X \times X$ with the following order $\leq_g$

$$(u, v) \leq_g (x, y) \Rightarrow g(u) < g(x), g(y) \leq g(v), \forall (x, y), (u, v) \in X \times X \tag{3.1}$$

In this case, we say that (u, v) and (x, y) are g -comparable if either $(u, v) \leq_g (x, y)$ or $(x, y) \leq_g (u, v)$. If $g = I_X$, then we simply say that (u, v) and (x, y) are comparable and denote this fact by $(u, v) \leq (x, y)$.

Lemma 3.1. *Let $F : X \times X \to X$ and $g : X \to X$ be compatible maps and suppose there exists an element $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Then $gF(x, y) = F(g(x), g(y))$ and $gF(y, x) = F(g(y), g(x))$.*

Proof. Since the pair (F, g) is compatible, it follows that

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = a$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = b$ for some $a, b \in X$.
Taking $x_n = x, y_n = y$ and using the fact that $g(x) = F(x, y), g(y) = F(y, x)$, it follows immediately that $d(gF(x, y), F(g(x), g(y))) = 0$ and $d(gF(y, x), F(g(y), g(x))) = 0$.
Hence, $gF(x, y) = F(g(x), g(y))$ and $gF(y, x) = F(g(y), g(x))$. □

Theorem 3.2. *In Theorem 2.1, assume, in addition, that, for all non g-comparable points $(x, y), (x^*, y^*) \in X \times X$, there exists a point $(a, b) \in X \times X$ such that $(F(a, b), F(b, a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), (y^*))$. Also assume that F and g are compatible. Then,F and g have a unique coupled common fixed point; that is, there exists a point $(u, v) \in X \times X$ such that*

$$u = g(u) = F(u, v), v = g(v) = F(v, u) \tag{3.2}$$

Proof. From Theorem 2.1 it follows that the set of coupled coincidence points of F and g is non-empty. We shall first show that, if (x, y) and (x^*, y^*) are coupled coincidence points, that is, if $g(x) = F(x, y)$, $g(y) = F(y, x)$ and $g(x^*) = F(x^*, y^*)$, $g(y^*) = F(y^*, x^*)$, then

$$g(x) = g(x^*) \text{and} g(y) = g(y*) \tag{3.3}$$

For this, we distinguish the following two cases.
First Case. (x, y) is g-comparable to (x^*, y^*) with respect to the ordering in $X \times X$, where

$$F(x, y) = g(x), F(y, x) = g(y), F(x^*, y^*) = g(x^*), F(y^*, x^*) = g(y^*) \tag{3.4}$$

Without loss of generality, we may assume that

$$g(x) = F(x, y) < F(x^*, y^*) = g(x^*), g(y) = F(y, x) \geq F(y^*, x^*) = g(y^*) \tag{3.5}$$

Using Lemma 1.15 we have

$$0 < d(g(x), g(x^*)) + d(g(y^*), g(y)) = d(F(x, y), F(x^*, y^*)) + d(F(y^*, x^*), F(y, x))$$

$$< d(g(x), g(x^*)) + d(g(y^*), g(y))$$

a contradiction. Therefore, we have $(g(x), g(y)) = (g(x^*), g(y^*))$. Hence (3.3) holds.
Second Case. (x, y) is not g-comparable to (x^*, y^*).

By assumption, there exists a point $(a,b) \in X \times X$ such that $(F(a,b), F(b,a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), g(y^*))$. Then we have

$$g(x) = F(x,y) < F(a,b), \quad F(x^*, y^*) = g(x^*) < F(a,b), \tag{3.6}$$

and

$$g(y) = F(y,x) \geq F(b,a), \quad F(y^*, x^*) = g(y^*) \geq F(b,a), \tag{3.7}$$

Further, setting $x = x_0, y = y_0, a = a_0, b = b_0$ and $x^* = x_0^*, y^* = y_0^*$ as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n), \forall n = 0,1,2,\dots \\ g(a_{n+1}) &= F(a_n, b_n), \quad g(b_{n+1}) = F(b_n, a_n), \forall n = 0,1,2,\dots \\ g(x_{n+1}^*) &= F(x_n^*, y_n^*), \quad g(y_{n+1}^*) = F(y_n^*, x_n^*), \forall n = 0,1,2,\dots \end{aligned} \tag{3.8}$$

Since $(F(x,y), F(y,x)) = (g(x), g(y)) = (g(x_1), g(y_1))$ is comparable with $(F(a,b), F(b,a)) = (g(a_1), g(b_1))$, we have $g(x) < g(a_1)$ and $g(y) \geq g(b_1)$.Using the fact that F has the mixed strict g -monotone property, $g(x) < g(a_n)$ and $g(b_n) < g(y)$ for all $n \geq 2$. Thus, by Lemma 1.15, we have

$$\begin{aligned} 0 < d(g(x), g(a_{n+1})) + d(g(y), g(b_{n+1})) &= d(F(x,y), F(a_n, b_n)) + d(F(y,x), F(b_n, a_n)) \\ &< d(g(x), g(a_n)) + d(g(y), g(b_n)) \end{aligned} \tag{3.9}$$

Let $\alpha_n = d(g(x), g(a_n)) + d(g(y), g(b_n))$. Then, by (3.9), it follows that $\{\alpha_n\}$ is a decreasing sequence, and hence converges to some $\alpha \geq 0$. We claim that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Then there exists a positive integer p such that, for $n \geq p$, we have

$$\epsilon \leq \frac{a_n}{2} = \frac{1}{2}[d(g(x), g(a_n)) + d(g(y), g(b_n))] < \epsilon + \delta(\epsilon), \tag{3.10}$$

where $\epsilon = \frac{\alpha}{2}$ and $\delta(\epsilon)$ is chosen by condition (b) of Theorem 2.1. In particular, for $n = p$,

$$\epsilon \leq \frac{a_p}{2} = \frac{1}{2}[d(g(x), g(a_p)) + d(g(y), g(b_p))] < \epsilon + \delta(\epsilon), \tag{3.11}$$

Then, by condition (b) of Theorem 2.1, we have

$$\frac{1}{2}[d(F(x,y), F(a_p, b_p)) + d(F(y,x), F(b_p, a_p))] < \epsilon, \tag{3.12}$$

and hence

$$\frac{1}{2}[d(g(x), g(a_{p+1})) + d(g(y), g(b_{p+1}))] < \epsilon, \tag{3.13}$$

a contradiction to (3.10) for $n = p+1$. Thus $\alpha = 0$, and hence

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}[d(g(x), g(a_n)) + d(g(y), g(b_n))] = 0 \tag{3.14}$$

Similarly, it follows that

$$\lim_{n\to\infty}[d(g(x^*), g(a_n)) + d(g(y^*), g(b_n))] = 0 \tag{3.15}$$

Using the triangle inequality, we get

$$\begin{aligned} d(g(x), g(x^*)) + d(g(y), g(y^*)) &\leq d(g(x), g(a_n)) + d(g(a_n), g(x^*)) \\ &\quad + d(g(y), g(b_n)) + d(g(b_n), g(y^*)) \\ &= [d(g(x), g(a_n)) + d(g(y), g(b_n))] \\ &\quad + [d(g(x^*), g(a_n)) + d(g(y^*), g(b_n))] \to 0 \end{aligned} \tag{3.16}$$

as $n \to \infty$.
Hence, it follows that $d(g(x), g(x^*)) = 0$ and $d(g(y), g(y^*)) = 0$. Therefore, (3.3) holds immediately. Thus, in both the cases, we have proved that (3.3) holds.
Now, since $g(x) = F(x, y), g(y) = F(y, x)$ and the pair (F, g) is compatible, by Lemma 3.1, it follows that

$$g(g(x)) = gF(x, y) = F(gx, gy) \qquad \text{and} \qquad g(g(y)) = gF(y, x) = F(gy, gx). \tag{3.17}$$

Denote $g(x) = z, g(y) = w$. Then by (3.17),

$$g(z) = F(z, w) \qquad \text{and} \qquad g(w) = F(w, z). \tag{3.18}$$

Thus (z, w) is a coupled coincidence point.
Then by (3.3) with $x* = z$ and $y* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \qquad \text{and} \qquad g(w) = w. \tag{3.19}$$

By (3.18) and (3.19),
$z = g(z) = F(z, w)$ and $w = g(w) = F(w, z)$. Therefore,(z, w) is a coupled common fixed point of F and g.
To prove uniqueness, assume that (p, q) is another coupled common fixed point of F and g. Then by (3.3) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. This completes the proof. □

Corollary 3.3. *Suppose that all the hypotheses of Corollary 2.2 hold, and further, for all $(x, y), (x^*, y^*) \in X \times X$, there exists a point $(a, b) \in X \times X$ that is comparable to (x, y) and (x^*, y^*). Then F has a unique coupled fixed point.*

4 Results of Integral Type

Inspired by the work of Suzuki [37], we prove the following result, which will be useful in developing some applications of the main results proved in Section 2.

Theorem 4.1. *Let $(X, d, \leq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. Assume that there exists a function $\theta : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:*
(I) $\theta(0) = 0$ and $\theta(t) > 0$ for any $t > 0$;
(II) θ is increasing and right continuous;
(III) for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

$$\begin{aligned} &\epsilon \leq \theta(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]) < \epsilon + \delta(\epsilon) \\ &\qquad \textit{implies} \qquad \theta(\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))]) < \epsilon \end{aligned} \tag{4.1}$$

Then F is a generalized symmetric g-Meir-Keeler type contraction.

Proof. For any $\epsilon > 0$ it follows from (I) that $\theta(\epsilon) > 0$, and so there exists an $\alpha > 0$ such that, for all $u, v, u^*, v^* \in X$ with $g(u) \leq g(u^*)$ and $g(v) \geq g(v^*)$,

$$\begin{aligned} &\theta(\epsilon) \leq \theta(\frac{1}{2}[d(g(u), g(u^*)) + d(g(v), g(^*v))]) < \theta(\epsilon) + \alpha \\ &\qquad \text{implies} \qquad \theta(\frac{1}{2}[d(F(u, v), F(u^*, v^*)) + d(F(v, u), F(v^*, u^*))]) < \theta(\epsilon) \end{aligned} \tag{4.2}$$

By the right continuity of θ , there exists $\delta > 0$ such that $\theta(\epsilon + \delta) < \theta(\epsilon) + \alpha$.
For any $x, y, u, v \in X$ such that $g(x) \leq g(u), g(y) \geq g(v)$ and

$$\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta. \tag{4.3}$$

Then, since θ is an increasing function, we get the following:

$$\theta(\epsilon) \leq \theta(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]) < \theta(\epsilon + \alpha) < \theta(\epsilon) + \alpha. \tag{4.4}$$

By (4.2), we have

$$\theta(\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))]) < \theta(\epsilon)$$

and hence,

$$\frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))]) < \epsilon.$$

Therefore, it follows that F is a generalized symmetric g-Meir-Keeler type contraction. This completes the proof. □

The following result is an immediate consequence of Theorems 2.1 and 4.1.

Corollary 4.2. *Let $(X, d, \leq)$ be a partially ordered metric space. Given $F : X \times X \to X$ and $g : X \to X$ such that $F(X \times X) \subset g(X), g(X)$ is a complete subspace and the following hypotheses hold:*
(IV) F has the mixed strict g-monotone property;
(V) for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \phi(t)dt < \epsilon + \delta(\epsilon)$$
$$\textit{implies} \qquad \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(t)dt < \epsilon \tag{4.5}$$

for all $gx \leq gu$ and $gy \geq gv$, where $\phi : [0, +\infty) \to [0, +\infty)$ is a locally integrable function satisfying $\int_0^s \phi(t)dt > 0$ for all $s > 0$;
(VI) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Assume that the hypotheses (i) and (ii) given in Theorem 2.1 hold. Then,F and g have a coupled coincidence point.

Corollary 4.3. *Let $(X, d, \leq)$ be a partially ordered metric space. Given $F : X \times X \to X$ and $g : X \to X$ such that $F(X \times X) \subset g(X), g(X)$ is a complete subspace and the following hypotheses hold:*
(VII) F has the mixed g-monotone property;
(VIII) for all $gx \leq gu$ and $gy \leq gv$

$$\int_0^{(1/2)[d(g(x),g(u))+d(g(y),g(v))]} \phi(t)dt \leq k \int_0^{(1/2)[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \phi(t)dt \tag{4.6}$$

where $k \in (0, 1)$ and ϕ is a locally integrable function from $[0, +\infty)$ into itself satisfying $\int_0^s \phi(t)dt > 0$ for all $s > 0$;
(IX) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$.
Assume that the hypotheses (i) and (ii) given in Theorem 2.1 hold. Then,F and g have a coupled coincidence point.

Proof. For each $\epsilon > 0$, take $\delta(\epsilon) = (\frac{1}{k} - 1)\epsilon$ and apply Corollary 4.2. □

5 Applications to Integral Equations

As an application of the results proved in Sections 2 and 3, we study the existence of solutions for the following system of integral equations:

$$\begin{aligned} x(t) &= \int_a^b (K_1(t,s)+K_2(t,s))(f(s,x(s))+g(s,y(s)))ds+h(t) \\ y(t) &= \int_a^b (K_1(t,s)+K_2(t,s))(f(s,y(s))+g(s,x(s)))ds+h(t) \end{aligned} \tag{5.1}$$

where $t \in I = [a,b]$.
Let $\Phi : [0,+\infty) \to [0,+\infty)$ denote the class of functions $\phi : [0,) \to [0,)$ which satisfies the following conditions:
(i) ϕ is increasing;
(ii) for each $x \geq 0$, there exists a $k \in (0,1)$ such that $\phi(x) \leq (\frac{k}{2})x$
We assume that K_1, K_2, f, g satisfy the following conditions.

Assumption 5.1. (i) $K_1(t,s) \geq 0$ and $K_2(t,s) \leq 0$ for all $t,s \in [a,b]$;
(ii) There exist $\lambda, \mu > 0$ and $\phi \in \Phi$ such that for all $x,y \in \mathbb{R}, x > y$,

$$0 < f(t,x) - f(t,y) \leq \lambda\phi(x-y) \tag{5.2}$$

and

$$-\mu\phi(x-y) \leq g(t,x) - g(t,y) < 0; \tag{5.3}$$

(iii)

$$(\lambda+\mu)\sup_{t\in I}\int_a^b (K_1(t,s)-K_2(t,s))ds \leq 1; \tag{5.4}$$

Definition 5.2. An element $(\alpha,\beta) \in X \times X$ with $X = C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equation (5.1) if for all $t \in I$,

$$\alpha(t) < \int_a^b (K_1(t,s)(f(s,\alpha(s))+g(s,\beta(s)))ds + \int_a^b K_2(t,s))(f(s,\beta(s))+g(s,\alpha(s)))ds+h(t)$$

and

$$\beta(t) \geq \int_a^b (K_1(t,s)(f(s,\beta(s))+g(s,\alpha(s)))ds + \int_a^b K_2(t,s))(f(s,\alpha(s))+g(s,\beta(s)))ds+h(t)$$

Theorem 5.3. *Consider the integral equation (5.1) with $K_1, K_2 \in C(I,\mathbb{R}), f,g \in C(I\times\mathbb{R},\mathbb{R})$ and $h \in C(I,\mathbb{R})$. Suppose that there exists a coupled lower and upper solution (α,β) of (5.1) with $\alpha \leq \beta$ and that Assumption 5.1 is satisfied. Then the integral equation (5.1) has a solution.*

Proof. Consider the natural order relation on $X = C(I,\mathbb{R})$; that is, for $x,y \in C(I,\mathbb{R})$

$$x \leq y \Rightarrow x(t) \leq y(t), \forall t \in I$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,y) = \sup_{t\in I}|x(t)-y(t)|, x,y \in C(I,\mathbb{R}).$$

Suppose that $\{u_n\}$ is a strictly increasing sequence in X that converges to a point $u \in X$. Then for every $t \in I$, the sequence of real numbers

$$u_1(t) < u_2(t) < \ldots < u_n(t) < \ldots$$

converges to $u(t)$. Therefore, for all $t \in I, n \in \mathbb{N}, u_n(t) < u(t)$. Hence, $u_n < u$ for all n. Similarly, it can be verified that, if for all $t \in I, v(t)$ is a limit of a strictly decreasing sequence

$v_n(t)$ in X, then $v(t) < v_n(t)$ for all n and hence $v < v_n$. Therefore conditions (i) and (ii) of Corollary 2.1 hold.
Also, $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set under the following order relation in $X \times X$

$$(x, y), (u, v) \in X \times X, (x, y) \leq (u, v) \Rightarrow x(t) \leq u(t) \quad \text{and} \quad y(t) \geq v(t), \forall t \in I.$$

For any $x, y \in X$, $\max\{x(t), y(t)\}$ and $\min\{x(t), y(t)\}$, for each $t \in I$, are in X and are the upper and lower bounds of x, y, respectively. Therefore, for every $(x, y), (u, v) \in X \times X$, there exists a $(\max\{x, u\}, \min\{y, v)\}) \in X \times X$ that is comparable to (x, y) and (u, v).
Define $F : X \times X \to X$ by
$F(x, y)(t) = \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y(s)))ds + \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x(s)))ds + h(t)$ for all $t \in [a, b]$. We now show that F has the mixed strict monotone property. For $x_1(t) < x_2(t)$ for all $t \in [a, b]$ we have

$$\begin{aligned}
F(x_1, y)(t) \quad &- \quad F(x_2, y)(t) = \int_a^b K_1(t, s)(f(s, x_1(s)) + g(s, y(s)))ds \\
&+ \quad \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_1(s)))ds + h(t) \\
&- \quad \int_a^b K_1(t, s)(f(s, x_2(s)) + g(s, y(s)))ds \\
&- \quad \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_2(s)))ds - h(t) \\
&= \quad \int_a^b K_1(t, s)(f(s, x_1(s)) - f(s, x_2(s)))ds \\
&+ \quad \int_a^b K_2(t, s)(g(s, x_1(s)) - g(s, x_2(s)))ds < 0
\end{aligned}$$

by Assumption 5.1. Hence $F(x_1, y)(t) < F(x_2, y)(t), \forall t \in I$; that is, $F(x_1, y) < F(x_2, y)$. Similarly, if $y_1 > y_2$, that is, $y_1(t) > y_2(t)$, for all $t \in [a, b]$, we have

$$\begin{aligned}
F(x, y_1)(t) \quad &- \quad F(x, y_2)(t) = \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_1(s)))ds \\
&+ \quad \int_a^b K_2(t, s)(f(s, y_1(s)) + g(s, x(s)))ds + h(t) \\
&- \quad \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_2(s)))ds \\
&- \quad \int_a^b K_2(t, s)(f(s, y_2(s)) + g(s, x(s)))ds - h(t) \\
&= \quad \int_a^b K_1(t, s)(g(s, y_1(s)) - g(s, y_2(s)))ds \\
&+ \quad \int_a^b K_2(t, s)(f(s, y_1(s)) - f(s, y_2(s)))ds < 0
\end{aligned}$$

by Assumption 5.1. Hence $F(x, y_1)(t) < F(x, y_2)(t), \forall t \in I$; that is, $F(x, y_1) < F(x, y_2)$. Therefore F satisfies mixed strict monotone property.
Next, we verify that F satisfies (1.3). For $x \geq u, y \leq v$, that is, $x(t) \geq u(t), y(t) \leq v(t)$ for all

$t \in I$, we have

$$\begin{aligned}
F(x,y)(t) - F(u,v)(t) &= \int_a^b K_1(t,s)(f(s,x(s)) + g(s,y_1(s)))ds \\
&+ \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds \\
&- \int_a^b K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds \\
&- \int_a^b K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds \\
&= \int_a^b K_1(t,s)(f(s,x(s)) - f(s,u(s)) - g(s,y(s)) - g(s,v(s)))]ds \\
&+ \int_a^b K_2(t,s)[(f(s,y(s)) - f(s,v(s))) - g(s,x(s)) - g(s,u(s)))ds \\
&= \int_a^b K_1(t,s)[(f(s,x(s)) - f(s,u(s)) - (g(s,v(s)) - g(s,y(s)))]ds \\
&- \int_a^b K_2(t,s)[f(s,v(s)) - f(s,y(s)) - (g(s,x(s)) - g(s,u(s))]ds \\
&\leq \int_a^b K_1(t,s)[\lambda\phi(x(s) - u(s)) + \mu\phi(v(s) - y(s))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(v(s) - y(s)) + \mu\phi(x(s) - u(s))]ds \qquad (5.5)
\end{aligned}$$

Since the function ϕ is increasing and $x \geq u$ and $y \leq v$, we have

$$\phi(x(s) - u(s)) \leq \phi(\sup_{t \in I} |x(t) - u(t)|) = \phi(d(x,u))$$

and

$$\phi(v(s) - y(s)) \leq \phi(\sup_{t \in I} |v(t) - y(t)|) = \phi(d(v,y))$$

Hence, using (5.5) and the fact that $K_2(t,s) \leq 0$, we obtain

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| &\leq \int_a^b K_1(t,s)[\lambda\phi(d(x,u)) + \mu\phi(d(v,y))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(d(v,y)) + \mu\phi(d(x,u))]ds \qquad (5.6)
\end{aligned}$$

Since all of the quantities on the right hand side of (5.5) are non-negative, inequality (5.6) is satisfied.

Similarly, we can show that

$$\begin{aligned}
|F(y,x)(t) - F(v,u)(t)| &\leq \int_a^b K_1(t,s)[\lambda\phi(d(v,y)) + \mu\phi(d(x,u))]ds \\
&- \int_a^b K_2(t,s)[\lambda\phi(d(x,u)) + \mu\phi(d(v,y))]ds \qquad (5.7)
\end{aligned}$$

Summing (5.6) and (5.7), dividing by 2 , and then taking the supremum with respect to t we

get, by using (5.4) that

$$\begin{aligned}&\frac{d(F(x,y)+F(u,v))+d(F(y,x)+F(v,u))}{2}\\&\quad\leq(\lambda+\phi)\sup_{t\in I}\int_a^b(K_1(t,s)-K_2(t,s))ds.\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\\&\quad\leq\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\end{aligned}$$

Since ϕ is increasing,

$$\phi(d(x,u))\leq\phi(d(x,u)+d(v,y)),\quad \phi(d(v,y))\leq\phi(d(x,u)+d(v,y))$$

and hence

$$\frac{\phi(d(v,y))+\phi(d(x,u))}{2}\leq\phi(d(x,u)+d(v,y))\leq(\frac{k}{2})[d(x,u)+d(v,y)]$$

by the definition of ϕ. Thus

$$\frac{d(F(x,y)+F(u,v))+d(F(y,x)+F(v,u))}{2}\leq(\frac{k}{2})[d(x,u)+d(v,y)]$$

which reduces to the symmetric contractive condition (1.3).
Then, by Proposition 1.14, F is a generalized symmetric Meir-Keeler type contraction.
Finally, let (α,β) be a coupled lower and upper solution of the integral equation (5.1), then we have $\alpha(t)<F(\alpha,\beta)(t)$ and $\beta(t)\geq F(\beta,\alpha)(t)$ for all $t\in[a,b]$, that is, $\alpha<F(\alpha,\beta)$ and $\beta\geq F(\beta,\alpha)$.
Therefore, Corollaries 2.2 and 3.2 yield that F has a unique coupled fixed point (x,y) and hence the system (5.1) has a unique solution. □

References

[1] S. Banach, Surles operations dans les ensembles et leur application aux equation sitegrales. Fund. Math. 3, 133-181 (1922).

[2] R. P. Agarwal, M. Meehan, and D. ORegan, Fixed Point Th. Appl., Camb. Univ. Press, 2001.

[3] R. P. Agarwal, M. A. El-Gebeily, and D. ORegan, Generalized contractions in partially ordered metric spaces, Applicable Anal., vol. 87, no. 1, pp. 109-116, 2008.

[4] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc., Vol. 20, pp. 458-464, 1969.

[5] A. Branciari, A xed point theorem for mappings satisfying a general contractive condition of integral type, Internat. Jour. Math. Math. Sc., Vol. 29, no. 9, pp. 531-536, 2002.

[6] L. B. Ciri'c , A generalization of Banachs contraction principle, Proc. Amer. Math. Soc., Vol. 45, pp. 267-273, 1974.

[7] J. Dugundji and A. Granas, Fixed Point Th., Springer, New York, NY, USA, 2003.

[8] D. J. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Boston, Mass, USA, 1988.

[9] A. Meir and E. Keeler, A theorem on contraction mappings, Jour. Math. Anal. Appl., Vol. 28, pp. 326-329, 1969.

[10] J. J. Nieto and R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary dierential equations, Order, Vol. 22, no. 3, pp. 223-239, 2005.

[11] B. E. Rhoades, A comparison of various denitions of contractive mappings, Trans. Amer. Math. Soc., vol. 226, pp. 257-290, 1977.

[12] D. R. Smart, Fixed Point Theorems, Camb. Univ. Press, London, UK, 1974.

[13] T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, Internat. Jour. Math. Math. Sc., Article ID 39281, 6 pages, 2007.

[14] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., Vol. 136, no. 5, pp. 1861-1869, 2008.

[15] E. Zeidler, Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems, Springer, Berlin, Germany, 1986.

[16] M. Turinici, Abstract comparison principles and multivariable Gronwall-Bellman inequalities, Jour. Math. Anal. Appl., Vol. 117, no. 1, pp. 100-127, 1986.

[17] A. C. M. Ran and M. C. B. Reurings, A xed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., Vol. 132, no. 5, pp. 1435-1443, 2004.

[18] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal.: TMA, Vol. 65, no. 7, pp. 1379-1393, 2006.

[19] V. Lakshmikantham and Lj. B. Ciri'c, Coupled xed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., Vol. 70, 4341-4349, 2009.

[20] Binayak S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal., Vol. 73, 2524-2531 (2010).

[21] A. Alotaibi, S. M. Alsulami, Coupled coincidence points for monotone operators in partially ordered metric spaces, Fixed Point Theory and Appl., 2011, 44 (2011).

[22] N. V. Luong, N. X. Thuan, Coupled xed point in partially ordered metric spaces and applications. Nonlinear Anal., Vol. 74, 983992 (2011).

[23] N. Hussain, A. Latif, M. H. Shah, Coupled and tripled coincidence point results without compatibility, Fixed Point Theory and Appl., 2012, 77(2012), doi: 10.1186/1687-1812-2012-77.

[24] R. Saadati, S.M. Vaezpour, P. Vetro, B.E. Rhoades, Fixed point theorems in generalized partially ordered G-metric spaces, Math. Comput. Modelling 52 (2010) 797-801.

[25] B. Samet, Calogero Vetro, Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Anal.: TMA, Vol. 74 (12) (2011) 4260-4268.

[26] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results Comput. Math. Appl., Vol. 60 (8) (2010) 2508-2515.

[27] W. Shatanawi, B. Samet, M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Math. Comput. Modelling, doi:10.1016/j.mcm.2011.08.042.

[28] W. Sintunavarat, Y. J. Cho, P. Kumam, Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces, Fixed Point Theory and Appl., 2011, 2011:81.

[29] W. Sintunavarat, Y. J. Cho, P. Kumam, Coupled fixed point theorems for weak contraction mapping under F-invariant set, Abstract and Appl. Anal., Vol. 2012, Article ID 324874, 15 pages, 2012.

[30] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed Point Theory and Appl., 2012, 2012:93.

[31] H. Aydi, E. Karapnar, and W. Shatanawi, Coupled xed point results for (ψ, ϕ) -weakly contractive condition in ordered partial metric spaces, Comput. Math. Appl., Vol. 62, no. 12, pp. 44494460, 2011

[32] M. E. Gordji, Y. J. Cho, S. Ghods, M. Ghods, and M. H. Dehkordi, Coupled xed point theorems for contractions in partially ordered metric spaces and applications, Mathematical Problems in Engineering, Vol. 2012, Article ID 150363, 20 pages, 2012.

[33] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., Vol. 72, no. 12, pp. 45084517, 2010.

[34] Thabet Abdeljawad, Hassen Aydi, and Erdal Karapnar, Coupled Fixed Points for Meir-Keeler Contractions in Ordered Partial Metric Spaces, Mathematical Problems in Engineering, Vol. 2012, Article ID 327273, 20 pages, doi:10.1155/2012/327273.

[35] V. Berinde, and M. Pacurar, Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces, Fixed Point Theory and Appl., 2012, 2012:115, doi:10.1186/1687-1812-2012-115.

[36] M. Jain, K. Tas, S. Kumar, N. Gupta, Coupled common fixed points involving a (ϕ, ψ) - contractive condition for mixed g-monotone operators in partially ordered metric spaces, Journal of Inequalities and Applications 2012, 2012:285 doi:10.1186/1029-242X-2012-285.

[37] T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, Internat. Jour. Math. Math. Sc., Vol. 2007, Article ID 39281, 6 pages, 2007.

Pointwise Superconvergence Patch Recovery for the Gradient of the Linear Tetrahedral Element

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We consider the finite element approximation to the solution of a self-adjoint, second-order elliptic boundary value problem in three dimensions over a fully uniform mesh of piecewise linear tetrahedral elements. First, the supercloseness of the gradients between the piecewise linear finite element solution u_h and the linear interpolation u_I is derived by using a weak estimate and an estimate of the discrete derivative Green's function. We then analyze a superconvergence patch recovery scheme for the gradient of the finite element solution, showing that the recovered gradient of u_h is superconvergent to the gradient of the true solution u.

1 Introduction

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic; see, for example, Babuška and Strouboulis [1], Chen [2], Chen and Huang [3], Lin and Yan [4], Wahlbin [5] and Zhu and Lin [6] for overviews of this field. Nevertheless, how to obtain the superconvergent numerical solution is an issue to researchers. In general, it needs to use post-processing techniques to get recovered gradients with high order accuracy from the finite element solution. Usual post-processing techniques include Interpolation technique, Projection technique, Average technique, Extrapolation technique, Superconvergence Patch Recovery (SPR) technique introduced by Zienkiewicz and Zhu [7–9] and Polynomial Patch Recovery (PPR) technique raised by Zhang and Naga [10]. In previous works, for the linear tetrahedral element, Chen and Wang [11] obtained the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm by using SPR. Using the L^2-projection technique, in the average sense of the L^2-norm, Chen [12] got the recovered gradient with $\mathcal{O}(h^{1+min(\sigma,\frac{1}{2})})$ order accuracy. Goodsell [13] derived by using the average technique the pointwise superconvergence estimate of the

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recovered gradient with $\mathcal{O}(h^{2-\varepsilon})$ order accuracy.

Unlike the results in [11–13], this article will show a pointwise superconvergence estimate with $\mathcal{O}(h^2|\ln h|^{\frac{4}{3}})$ order accuracy for the recovered gradient by using SPR. In this article, we shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

2 Model Problem and Finite Element Space

Suppose $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. We consider the self-adjoint, variable coefficients second-order elliptic problem

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Here we assume f is smooth enough, and $A = (a_{ij})$ is a 3×3 symmetric matrix function in $(L^\infty(\Omega))^{3\times 3}$ and uniformly positive definite. Set $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$. Thus, the variational formulation of (2.1) is

$$a(u\,,\, v) = (f\,,\, v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u\,,\, v) \equiv \int_\Omega \sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v \, dxdydz$$

and

$$(f\,,\, v) = \int_\Omega fv \, dxdydz.$$

To discretize the problem (2.2), one proceeds as follows. The domain Ω is firstly partitioned into cubes of side h, and each of these is then subdivided into six tetrahedra (see Fig. 1). We denote by $\mathcal{T}^h$ this partition.

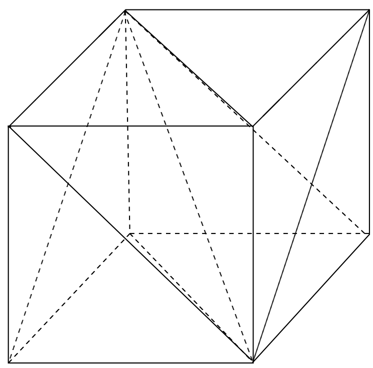

Figure 1: A tetrahedral partition

For this fully uniform mesh of tetrahedral elements, let $S_0^h(\Omega) \subset H_0^1(\Omega)$ be the piecewise linear tetrahedral finite element space, and $u_I \in S_0^h(\Omega)$ the Lagrange interpolant to the solution u of (2.2).

Discretizing (2.2) using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\,v) = (f\,,\,v)$ for all $v \in S_0^h$. Here u_h is a finite element approximation to u. Thus we have the Galerkin orthogonality relation

$$a(u - u_h\,,\,v) = 0 \;\; \forall\, v \in S_0^h(\Omega). \tag{2.3}$$

To derive the main result of this article, for every $Z \in \Omega$, we need to introduce the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h \in S_0^h(\Omega)$ defined by

$$a(v, \partial_{Z,\ell} G_Z^h) = \partial_\ell v(Z) \;\forall\, v \in S_0^h(\Omega). \tag{2.4}$$

Here, for any direction $\ell \in R^3$, $|\ell| = 1$, $\partial_{Z,\ell} G_Z^h$ and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell} G_Z^h = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},\; \partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|},\; \Delta Z = |\Delta Z|\ell.$$

Remark 1. Since $\Delta Z = |\Delta Z|\ell$, that is, ΔZ is of the same direction as ℓ. Thus, provided that the direction ℓ is given, the above limits exist. Hence, no matter what direction is given, the above definition has good meaning.

3 Gradients Recovered by SPR and Superconvergence

SPR is a gradient recovery method introduced by Zienkiewicz and Zhu. This method is now widely used in engineering practices for its robustness in a posterior error estimation and its efficiency in computer implementation.

For $v \in S_0^h(\Omega)$, we denote by R_h the SPR-recovery operator and begin by defining the point values of $R_h v$ at the element nodes. After the recovered gradient values of all nodes are obtained, we give a linear interpolation by using these values, namely SPR-recovery gradient $R_h v$. Obviously $R_h v \in S_0^h(\Omega)$.

Let us firstly assume N is a interior node of the partition $\mathcal{T}^h$, and denote by ω the element patch around N containing 24 tetrahedra. Under the local coordinate system centered N, we let Q_i be the barycenter of a tetrahedron $e_i \subset \omega$, $i = 1, 2, \cdots, 24$. SPR uses the discrete least-squares fitting to seek linear vector function $\mathbf{p} \in (P_1(\omega))^3$, such that

$$\sum_{i=1}^{24} [\mathbf{p}(Q_i) - \nabla v(Q_i)] q(Q_i) = \mathbf{0} \;\;\; \forall q \in P_1(\omega), \tag{3.1}$$

where $v \in S_0^h(\Omega)$. The existence and uniqueness of the minimizer in (3.1) can be found in [14].

We define $R_h v(N) = \mathbf{p}(\mathbf{0})$. Then the following Lemma 3.1 and Lemma 3.2 hold.

Lemma 3.1. Let ω be the element patch around an interior node N, and $u \in W^{3,\infty}(\omega)$. For $u_I \in S_0^h(\Omega)$ the interpolant to u, we have

$$|\nabla u(N) - R_h u_I(N)| \le Ch^2 \|u\|_{3,\infty,\omega}.$$

Lemma 3.2. The recovery operator R_h satisfies

$$R_h v(N) = \frac{1}{24} \sum_{i=1}^{24} \nabla v(Q_i).$$

Lemma 3.3. For $v \in S_0^h(\Omega)$, we have the weak estimate

$$|a(u - u_I, v)| \le Ch^2 \|u\|_{3,\infty,\Omega} |v|_{1,1,\Omega}.$$

Lemma 3.4. For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function defined in (2.4), we have the following estimate

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} \le C|\ln h|^{\frac{4}{3}}.$$

Remark 2. The proofs of Lemma 3.1 and Lemma 3.2 can be seen in [11], Lemma 3.3 in [13], and Lemma 3.4 in [15].

Theorem 3.1. For u_I and u_h, the linear interpolant and the linear tetrahedral finite element approximation to u, respectively. Thus we have the following supercloseness estimate

$$|u_h - u_I|_{1,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{4}{3}} \|u\|_{3,\infty,\Omega}.$$

Proof. For every $Z \in \Omega$ and any direction ℓ, from (2.3) and (2.4),

$$\partial_\ell (u_h - u_I)(Z) = a(u_h - u_I, \partial_{Z,\ell} G_Z^h) = a(u - u_I, \partial_{Z,\ell} G_Z^h).$$

Hence, using Lemma 3.3,

$$|\partial_\ell (u_h - u_I)(Z)| \le Ch^2 \|u\|_{3,\infty,\Omega} \left|\partial_{Z,\ell} G_Z^h\right|_{1,1,\Omega},$$

which combined with Lemma 3.4 completes the proof of Theorem 3.1.

Theorem 3.2. For $u_I \in S_0^h(\Omega)$ the linear interpolant to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_I|_{0,\infty,\Omega} \le Ch^2 \|u\|_{3,\infty,\Omega}.$$

Proof. Denote by $F : \hat{e} \to e$ an affine transformation. Obviously, there exists an element $e \in \mathcal{T}^h$, using the triangle inequality and the Sobolev Embedding Theorem [16], and considering Lemma 3.2, such that

$$\begin{aligned}
|\nabla u - R_h u_I|_{0,\infty,\Omega} &= |\nabla u - R_h u_I|_{0,\infty,e} \\
&\le Ch^{-1} |\nabla \hat{u} - R_h \hat{u}_I|_{0,\infty,\hat{e}} \\
&\le Ch^{-1} \left[|\nabla \hat{u}|_{0,\infty,\hat{e}} + |R_h \hat{u}_I|_{0,\infty,\hat{e}} \right] \\
&\le Ch^{-1} \left[|\nabla \hat{u}|_{0,\infty,\hat{\chi}} + |\hat{u}_I|_{1,\infty,\hat{\chi}} \right] \\
&\le Ch^{-1} \|\hat{u}\|_{3,\infty,\hat{\chi}},
\end{aligned}$$

where $\hat{\chi}$ is a small patch of elements surrounding the tetrahedron, $\hat{e}$. Due to the fact that, for $\hat{u}$ quadratic over $\hat{\chi}$,

$$\nabla\hat{u} - R_h\hat{u_I} = 0 \quad \text{in } \hat{e},$$

so, from the Bramble-Hilbert Lemma [17],

$$|\nabla u - R_h u_I|_{0,\,\infty,\,\Omega} \le Ch^{-1}\,|\hat{u}|_{3,\,\infty,\,\hat{\chi}} \le Ch^2\,|u|_{3,\,\infty,\,\Omega}\,,$$

which completes the proof of Theorem 3.2.

Theorem 3.3. For $u_h \in S_0^h(\Omega)$ the linear finite element approximation to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

Proof. Using the triangle inequality, we have

$$\begin{aligned}|\nabla u - R_h u_h|_{0,\,\infty,\,\Omega} &\le |R_h(u_h - u_I)|_{0,\,\infty,\,\Omega} \\ &\quad + |\nabla u - R_h u_I|_{0,\,\infty,\,\Omega} \\ &\le |u_h - u_I|_{1,\,\infty,\,\Omega} \\ &\quad + |\nabla u - R_h u_I|_{0,\,\infty,\,\Omega}\,,\end{aligned}$$

which combined with Theorems 3.1 and 3.2 completes the proof of Theorem 3.3.

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References

[1] I. Babuška and T. Strouboulis,*The finite element method and its reliability*, Numerical Mathematics and Scientific Computation, Oxford Science Publications, 2001.

[2] C. M. Chen, *Construction theory of superconvergence of finite elements*, Hunan Science and Technology Press, Changsha, China, 2001 (in Chinese).

[3] C. M. Chen and Y. Q. Huang, *High accuracy theory of finite element methods*, Hunan Science and Technology Press, Changsha, China, 1995 (in Chinese).

[4] Q. Lin and N. N. Yan,*Construction and analysis of high efficient finite elements*, Hebei University Press, Baoding, China, 1996 (in Chinese).

[5] L. B. Wahlbin,*Superconvergence in Galerkin finite element methods*, Springer Verlag, Berlin, 1995.

[6] Q. D. Zhu and Q. Lin,*Superconvergence theory of the finite element methods*, Hunan Science and Technology Press, Changsha, China, 1989 (in Chinese).

[7] O. C. Zienkiewicz and J. Z. Zhu, A simple estimator and adaptive procedure for practical engineering analysis, *International Journal for Numerical Methods in Engineering,* vol. 24, pp. 337–357, 1987.

[8] O. C. Zienkiewicz and J. Z. Zhu, The superconvergence patch recovery and a posteriori error estimates. Part 1: The recovery techniques, *International Journal for Numerical Methods in Engineering,* vol. 33, pp. 1331–1364, 1992.

[9] O. C. Zienkiewicz and J. Z. Zhu, The superconvergence patch recovery and a posteriori error estimates. Part 2: Error estimates and adaptivity, *International Journal for Numerical Methods in Engineering,* vol. 33, pp. 1365–1382, 1992.

[10] Z. M. Zhang and A. Naga, A new finite element gradient recovery method: Superconvergence property, *SIAM Journal on Scientic Computing,* vol. 26, pp. 1192–1213, 2005.

[11] J. Chen and D. S. Wang, Three-dimensional finite element superconvergent gradient recovery on Par6 patterns, *Numerical Mathematics: Theory, Methods and Applications,* vol. 3, no. 2, pp. 178–194, 2010.

[12] L. Chen, Superconvergence of tetrahedral linear finite elements, *International Journal of Numerical Analysis and Modeling,* vol. 3, no. 3, pp. 273–282, 2006.

[13] G. Goodsell, Pointwise superconvergence of the gradient for the linear tetrahedral element, *Numerical Methods for Partial Differential Equations,* vol. 10, pp. 651–666, 1994.

[14] B. Li and Z. M. Zhang, Analysis of a class of superconvergence patch recovery techniques for linear and bilinear Finite elements,*Numerical Methods for Partial Differential Equations,* vol. 15, pp. 151–167, 1999.

[15] J. H. Liu and Q. D. Zhu, The estimate for the $W^{1,1}$-seminorm of discrete derivative Green's function in three dimensions, *Journal of Hunan University of Arts and Sciences,* vol. 16, pp. 1–3, 2004 (in Chinese).

[16] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[17] J. H. Bramble and S. R. Hilbert, Estimation of Linear Functionals on Sobolev Spaces with Applications to Fourier Transforms and Spline Interpolation, *SIAM Journal on Numerical Analysis,* vol. 7, pp. 112, 1970.

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Hyers-Ulam stability of quadratic functional equations in paranormed spaces

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Abstract. Lin [18, 19] introduced and investigated the following quadratic functional equations

$$cf\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i-(n+c-1)x_j\right) \tag{0.1}$$
$$=(n+c-1)\left(f(x_1)+c\sum_{i=2}^{n} f(x_i)+\sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i-x_j)\right)\right),$$

$$Q\left(\sum_{i=1}^{n} d_i x_i\right)+\sum_{1\le i<j\le n} d_i d_j Q(x_i-x_j)=\left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right). \tag{0.2}$$

In this paper, we prove the Hyers-Ulam stability of the above quadratic functional equations in paranormed spaces.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [7] and Steinhaus [31] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [8, 16, 20, 21, 29]). This notion was defined in normed spaces by Kolk [17].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [33] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;

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(2) $P(-x) = P(x)$;

(3) $P(x+y) \leq P(x) + P(y)$ (triangle inequality)

(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [24] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [9] following the same approach as in Th.M. Rassias [23], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Th.M. Rassias and Šemrl [28] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [12]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [30] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [4] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 6], [13]–[15], [22], [25]–[27]).

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in paranormed spaces. In Section 3, we prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in paranormed spaces.

2. Hyers-Ulam stability of the functional equation (0.1) in paranormed spaces

For a given mapping f, we define

$$\begin{aligned} Df(x_1, x_2, \cdots, x_n) &= cf\left(\sum_{i=1}^n x_i\right) + \sum_{j=2}^n f\left(\sum_{i=1}^n x_i - (n+c-1)x_j\right) \\ &\quad -(n+c-1)\left(f(x_1) + c\sum_{i=2}^n f(x_i) + \sum_{i<j,j=3}^n \left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right). \end{aligned}$$

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x_1, \cdots, x_n) = 0$ in paranormed spaces.

Throughout this section, assume that $v := 2 - n - c$ is an integer greater than one.

Note that $P(vx) \leq vP(x)$ for all $x \in Y$.

Theorem 2.1. *Let r, θ be positive real numbers with $r > 2$. Let $f : Y \to X$ be a mapping such that*

$$P(Df(x_1, \cdots, x_n)) \leq \theta \sum_{i=1}^{n} \|x_i\|^r \tag{2.1}$$

for all $x_1, \cdots, x_n \in Y$. Then there exists a unique quadratic mapping $R : Y \to X$ such that

$$P(f(x) - R(x)) \leq \frac{\theta}{v^r - v^2} \|x\|^r \tag{2.2}$$

for all $x \in Y$.

Proof. Putting $x_2 = \frac{x}{v}$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.1), we get

$$P\left(f(x) - v^2 f\left(\frac{x}{v}\right)\right) \leq \frac{\theta \|x\|^r}{v^r}$$

for all $x \in Y$

Hence

$$P\left(v^{2l} f\left(\frac{x}{v^l}\right) - v^{2m} f\left(\frac{x}{v^m}\right)\right) \leq \sum_{j=l}^{m-1} \frac{\theta \|x\|^r}{v^{(r-2)j+r}} \tag{2.3}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in Y$. It follows from (2.3) that the sequence $\{v^{2k} f(\frac{x}{v^k})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{v^{2k} f(\frac{x}{v^k})\}$ converges. So the mapping $R : Y \to X$ can be defined as

$$R(x) := \lim_{k \to \infty} v^{2k} f\left(\frac{x}{v^k}\right)$$

for all $x \in Y$.

By (2.1),

$$\begin{aligned}
P(DR(x_1, \cdots, x_n)) &= \lim_{k\to\infty} P\left(v^{2k} DR\left(\frac{x_1}{v^k}, \cdots, \frac{x_n}{v^k}\right)\right) \leq \lim_{k\to\infty} v^{2k} P\left(DR\left(\frac{x_1}{v^k}, \cdots, \frac{x_n}{v^k}\right)\right) \\
&\leq \lim_{k\to\infty} v^{2k}\theta \left(\sum_{i=1}^{n} ||\frac{x_i}{v^k}||^r\right) = \lim_{k\to\infty} \frac{\theta\left(\sum_{i=1}^{n} ||x_i|||^r\right)}{v^{(r-2)k}} = 0
\end{aligned}$$

for all $x_1, \cdots, x_n \in Y$. So $DR(x_1, \cdots, x_n) = 0$. So the mapping $R : Y \to X$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2). So there exists a quadratic mapping $R : Y \to X$ satisfying (2.2).

Now, let $T : Y \to X$ be another quadratic mapping satisfying (2.2). Then we have

$$\begin{aligned}
P(R(x) - T(x)) &= P\left(v^{2q} R\left(\frac{x}{v^q}\right) - v^{2q} T\left(\frac{x}{v^q}\right)\right) \\
&\leq P\left(v^{2q}\left(R\left(\frac{x}{v^q}\right) - f\left(\frac{x}{v^q}\right)\right)\right) + P\left(v^{2q}\left(T\left(\frac{x}{v^q}\right) - f\left(\frac{x}{v^q}\right)\right)\right) \\
&\leq \frac{2\theta}{v^r - v^2}\|x\|^r \cdot \frac{v^{2q}}{v^{rq}},
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $R(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of R. Thus the mapping $R : Y \to X$ is the unique quadratic mapping satisfying (2.2). □

Theorem 2.2. *Let r, θ be positive real numbers with $r < 2$. Let $f : X \to Y$ be a mapping such that*

$$||Df(x_1, \cdots, x_n)|| \leq \theta \sum_{i=1}^{n} P(x_i)^r \tag{2.4}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique quadratic mapping $R : X \to Y$ such that

$$||f(x) - R(x)|| \leq \frac{\theta}{v^2 - v^r} P(x)^r \tag{2.5}$$

for all $x \in X$.

Proof. Putting $x_2 = x$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.4), we get

$$||f(vx) - v^2 f(x)|| \leq \theta P(x)^r$$

and so

$$\left\| f(x) - \frac{1}{v^2} f(vx) \right\| \leq \theta \frac{1}{v^2} P(x)^r$$

for all $x \in X$

Hence

$$\left\| \frac{1}{v^{2l}} f\left(v^l x\right) - \frac{1}{v^{2m}} f\left(v^m x\right) \right\| \leq \sum_{j=l}^{m-1} \theta \frac{1}{v^2} \frac{v^{rj} P(x)^r}{v^{2j}} \tag{2.6}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{v^{2k}} f(v^k x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{v^{2k}} f(v^k x)\}$ converges. So the mapping $R : X \to Y$ can be defined as

$$R(x) := \lim_{k \to \infty} \frac{1}{v^{2k}} f\left(v^k x\right)$$

for all $x \in X$.

By (2.4),

$$||DR(x_1, \cdots, x_n)|| = \lim_{k \to \infty} \left\| \frac{1}{v^{2k}} DR(v^k x_1, \cdots, v^k x_n) \right\| \leq \lim_{k \to \infty} \frac{1}{v^{2k}} \|DR(v^k x_1, \cdots, v^k x_n)\|$$
$$\leq \lim_{k \to \infty} \frac{v^{rk}}{v^{2k}} \theta \sum_{i=1}^{n} P(x_i)^r = 0$$

for all $x_1, \cdots, x_n \in X$. So $DR(x_1, \cdots, x_n) = 0$. So the mapping $R : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5). So there exists a quadratic mapping $R : X \to Y$ satisfying (2.5).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} ||R(x) - T(x)|| &= \left\| \frac{1}{v^{2q}} R\left(v^q x\right) - \frac{1}{v^{2q}} T\left(v^q x\right) \right\| \\ &\leq \left\| \frac{1}{v^{2q}} \left(R\left(v^q x\right) - f\left(v^q x\right)\right) \right\| + \left\| \frac{1}{v^{2q}} \left(T\left(v^q x\right) - f\left(v^q x\right)\right) \right\| \\ &\leq \frac{2\theta}{v^2 - v^r} P(x)^r \cdot \frac{v^{rq}}{v^{2q}}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $R(x) = T(x)$ for all $x \in X$. This proves the uniqueness of R. Thus the mapping $R : X \to Y$ is the unique quadratic mapping satisfying (2.5). □

3. Hyers-Ulam stability of the functional equation (0.2) in paranormed spaces

For a given mapping f, we define

$$DQ(x_1, \cdots, x_n) := Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) - \sum_{i=1}^{n} d_i \left(\sum_{i=1}^{n} d_i Q(x_i)\right).$$

In this section, we prove the Hyers-Ulam stability of the functional equation $DQ(x_1, \cdots, x_n) = 0$ in paranormed spaces.

Throughout this section, assume that $d := \sum_{j=1}^{n} d_j$ is an integer greater than one.

Note that $P(dx) \le dP(x)$ for all $x \in Y$.

Theorem 3.1. *Let r, θ be positive real numbers with $r > 2$. Let $Q : Y \to X$ be a mapping such that*

$$P(DQ(x_1, \cdots, x_n)) \le \theta \sum_{i=1}^{n} \|x_i\|^r \tag{3.1}$$

for all $x_1, \cdots, x_n \in Y$. Then there exists a unique quadratic mapping $R : Y \to X$ such that

$$P(Q(x) - R(x)) \le \frac{n\theta}{d^r - d^2}\|x\|^r \tag{3.2}$$

for all $x \in Y$.

Proof. Putting $x_1 = \cdots = x_n = \frac{x}{d}$ in (3.1), we get

$$P\left(Q(x) - d^2 Q\left(\frac{x}{d}\right)\right) \le \frac{\theta(n\|x\|^r)}{d^r}$$

for all $x \in X$

Hence

$$P\left(d^{2l} Q\left(\frac{x}{d^l}\right) - d^{2m} Q\left(\frac{x}{d^m}\right)\right) \le \sum_{j=l}^{m-1} \frac{\theta(n\|x\|^r)}{d^{(r-2)j+r}} \tag{3.3}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in Y$. It follows from (3.3) that the sequence $\{d^{2k}Q(\frac{x}{d^k})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{d^{2k}Q(\frac{x}{d^k})\}$ converges. So the mapping $R : Y \to X$ can be defined as

$$R(x) := \lim_{k\to\infty} d^{2k} Q\left(\frac{x}{d^k}\right)$$

for all $x \in Y$.

By (3.1),

$$\begin{aligned} P(DR(x_1, \cdots, x_n)) &= \lim_{k\to\infty} P\left(d^{2k} DR\left(\frac{x_1}{d^k}, \cdots, \frac{x_n}{d^k}\right)\right) \le \lim_{k\to\infty} d^{2k} P\left(DR\left(\frac{x_1}{d^k}, \cdots, \frac{x_n}{d^k}\right)\right) \\ &\le \lim_{k\to\infty} d^{2k}\theta\left(\sum_{i=1}^{n}\left\|\frac{x_i}{d^k}\right\|^r\right) = \lim_{k\to\infty} \frac{\theta\left(\sum_{i=1}^{n} ||x_i|||^r\right)}{d^{(r-2)k}} = 0 \end{aligned}$$

for all $x_1, \cdots, x_n \in Y$. So $DR(x_1, \cdots, x_n) = 0$. So the mapping $R : Y \to X$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2). So there exists a quadratic mapping $R : Y \to X$ satisfying (3.2).

Now, let $T : Y \to X$ be another quadratic mapping satisfying (3.2). Then we have

$$\begin{aligned} P(R(x) - T(x)) &= P\left(d^{2q} R\left(\frac{x}{d^q}\right) - d^{2q} T\left(\frac{x}{d^q}\right)\right) \\ &\le P\left(d^{2q}\left(R\left(\frac{x}{d^q}\right) - Qf\left(\frac{x}{d^q}\right)\right)\right) + P\left(d^{2q}\left(T\left(\frac{x}{d^q}\right) - Q\left(\frac{x}{d^q}\right)\right)\right) \\ &\le \frac{2n\theta}{d^r - d^2}\|x\|^r \cdot \frac{d^{2q}}{d^{rq}}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $R(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of R. Thus the mapping $R : Y \to X$ is the unique quadratic mapping satisfying (3.2). □

Theorem 3.2. *Let r, θ be positive real numbers with $r < 2$. Let $Q : X \to Y$ be a mapping such that*

$$||DQ(x_1, \cdots, x_n)|| \leq \theta \sum_{i=1}^{n} P(x_i)^r \tag{3.4}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique quadratic mapping $R : X \to Y$ such that

$$||Q(x) - R(x)|| \leq \frac{n\theta}{d^2 - d^r} P(x)^r$$

for all $x \in X$.

Proof. Putting $x_1 = \cdots = x_n = x$ in (3.4), we get

$$||Q(dx) - d^2 Q(x)|| \leq n\theta P(x)^r$$

and so

$$\left\| Q(x) - \frac{1}{d^2} Q(dx) \right\| \leq \frac{n\theta}{d^2} P(x)^r$$

for all $x \in X$

Hence

$$\left\| \frac{1}{d^{2l}} Q\left(d^l x\right) - \frac{1}{d^{2m}} Q\left(d^m x\right) \right\| \leq \frac{n\theta}{d^2} \sum_{j=l}^{m-1} \frac{d^{rj}}{d^{2j}} P(x)^r \tag{3.5}$$

holds for all non-negative integers l and m with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{d^{2k}} Q(d^k x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{d^{2k}} Q(d^k x)\}$ converges. So the mapping $R : X \to Y$ can be defined as

$$R(x) := \lim_{k \to \infty} \frac{1}{d^{2k}} Q\left(d^k x\right)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

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References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] P.W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[5] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.

[6] M. Eshaghi Gordji and M.B. Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces,* Appl. Math. Letters **23** (2010), 1198–1202.

[7] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[8] J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[9] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[10] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[11] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[13] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of ψ-additive mappings,* J. Approx. Theory **72** (1993), 131–137.

[14] K. Jun and Y. Lee, *A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations*, J. Math. Anal. Appl. **297** (2004), 70–86.

[15] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.

[16] S. Karakus, *Statistical convergence on probabilistic normed spaces*, Math. Commun. **12** (2007), 11–23.

[17] E. Kolk, *The statistical convergence in Banach spaces*, Tartu Ul. Toime. **928** (1991), 41–52.

[18] C.S. Lin, *A-bilinear forms and generalized A-quadratic forms on unitary left A-modules*, Bull. Austral. Math. Soc. **39** (1989), 49–53.

[19] C.S. Lin, *Sesquilinear and quadratic forms on modules over $*$-algebra*, Publ. de L'Inst. Math. **51** (1992), 81–86.

[20] M. Mursaleen, *λ-statistical convergence*, Math. Slovaca **50** (2000), 111–115.

[21] M. Mursaleen and S.A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Computat. Anal. Math. **233** (2009), 142–149.

[22] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[23] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[24] Th.M. Rassias, *Problem 16; 2*, Report of the 27^{th} International Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.

[25] Th.M. Rassias (ed.), *Functional Equations and Inequalities,* Kluwer Academic, Dordrecht, 2000.

[26] Th.M. Rassias, *On the stability of functional equations in Banach spaces,* J. Math. Anal. Appl. **251** (2000), 264–284.

[27] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam,* Acta Math. Appl. **62** (2000), 23–130.

[28] Th.M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[29] T. Šalát, *On the statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[30] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[31] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–34.

[32] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[33] A. Wilansky, *Modern Methods in Topological Vector Space*, McGraw-Hill International Book Co., New York, 1978.

Union soft sets applied to commutative BCI-ideals

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union soft commutative BCI-ideals is introduced, and related properties are investigated. Relations between a union soft commutative BCI-ideal and a (closed) union soft BCI-ideal are displayed. Conditions for a union soft BCI-ideal to be a union soft commutative BCI-ideal are established. Characterizations of a union soft commutative BCI-ideal are considered, and a new union soft commutative BCI-ideal from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [26]. In response to this situation Zadeh [27] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [28]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [23]. Maji et al. [19] and Molodtsov [23] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [23] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory

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and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [19] described the application of soft set theory to a decision making problem. Maji et al. [18] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun and Park [17] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [14, 15] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras. Also, Jun [10] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 5, 6, 7, 9, 11, 12, 13, 16, 24, 25, 29] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the union soft sets in a commutative BCI-ideals of BCI-algebras. We introduce the notion of union soft commutative BCI-ideals, and investigated related properties. We consider relations between a union soft commutative BCI-ideal and a (closed) union soft BCI-ideal. We provide conditions for a union soft BCI-ideal to be a union soft commutative BCI-ideal, and establish characterizations of a union soft commutative BCI-ideal. We construct a new union soft commutative BCI-ideal from an old one.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(b1) $(\forall x, y \in X)\,(x * (x * (x * y)) = x * y)$,
(b2) $(\forall x, y \in X)\,(0 * (x * y) = (0 * x) * (0 * y))$.

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A BCI-algebra X is said to be *commutative* (see [22]) if

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; x = y * (y * x)). \tag{2.1}$$

Proposition 2.1. *A BCI-algebra X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * (x * (x * y)))). \tag{2.2}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCI-algebra X is called a *BCI-ideal* of X if it satisfies:

$$0 \in I, \tag{2.3}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I). \tag{2.4}$$

A BCI-ideal I of a BCI-algebra X satisfies:

$$(\forall x \in X)(\forall y \in I)\,(x \leq y \;\Rightarrow\; x \in I) \tag{2.5}$$

A BCI-ideal I of a BCI-algebra X is said to be *closed* if it satisfies:

$$(\forall x \in X)\,(x \in I \;\Rightarrow\; 0 * x \in I)$$

A subset I of a BCI-algebra X is called a *commutative BCI-ideal* (briefly, *c-BCI-ideal*) of X (see [20]) if it satisfies (2.3) and

$$(x * y) * z \in I,\; z \in I \;\Rightarrow\; x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I \tag{2.6}$$

for all $x, y, z \in X$.

Proposition 2.2 ([20])**.** *A BCI-ideal I of a BCI-algebra X is commutative if and only if $x * y \in I$ implies $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I$.*

Proposition 2.3 ([20])**.** *Let I be a closed BCI-ideal of a BCI-algebra X. Then I is commutative if and only if it satisfies:*

$$(\forall x, y \in X)\,(x * y \in I \;\Rightarrow\; x * (y * (y * x)) \in I)$$

Observe that every c-BCI-ideal is a BCI-ideal, but the converse is not true (see [20]).

We refer the reader to the books [8, 21] for further information regarding BCK/BCI-algebras.

A soft set theory is introduced by Molodtsov [23], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.4 ([4, 23])**.** A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\; f_A(x) \in \mathscr{P}(U)\},$$

where $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Let $\mathscr{F}_A \in S(U)$ and let $\tau \subseteq U$. Then the *τ-exclusive set* of $\mathscr{F}_A$ is defined to be the set

$$e(\mathscr{F}_A;\tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

(1) $e(\mathscr{F}_A;U) = A$.
(2) $f_A(x) = \cap\{\tau \subseteq U \mid x \in e(\mathscr{F}_A;\tau)\}$.
(3) $(\forall \tau_1,\tau_2 \subseteq U)\,(\tau_1 \subseteq \tau_2 \;\Rightarrow\; e(\mathscr{F}_A;\tau_1) \subseteq e(\mathscr{F}_A;\tau_2))$.

3. Union soft c-BCI-ideals

Definition 3.1 ([10]). Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft BCI-ideal* (briefly, *U-soft BCI-ideal*) over U if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{3.1}$$

$$(\forall x,y \in A)\,(f_A(x) \subseteq f_A(x*y) \cup f_A(y)). \tag{3.2}$$

Definition 3.2. Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft commutative BCI-ideal* (briefly, *U-soft c-BCI-ideal*) over U if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$f_A(x*((y*(y*x))*(0*(0*(x*y))))) \subseteq f_A((x*y)*z) \cup f_A(z) \tag{3.3}$$

for all $x,y,z \in A$.

Example 3.3. Let $(U,E) = (U,X)$ where $X = \{0,a,1,2,3\}$ is a BCI-algebra with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0,\tau_1),(a,\tau_2),(1,\tau_3),(2,\tau_3),(3,\tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft c-BCI-ideal over U.

Theorem 3.4. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Then every U-soft c-BCI-ideal is a U-soft BCI-ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft c-BCI-ideal over U where A is a subalgebra of E. Taking $y = 0$ in (3.3) and using (a1) and (III) imply that

$$\begin{aligned} f_A(x) &= f_A(x * 0) = f_A(x * ((0 * (0 * x)) * (0 * (0 * (x * 0))))) \\ &\subseteq f_A((x * 0) * z) \cup f_A(z) = f_A(x * z) \cup f_A(z) \end{aligned}$$

for all $x, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft BCI-ideal over U. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (3, \tau_3), (4, \tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft BCI-ideal over U. But it is not a U-soft c-BCI-ideal over U since

$$f_E(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = \tau_3 \nsubseteq \tau_1 = f_E((2 * 3) * 0) \cup f_E(0).$$

We provide conditions for a U-soft BCI-ideal to be a U-soft c-BCI-ideal.

Theorem 3.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *$\mathscr{F}_A$ is a U-soft BCI-ideal over U and its approximate function f_A satisfies:*

$$(\forall x, y \in A)\,(f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq f_A(x * y)). \tag{3.4}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Then $\mathscr{F}_A$ is a U-soft BCI-ideal over U (see Theorem 3.4). If we take $z = 0$ in (3.3) and use (a1) and (3.1), then we have (3.4).

Conversely, let $\mathscr{F}_A$ be a U-soft BCI-ideal over U such that its approximate function f_A satisfies (3.4). Then $f_A(x * y) \subseteq f_A((x * y) * z) \cup f_A(z)$ for all $x, y, z \in A$ by (3.2), which implies from (3.4) that

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq f_A((x * y) * z) \cup f_A(z)$$

for all $x, y, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Definition 3.7 ([10])**.** Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. A U-soft BCI-ideal $\mathscr{F}_A$ over U is said to be *closed* if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0 * x) \subseteq f_A(x))\,. \tag{3.5}$$

Lemma 3.8 ([10])**.** *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$.*

(1) *If $\mathscr{F}_A$ is a U-soft BCI-ideal over U, then the approximate function f_A satisfies the following condition:*

$$(\forall x, y, z \in A)\,(x * y \leq z \;\Rightarrow\; f_A(x) \subseteq f_A(y) \cup f_A(z))\,. \tag{3.6}$$

(2) *If the approximate function f_A of $\mathscr{F}_A$ satisfies* (3.1) *and* (3.6)*, then $\mathscr{F}_A$ is a U-soft BCI-ideal over U.*

Theorem 3.9. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A$ be a closed U-soft BCI-ideal over U. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *The approximate function f_A of $\mathscr{F}_A$ satisfies:*

$$(\forall x, y \in A)\,(f_A(x * (y * (y * x))) \subseteq f_A(x * y))\,. \tag{3.7}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Note that

$$\begin{aligned}
&(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y)))))\\
&\quad \leq ((y * (y * x)) * (0 * (0 * (x * y)))) * (y * (y * x))\\
&\quad = ((y * (y * x)) * (y * (y * x))) * (0 * (0 * (x * y)))\\
&\quad = 0 * (0 * (0 * (x * y))) = 0 * (x * y)
\end{aligned}$$

for all $x, y \in A$. Using Lemma 3.8(1), (3.4) and (3.5), we have

$$\begin{aligned}
&f_A(x * (y * (y * x)))\\
&\quad \subseteq f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \cup f_A(0 * (x * y))\\
&\quad \subseteq f_A(x * y) \cup f_A(0 * (x * y)) = f_A(x * y)
\end{aligned}$$

for all $x, y \in A$. Now suppose that the approximate function f_A of $\mathscr{F}_A$ satisfies (3.7). Since

$$\begin{aligned}
&(x * ((y * (y * x)) * (0 * (0 * (x * y))))) * (x * (y * (y * x)))\\
&\quad \leq (y * (y * x)) * ((y * (y * x)) * (0 * (0 * (x * y))))\\
&\quad \leq 0 * (0 * (x * y)),
\end{aligned}$$

it follows from Lemma 3.8(1), (3.5) and (3.7) that

$$\begin{aligned} & f_A(x*((y*(y*x))*(0*(0*(x*y))))) \\ & \quad \subseteq f_A(x*(y*(y*x))) \cup f_A(0*(0*(x*y))) \\ & \quad \subseteq f_A(x*y) \cup f_A(0*(0*(x*y))) = f_A(x*y) \end{aligned}$$

for all $x, y \in A$. By Theorem 3.6, $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Theorem 3.10. *Let $(U,E) = (U,X)$ where X is a commutative BCI-algebra. Then every closed U-soft BCI-ideal is a U-soft c-BCI-ideal.*

Proof. Let $\mathscr{F}_A$ be a closed U-soft BCI-ideal over U where A is a subalgebra of E. Using (a3), (b1), (I), (III) and Proposition 2.1, we have

$$\begin{aligned} (x*(y*(y*x)))*(x*y) &= (x*(x*y))*(y*(y*x)) \\ &= (y*(y*(x*(x*y))))*(y*(y*x)) \\ &= (y*(y*(y*x)))*(y*(x*(x*y))) \\ &= (y*x)*(y*(x*(x*y))) \\ &\leq (x*(x*y))*x = 0*(x*y) \end{aligned}$$

It follows from Lemma 3.8(1) and (3.5) that

$$f_A(x*(y*(y*x))) \subseteq f_A(x*y) \cup f_A(0*(x*y)) = f_A(x*y),$$

for all $x, y \in A$. Therefore, by Theorem 3.9, $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

Using the notion of τ-exclusive sets, we consider a characterization of a U-soft c-BCI-ideal.

Lemma 3.11 ([10])**.** *Let $(U,E) = (U,X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent.*

(1) *$\mathscr{F}_A$ is a U-soft BCI-ideal over U.*
(2) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a BCI-ideal of A for any $\tau \subseteq U$.*

Theorem 3.12. *Let $(U,E) = (U,X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent.*

(1) *$\mathscr{F}_A$ is a U-soft c-BCI-ideal over U.*
(2) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a c-BCI-ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. Then $\mathscr{F}_A$ is a U-soft BCI-ideal over U by Theorem 3.4. Hence $e(\mathscr{F}_A;\tau)$ is a BCI-ideal of A for all $\tau \subseteq U$ by Lemma 3.11. Let $\tau \subseteq U$ and $x, y \in A$ be such that $x*y \in e(\mathscr{F}_A;\tau)$. Then $f_A(x*y) \subseteq \tau$, and so

$$f_A(x*((y*(y*x))*(0*(0*(x*y))))) \subseteq f_A(x*y) \subseteq \tau$$

by Theorem 3.6. Thus

$$x*((y*(y*x))*(0*(0*(x*y)))) \in e(\mathscr{F}_A;\tau).$$

It follows from Proposition 2.2 that $e(\mathscr{F}_A;\tau)$ is a c-BCI-ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a c-BCI-ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is a BCI-ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft BCI-ideal over U by Lemma 3.11. Let $x, y \in A$ be such that $f_A(x*y) = \tau$. Then $x*y \in e(\mathscr{F}_A;\tau)$, and so

$$x*((y*(y*x))*(0*(0*(x*y)))) \in e(\mathscr{F}_A;\tau)$$

by Proposition 2.2. Hence

$$f_A\left(x*((y*(y*x))*(0*(0*(x*y))))\right) \subseteq \tau = f_A(x*y).$$

It follows from Theorem 3.6 that $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U. □

The c-BCI-ideals $e(\mathscr{F}_A;\tau)$ in Theorem 3.12 are called the *exclusive c-BCI-ideals* of $\mathscr{F}_A$.

Theorem 3.13. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Let $\mathscr{F}_E, \mathscr{G}_E \in S(U)$ such that*

(i) $(\forall x \in E)\,(f_E(x) \subseteq g_E(x))$,

(ii) *$\mathscr{F}_E$ and $\mathscr{G}_E$ are U-soft BCI-ideals over U.*

If $\mathscr{F}_E$ is closed and $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U, then $\mathscr{F}_E$ is also a U-soft c-BCI-ideal over U.

Proof. Assume that $\mathscr{F}_E$ is closed and $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U. Let τ be a subset of U such that $e(\mathscr{F}_E;\tau) \neq \emptyset \neq e(\mathscr{G}_E;\tau)$. Then $e(\mathscr{F}_E;\tau)$ and $e(\mathscr{G}_E;\tau)$ are BCI-ideals of E and obviously $e(\mathscr{F}_E;\tau) \supseteq e(\mathscr{G}_E;\tau)$. Let $x \in e(\mathscr{F}_E;\tau)$. Then $f_E(x) \subseteq \tau$, and so $f_E(0*x) \subseteq f_E(x) \subseteq \tau$ since $\mathscr{F}_E$ is closed. Thus $0*x \in e(\mathscr{F}_E;\tau)$, and thus $e(\mathscr{F}_E;\tau)$ is a closed BCI-ideal of E. Since $\mathscr{G}_E$ is a U-soft c-BCI-ideal over U, it follows from Theorem 3.12 that $e(\mathscr{G}_E;\tau)$ is a c-BCI-ideal of E. Let $x, y \in E$ be such that $x*y \in e(\mathscr{F}_E;\tau)$. Then $0*(x*y) \in e(\mathscr{F}_E;\tau)$. Since $(x*(x*y))*y = 0 \in e(\mathscr{G}_E;\tau)$, it follows from Proposition 2.2 that

$$\begin{aligned}
&(x*(x*y))*(y*(y*(x*(x*y))))\\
&\quad = (x*(x*y))*((y*(y*(x*(x*y))))*(0*(0*((x*(x*y))*y))))\\
&\quad \in e(\mathscr{G}_E;\tau) \subseteq e(\mathscr{F}_E;\tau)
\end{aligned}$$

so from (a3) that

$$(x*(y*(y*(x*(x*y)))))*(x*y) \in e(\mathscr{F}_E;\tau).$$

Hence $x*(y*(y*(x*(x*y)))) \in e(\mathscr{F}_E;\tau)$ by (2.4). Note that

$$\begin{aligned}
&(x*(y*(y*x)))*(x*(y*(y*(x*(x*y)))))\\
&\quad \leq (y*(y*(x*(x*y))))*(y*(y*x))\\
&\quad \leq (y*x)*(y*(x*(x*y)))\\
&\quad \leq (x*(x*y))*x = 0*(x*y) \in e(\mathscr{F}_E;\tau).
\end{aligned}$$

Using (2.5) and (2.4), we have $x*(y*(y*x)) \in e(\mathscr{F}_E;\tau)$. Hence $e(\mathscr{F}_E;\tau)$ is a c-BCI-ideal of E by Proposition 2.3. Therefore $\mathscr{F}_E$ is a U-soft c-BCI-ideal over U by Theorem 3.12. □

Theorem 3.14. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, then $e(\mathscr{F}_A; \tau)$ is a c-BCI-ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A; \tau)$, and so $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $(x * y) * z \in e(\mathscr{F}_A; \tau)$ and $z \in e(\mathscr{F}_A; \tau)$, then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in e(\mathscr{F}_A; \tau)$ and so

$$\begin{aligned} & f_A^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & = f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & \subseteq f_A((x * y) * z) \cup f_A(z) = f_A^*((x * y) * z) \cup f_A^*(z). \end{aligned}$$

If $(x * y) * z \notin e(\mathscr{F}_A; \tau)$ or $z \notin e(\mathscr{F}_A; \tau)$, then $f_A^*((x * y) * z) = U$ or $f_A^*(z) = U$. Hence

$$f_A^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq U = f_A^*((x * y) * z) \cup f_A^*(z).$$

This shows that $\mathscr{F}_A^*$ is a U-soft c-BCI-ideal over U. □

Theorem 3.15. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any c-BCI-ideal of E can be realized as an exclusive c-BCI-ideal of some U-soft c-BCI-ideal over U.*

Proof. Let A be a c-BCI-ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * y) * z \in A$ and $z \in A$ then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in A$. Hence

$$f_A((x * y) * z) \cup f_A(z) = \tau = f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))).$$

If $(x * y) * z \notin A$ or $z \notin A$ then $f_A((x * y) * z) = U$ or $f_A(z) = U$. It follows that

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \subseteq U = f_A((x * y) * z) \cup f_A(z).$$

Therefore $\mathscr{F}_A$ is a U-soft c-BCI-ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

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References

[1] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings,* Comput. Math. Appl. **59**(2010) 3458-3463.

[2] H. Aktaş and N. Çağman, *Soft sets and soft groups,* Inform. Sci. **177**(2007) 2726-2735.

[3] A. O. Atagün and A. Sezgin, *Soft substructures of rings, fields and modules,* Comput. Math. Appl. **61** (2011) 592-601.

[4] N. Çağman, F. Çitak and S. Enginoğlu, *Soft set theory and uni-int decision making,* Eur. J. Oper. Res. **207** (2010) 848-855.

[5] N. Çağman and S. Enginoğlu, *FP-soft set theory and its applications,* Ann. Fuzzy Math. Inform. **2** (2011) 219-226.

[6] F. Feng, *Soft rough sets applied to multicriteria group decision making,* Ann. Fuzzy Math. Inform. **2** (2011) 69-80.

[7] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings,* Comput. Math. Appl. **56** (2008) 2621-2628.

[8] Y. Huang, *BCI-algebra,* Science Press, Beijing 2006.

[9] Y. B. Jun, *Soft BCK/BCI-algebras,* Comput. Math. Appl. **56** (2008) 1408-1413.

[10] Y. B. Jun, *Union soft sets with applications in BCK/BCI-algebras,* Inform. Sci. (submitted).

[11] Y. B. Jun, H. S. Kim and J. Neggers, *Pseudo d-algebras,* Inform. Sci. **179** (2009) 1751-1759.

[12] Y. B. Jun, K. J. Lee and A. Khan, *Soft ordered semigroups,* Math. Logic Q. **56** (2010) 42-50.

[13] Y. B. Jun, K. J. Lee and C. H. Park, *Soft set theory applied to ideals in d-algebras,* Comput. Math. Appl. **57** (2009) 367-378.

[14] Y. B. Jun, M¿ S. Kang and K. J. Lee, Intersectional soft sets and applications to BCK/BCI-algebras, Commun. Koean Math. Soc. **28** (2013) 11-24.

[15] Y. B. Jun, K. J. Lee and E. H. Roh, Intersectional soft BCK/BCI-ideals, Ann. Fuzzy Math. Inform. **4** (2012) 1-7.

[16] Y. B. Jun, K. J. Lee and J. Zhan, Soft p-ideals of soft BCI-algebras, Comput. Math. Appl. **58** (2009) 2060-2068.

[17] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. **178** (2008) 2466-2475.

[18] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. **45** (2003) 555-562.

[19] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. **44** (2002) 1077-1083.

[20] J. Meng, An ideal characterization of commutative BCI-algebras, Pusan Kyongnam Math. J. **9** (1993) 1-6.

[21] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co. Seoul 1994.

[22] J. Meng and X. L. Xin, Commutative BCI-algebras, Math. Japon. **37** (1992) 569-572.

[23] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. **37** (1999) 19-31.

[24] C. H. Park, Y. B. Jun and M. A. Öztürk, Soft WS-algebras, Commun. Korean Math. Soc. **23** (2008) 313-324.

[25] S. Z. Song, K. J. Lee and Y. B. Jun, Intersectional soft sets applied to subalgebras/ideals in BCK/BCI-algebras, Knowledge-Based Systems (submitted).

[26] L. A. Zadeh, From circuit theory to system theory, Proc. Inst. Radio Eng. **50** (1962) 856-865.

[27] L. A. Zadeh, Fuzzy sets, Inform. Control **8** (1965) 338-353.

[28] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) - an outline, Inform. Sci. **172** (2005) 1-40.

[29] J. Zhan and Y. B. Jun, Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. **59** (2010) 2037-2046.

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SOME INEQUALITIES WHICH HOLD FOR STARLIKE LOG-HARMONIC MAPPINGS OF ORDER α

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Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z| \quad |z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D. On the other hand, if f vanishes at $z = 0$ but it is not identically zero then f admits following representation

$$f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$$

where $Re\beta > -\frac{1}{2}$, h and g are analytic in D, $g(0) = 1$, $h(0) \neq 0$. Let $f = z\,|z|^{2\beta}\,h\overline{g}$ be a univalent log-harmonic mapping.

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We say that f is a starlike log-harmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha\ ,\ 0 \leq \alpha < 1.\ (\forall z \in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α.

The aim of this paper is to define some inequalities of starlike log-harmonic functions of order α $(0 \leq \alpha \leq 1)$.

I. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the unit disc D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in D$.

Next, denote by $P(\alpha)(0 \leq \alpha < 1)$ the family of functions

$$p(z) = 1 + p_1 z + ...$$

regular in D and such that $p(z)$ in $P(\alpha)$ if and only if

$$p(z) = \frac{1 + (1 - 2\alpha)\ \phi(z)}{1 - \phi(z)}$$

for some functions $z \in \Omega$ and every $z \in D$.

Let $S_1(z)$ and $S_2(z)$ be analytic functions in the open unit disc, with $S_1(0) = S_2(0)$, if $S_1(z) = S_2(\phi(z))$ then we say that $S_1(z)$ is subordinate to $S_2(z)$, where $\phi(z) \in \Omega([4])$, and we write $S_1(z) \prec S_2(z)$.

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z|\,|z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$.

It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D.

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On the other hand, if f vanishes at $z=0$ but it is not identically zero then f admits following representation

$$f(z)=z\left|z\right|^{2\beta}h(z)\overline{g(z)}$$

where $Re\beta>-\frac{1}{2}$, h and g are analytic in D, $g(0)=1$, $h(0)\neq 0$.

Let $f=z\left|z\right|^{2\beta}h\overline{g}$ be a univalent log-harmonic mapping. We say that f is a starlike logharmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta}=Re\frac{zf_z-\overline{z}f_{\overline{z}}}{f}>\alpha\ ,\ 0\leq\alpha<1.\ \ (\forall z\in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α([3]).

If $\alpha=0$, we get the class of starlike log-harmonic mappings. Also, let

$$ST(\alpha)=\left\{f\in S^*_{lh}(\alpha)\,and\,f\in H(U)\right\}.$$

If $f\in S^*_{lh}(0)$ then $F(\varsigma)=\log(f(e^{\varsigma}))$ is univalent and harmonic on the half plane $\{\varsigma\mid\ \ Re\{\varsigma\}<0\}$. It is known that F is closely related with the theory of nonparametric minimal surfaces over domains of the form $-\infty<u<u_0(v)$, $u_0(v+2\pi)=u_0(v)$, see ([1],[2]).

In this paper, we obtain Marx-Strohhacker Inequality and new distortion theorems using the subordination prinsiple for the starlike log-harmonic mappings of order α, previously studied by Z. Abdulhadi and Y. Abu Muhanna [3] who obtained the representation theorem and a different distortion theorem for the same class.

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II. Main Results

Theorem 2.1. Let $f(z) = zh(z)\overline{g(z)}$ be an analytic logaritmic harmonic function in the open unit disc U. If $f(z)$ satisfies the condition

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z}{1-z} = F(z) \tag{1}$$

then $f \in S^*_{lh}(\alpha)$.

Proof. We define the function by

$$\frac{h}{g} = (1-\phi(z))^{-2(1-\alpha)} \tag{2}$$

where $(1-\phi(z))^{-2(1-\alpha)}$ has the value 1 at $z = 0$. Then $w(z)$ is analytic and $\phi(0) = 0$. If we take the logarithmic derivative of (2) and the after brief calculations, we get

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z\phi'(z)}{1-\phi(z)}$$

Now it is easy to realize that the subordination (1) is equivalent to $|\phi(z)| < 1$ for all $z \in U$. Indeed assume the contrary: then there is a $z_1 \in U$ such that $|\phi(z_1)| = 1$, so by I.S. Jack Lemma $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$ and for such $z_1 \in U$, we have

$$z_1\frac{h'(z_1)}{h(z_1)} - z_1\frac{g'(z_1)}{g(z_1)} \prec \frac{2(1-\alpha)k\phi(z_1)}{1-\phi(z_1)} = F(\phi(z_1)) \notin F(U)$$

but this contradicts (1); so our assumption is wrong, i.e, $|\phi(z)| < 1$ for all $z \in u$. By using condition (1) we get

$$1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)} \quad . \tag{3}$$

The equality (3) shows that $f(z) \in S^*_{lh}(\alpha)$.

Corollary 2.2. For the starlike logharmonic functions of order α, we have Marx-Strohhacker Inequality is

$$\left|1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|<1$$

g and $h$are analytic in u and $0 \notin hg(u)$.

Proof. Using theorem 2.1 we have

$(1-\phi(z))^{\frac{1-2\alpha+1}{-1}}=\frac{h}{g} \Rightarrow (1-\phi(z))^{-2(1-\alpha)}=\frac{h}{g} \Rightarrow \frac{1}{(1-\phi(z))^{2(1-\alpha)}}=\frac{h}{g} \Rightarrow \frac{1}{1-\phi(z)}=\left(\frac{h}{g}\right)^{\frac{1}{2(1-\alpha)}} \Rightarrow$
$1-\phi(z)=\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}} \Rightarrow 1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}=\phi(z) \Rightarrow \left|1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|=|\phi(z)|<1.$

Theorem 2.3. If $f \in S^*_{lh}(\alpha)$ then

$$\frac{1}{(1+r)^{2(1-\alpha)}} \le \left|\frac{h}{g}\right| < \frac{1}{(1-r)^{2(1-\alpha)}} \tag{4}$$

Proof. The set of the values of the function $\left(\frac{2(1-\alpha)z}{(1-z)}\right)$ is the closed disc with the centre C and the radius ρ, where

$$C=C(r)=\left(\frac{2(1-\alpha)r^2}{1-r^2},0\right) \quad , \quad \rho=\rho(r)=\frac{2(1-\alpha)r}{1-r^2}.$$

Using the subordination, we can write

$$\left|\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)-\frac{2(1-\alpha)r^2}{1-r^2}\right| \le \frac{2(1-\alpha)r}{1-r^2}. \tag{5}$$

Therefore we have

$$-\frac{2(1-\alpha)r}{1+r} \le Re\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right) \le \frac{2(1-\alpha)r}{1-r}. \tag{6}$$

On the other hand

$$Re\left(z\frac{h'}{h}\right)-Re\left(z\frac{g'}{g}\right)=r\frac{\partial}{\partial r}\left(\log|h|-\log|g|\right). \tag{7}$$

If we consider the relations (5), (6), (7) together we obtain

$$-\frac{2(1-\alpha)}{1+r} \le \frac{\partial}{\partial r}\left(\log|h|-\log|g|\right) \le \frac{2(1-\alpha)}{1-r} \tag{8}$$

After the integrating we obtain (4).

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Theorem 2.4. If $f \in S^*_{lh}(\alpha)$ then

$$\frac{|b_1| - |a_1|r}{|a_1| - |b_1|r}(1-r)^{2(1-\alpha)} \leq \frac{|g'(z)|}{|h'(z)|} \leq \frac{|b_1| + |a_1|r}{|a_1| + |b_1|r}(1+r)^{2(1-\alpha)}. \tag{9}$$

Proof. Using theorem 2.3 we can write

$$(1-r)^{2(1-\alpha)} \leq \frac{|g(z)|}{|h(z)|} \leq (1+r)^{2(1-\alpha)} \tag{10}$$

On the other hand, since f is solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

then we obtain

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} = \frac{b_1}{a_1} + \ldots \tag{11}$$

Now we define the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, z \in D. \tag{12}$$

Therefore $\phi(z)$ satisfies the condition of Schwarz lemma. Using the estimate the Schwarz lemma $|\phi(z)| \leq r$,which given

$$|\phi(z)| = \left|\frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}\right| \leq r \tag{13}$$

The inequality (13) can be written in the following form

$$\left|\frac{w(z) - \frac{b_1}{a_1}}{1 - \frac{\overline{b_1}}{\overline{a_1}}}w(z)\right| \leq r \Rightarrow \left|w(z) - \frac{b_1}{a_1}\right| \leq r\left|1 - \frac{\overline{b_1}}{\overline{a_1}}w(z)\right| \tag{14}$$

The inequality (14) is equivalent

$$\left|w(z) - \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1 - (\frac{b_1}{a_1})^2 r^2}\right| \leq \frac{\left(1 - \left|\frac{b_1}{a_1}\right|^2\right)r}{1 - \left|\frac{b_1}{a_1}\right|^2 r^2} \tag{15}$$

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The equality holds in the inequality (15) only for the function

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} \tag{16}$$

From the inequality (15) we have

$$\frac{\left|\frac{b_1}{a_1}\right| - r}{1 - \left|\frac{b_1}{a_1}\right| r} \leq |w(z)| \leq \frac{\left|\frac{b_1}{a_1}\right| + r}{1 + \left|\frac{b_1}{a_1}\right| r} \tag{17}$$

Considering the relation (10), (17) together, end after the simple calculations,

$$\frac{|b_1| - |a_1| r}{|a_1| - |b_1| r} \left|\frac{g(z)}{h(z)}\right| \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + |a_1| r}{|a_1| + |b_1| r} \left|\frac{g(z)}{h(z)}\right| \tag{18}$$

using inequality (4) in the inequality (18) we get (8).

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References

[**1**]Abdulhadi, Z., Bshouty, D. : *Univalent functions in* $H.\overline{H}(D)$, Trans. Amer. Math. Soc., 305, pp. 841-849, 1988.

[**2**]Abdulhadi, Z., Hengartner, W.:*One pointed univalent log-harmonic mappings*, J. Math. Anal. Apply. 203(2), pp. 333-351, 1996.

[**3**]Abdulhadi, Z., Abu Muhanna, Y. : *Starlike log-harmonic mappings of order* α, JIPAM.Vol.7, Issue 4, Article 123, 2006.

[**4**]Goodman, A. W. : *Univalent functions, Vol I,* Mariner Publishing Company, Inc., Washington, New Jersey, 1983.

N-differentiation composition operators from weighted Banach spaces of holomorphic function to weighted Bloch spaces

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[1]

Abstract

In this paper, we characterize nth differentiation composition operators from weighted Banach space of holomorphic function to weighted Bloch space, and give some necessary and sufficient conditions for the boundedness and compactness of the operators.

1 Introduction

Let $H(\mathbb{D})$ and $S(\mathbb{D})$ denote the class of analytic functions and analytic self-maps on the unit disk $\mathbb{D}$ of the complex plane of $\mathbb{C}$, respectively. Let v and w be strictly positive continuous and bounded functions (weight) on $\mathbb{D}$.

Weighted Banach spaces of holomorphic functions is defined by

$$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

endowed with the weighted sup-norm $\|.\|_v$.

An $f \in H(\mathbb{D})$ belongs to weighted Bloch spaces $\mathcal{B}_w$ if

$$b_w(f) = \sup_{z\in\mathbb{D}} w(z)|f'(z)| < \infty.$$

The quantity $b_w(f)$ defines a seminorm on $\mathcal{B}_w$, while a natural norm is given by

$$\|f\|_{\mathcal{B}_w} = |f(0)| + b_w(f).$$

This norm makes $\mathcal{B}_w$ into a Banach space.

By $\mathcal{B}_{w,0}$ we denote the little weighted Bloch space, the subspace of $\mathcal{B}_w$, consisting of all $f \in \mathcal{B}_w$ such that

$$\lim_{|z|\to 1} w(z)|f'(z)| = 0.$$

Each ϕ in $S(\mathbb{D})$ induces through composition a linear composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), f \mapsto f \circ \phi$. And n-differentiating composition operator is a linear operator defined by

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$D_\phi^n : H(\mathbb{D}) \to H(\mathbb{D}), f \mapsto f^{(n)}(\phi)$. We are interested in D_ϕ^n acting from weighted Banach spaces of holomorphic functions to weighted Bloch spaces.

In the setting of weighted spaces the so-called associated weight plays an important role. For a weight v its associated weight $\widetilde{v}$ is defined follows:

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty}},$$

where δ_z denotes the point evaluation of z. By [1] the associated weight $\widetilde{v}$ is continuous, $\widetilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = \frac{1}{\widetilde{v}(z)}$.

We say that a weight v is radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A positive continuous function v is called normal if there exist $\delta \in [0, 1)$ and $s, t (0 < s < t)$ such that for every $z \in \mathbb{D}$ with $|z| \in [\delta, 1)$,

$$\frac{v(|z|)}{(1-|z|)^s} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{|z|\to 1} \frac{v(|z|)}{(1-|z|)^s} = 0;$$

$$\frac{v(|z|)}{(1-|z|)^t} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{|z|\to 1} \frac{v(|z|)}{(1-|z|)^t} = \infty.$$

A radial, non-increasing weight is called typical if $\lim_{|z|\to 1} v(z) = 0$. When studying the structure and isomorphism classes of the space H_v^∞ (see [6, 7]), Lusky introduced the following condition $(L1)$ (renamed after the author) for radial weights:

$$(L1) \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{1-2^{-n}} > 0,$$

which will play a great role in this article. Moreover, radial weights with (L1) (for example, see [2]) are essential, that is, we can find a constant $k > 0$ such that

$$v(z) \le \widetilde{v}(z) \le kv(z) \text{ for every } z \in \mathbb{D}.$$

Now, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$, be the Möbius transformation that interchanges a and 0. We will use the fact that derivative of φ_a is given by

$$\varphi_a'(z) = -\frac{1-|a|^2}{(1-\bar{a}z)^2} \text{ for every } z \in \mathbb{D}.$$

Our aim in this note is to characterize boundedness and compactness of operator D_n^ϕ from weighted Banach spaces of holomorphic functions to weighted Bloch spaces in terms of the involved weights as well as the inducing map. For $n = 0$ and $n = 1$, as corollaries we get a characterization of boundedness and compactness of C_ϕ and $C_\phi D$ that act from weighted Banach spaces of holomorphic functions to weighted Bloch spaces.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to the other.

2 Background and Some Lemmas

Now let us state a couple of lemmas, which are used in the proof of the main results in the next sections. The first lemma is taken from [9].

Lemma 1. *Let v be a radial weight satisfying condition (L1). There is a constant $C > 0$ (depending only on the weight v) such that for all $f \in H_v^\infty$,*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_v}{v(z)(1-|z|^2)^n}, \tag{1}$$

for every $z \in \mathbb{D}$ and $n \in \mathbb{N}$.

Proof. We will prove the theorem by mathematical induction.

For $n = 1$, see Lemma 2 in [9].

If (1) is true for $n-1$. Then for n, let $u(z) = v(z)(1-|z|^2)^{n-1}$, since

$$|f^{(n-1)}(z)| \leq C \frac{\|f\|_v}{v(z)(1-|z|^2)^{n-1}},$$

then $f^{(n-1)} \in H_u^\infty$.

For $f^{(n-1)}$ using the result of $n = 1$ the lemma is proved. □

The following result is well-known (see, e.g. [3, 8])

Lemma 2. *Suppose that w is a normal weight and v is a radial weight satisfying (L1). Then the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$) is compact if and only if whenever $\{f_m\}$ is a bounded sequence in H_v^∞ with $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$, and then $\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0$.*

The following lemma can be proved similarly to Lemma 1 in [4] (see, also [5]). It will be useful to give a criterion for compactness in $\mathcal{B}_{w,0}$.

Lemma 3. *Assume w is normal. A closed set K in $\mathcal{B}_{w,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} w(z)|f'(z)| = 0. \tag{2}$$

3 The Boundedness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$)

In this section we formulate and prove results regarding the boundedness of the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(*or* $\mathcal{B}_{w,0}$).

Theorem 1. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \infty, \tag{3}$$

Proof. First, we assume that the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is bounded. Fix a point $a \in \mathbb{D}$, and consider the function

$$f_a(z) = \varphi_a^{n+1}(z) g_a(z) \text{ for every } z \in \mathbb{D},$$

where g_a is a function in the unit ball of H_v^∞ such that $g_a(a) = \frac{1}{\tilde{v}(a)}$. Then

$$\|f_a\|v = \sup_{z\in\mathbb{D}} v(z)|f_a(z)| \leq \sup_{z\in\mathbb{D}} v(z)|g_a(z)| \leq 1.$$

It is easy to check that

$$(\varphi_a^{n+1})^{(k)}(a) = 0, \quad k = 0, 1, ..., n;$$

$$(\varphi_a^{n+1})^{(n+1)}(a) = \frac{(-1)^{n+1}(n+1)!}{(1-|a|^2)^{n+1}}.$$

So

$$f_a^{(n+1)}(a) = \sum_{k=0}^{n+1} C_{n+1}^k (\varphi_a^{n+1})^{(k)}(a) g_a^{(n+1-k)}(a) = \frac{(-1)^{n+1}(n+1)!}{(1-|a|^2)^{n+1}\widetilde{v}(a)}.$$

Then by the boundedness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$, we have

$$\begin{aligned} \infty &> \|D_\phi^n f_{\phi(a)}\|_{\mathcal{B}_w} \geq \sup_{z\in\mathbb{D}} w(z)|f_{\phi(a)}^{(n+1)}(\phi(z))\phi'(z)| \\ &\geq w(a)|f_{\phi(a)}^{(n+1)}(\phi(a))\phi'(a)| = \frac{(n+1)!w(a)|\phi'(a)|}{(1-|\phi(a)|^2)^{n+1}\widetilde{v}(\phi(a))}. \end{aligned}$$

Since v has (L1), the weights v and $\widetilde{v}$ are equivalent then $\widetilde{v}$ can be replaced by v, and combine with the arbitrariness of $a \in \mathbb{D}$, we obtain (3).

Conversely, an application of Lemma 1 yields

$$w(z)|f^{(n+1)}(\phi(z))\phi'(z)| \leq C\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}}\|f\|_v, \tag{4}$$

and

$$|f^{(n)}(\phi(0))| \leq C\frac{\|f\|_v}{v(\phi(0))(1-|\phi(0)|^2)^n}.$$

Combine with this and taking the supremum in (4) over $\mathbb{D}$, then employing condition (3), we see that $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ must be bounded. □

By the similar proof of Theorem 1 we see that the following result is true.

Theorem 2. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{5}$$

Especially, for $n = 0$, we obtain necessary and sufficient conditions for the boundedness of the operators $C_\phi : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$).

Corollary 1. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then the following statements hold:*

(i) $C_\phi : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} < \infty.$$

(ii) $C_\phi : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

For $n = 1$, D_ϕ^n is the operator $C_\phi D$, then we have the following corollary .

Corollary 2. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1), then the following statements hold:*

(i) $C_\phi D : H_v^\infty \to \mathcal{B}_w$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} < \infty.$$

(ii) $C_\phi D : H_v^\infty \to \mathcal{B}_{w,0}$ is bounded if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

4 The Compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_w$(or $\mathcal{B}_{w,0}$)

In this section, we turn our attention to the question of compactness.

Theorem 3. *Suppose that w be arbitrary weight, v be a radial weight satisfying condition (L1). Then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact if and only if*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{6}$$

Proof. First, we assume that the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact. Let $\{z_m\}_m \subset \mathbb{D}$ be a sequence with $|\phi(z_m)| \to 1$ such that

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = \lim_{m\to\infty} \frac{w(z_m)|\phi'(z_m)|}{v(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}.$$

By passing to a subsequence and still denoted by $\{z_m\}_m$, we assume that there is $N \in \mathbb{N}$, such that $|\phi(z_m)|^m \geq \frac{1}{2}$ for every $m \geq N$. For every $m \in \mathbb{N}$, we consider functions

$$f_m(z) = z^m \varphi_{\phi(z_m)}^{n+1}(z) g_{\phi(z_m)}(z) \ \text{ for every } \ z \in \mathbb{D},$$

where $g_{\phi(z_m)}$ is a function in the unit ball of H_v^∞ such that $|g_{\phi(z_m)}(\phi(z_m))| = \frac{1}{\widetilde{v}(\phi(z_m))}$. Again since v has (L1), $\widetilde{v}$ may be replaced by v. Obviously, $\{f_m\}_m \subset H_v^\infty$ is a bounded sequence that tends to zero uniformly on the compact subsets of $\mathbb{D}$. Hence by Lemma 2, we have that $\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0$. Moreover,

$$(z^m \varphi_{\phi(z_m)}^{n+1})^{(k)}(\phi(z_m)) = 0, \quad k = 0, 1, ..., n;$$

$$(z^m \varphi_{\phi(z_m)}^{n+1})^{(n+1)}(\phi(z_m)) = \frac{(-1)^{n+1}(n+1)!\phi^m(z_m)}{(1-|\phi(z_m)|^2)^{n+1}}.$$

Since

$$f_m^{(n+1)}(\phi(z_m)) = \sum_{k=0}^{n+1} C_{n+1}^k (z^m \varphi_{\phi(z_m)}^{n+1})^{(k)} g_{\phi_{z_m}}^{(n+1-k)}(\phi(z_m)).$$

Therefore $|f_m^{(n+1)}(\phi(z_m))| = \frac{(n+1)!|\phi(z_m)|^m}{\widetilde{v}(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}$, and for $m \geq N$

$$\begin{aligned} 0 \ \leftarrow \ & \|D_\phi^n f_m\|_{\mathcal{B}_w} \geq w(z_m)|f_m^{(n+1)}(\phi(z_m))\phi'(z_m)| \\ = \ & \frac{(n+1)!w(z_m)|\phi'(z_m)||\phi(z_m)|^m}{\widetilde{v}(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}} \\ \geq \ & \frac{1}{2} \frac{w(z_m)|\phi'(z_m)|}{v(\phi(z_m))(1-|\phi(z_m)|^2)^{n+1}}, \end{aligned}$$

and the claim follows.

Conversely, suppose that (6) holds. Let $\{f_m\}_m \subset H_v^\infty$ be a bounded sequence which converges to zero uniformly on the compact subsets of $\mathbb{D}$, we may assume that $\|f_m\|_v \leq 1$ for every $m \in \mathbb{N}$. By Lemma 2 we have to show that

$$\|D_\phi^n f_m\|_{\mathcal{B}_w} \to 0 \text{ if } m \to \infty.$$

Let us fix $\varepsilon > 0$. By hypothesis there is $0 < r < 1$ such that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \frac{\varepsilon}{2C} \text{ if } |\phi(z)| > r,$$

where C is the constant given in Lemma 1. Thus, if $|\phi(z)| > r$, by Lemma 1,

$$w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \leq C\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}}\|f_m\|_v < \frac{\varepsilon}{4}. \tag{7}$$

Now, we can find $M > 0$ such that

$$\sup_{|\phi(z)|\leq r} w(z)|\phi'(z)| \leq M. \tag{8}$$

Moreover, since $\{f_m\}_m$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. Cauchy's integral formula gives that $\{f_m^{(n+1)}\}_m$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. So there is $N_1 \in \mathbb{N}$ such that

$$\sup_{|\phi(z)|\leq r} |f_m^{(n+1)}(\phi(z))| \leq \frac{\varepsilon}{4M} \text{ for every } m \geq N_1. \tag{9}$$

Also, $\{f_m^{(n)}(\phi(0))\}_m$ converges to 0 as $m \to \infty$, then there exists $N_2 > 0$ such that $|f_m^{(n)}(\phi(0))| < \frac{\varepsilon}{2}$ for every $m > N_2$. Finally, together with (7) (8) and (9) we can conclude that

$$\begin{aligned}
\|D_\phi^n f_m\|_{\mathcal{B}_w} &= |f_m^{(n)}(\phi(0))| + \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \\
&\leq |f_m^{(n)}(\phi(0))| + \sup_{|\phi(z)|\leq r} w(z)|\phi'(z)| \sup_{|\phi(z)|\leq r} |f_m^{(n+1)}(\phi(z))| \\
&\quad + \sup_{|\phi(z)|>r} w(z)|\phi'(z)||f_m^{(n+1)}(\phi(z))| \\
&< \varepsilon,
\end{aligned}$$

for every $m \geq N$, where $N := \max\{N_1, N_2\}$. Hence the claim follows. □

Theorem 4. *Suppose that w be a normal weight, v be a radial weight satisfying condition (L1). Then $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} = 0. \tag{10}$$

Proof. Suppose that $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact. Then $D_\phi^n : H_v^\infty \to \mathcal{B}_w$ is compact. Hence, by Theorem 3 we see that (6) holds. Then for every $\varepsilon > 0$ there exists a $r \in (0,1)$ such that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \varepsilon \text{ if } r < |\phi(z)| < 1.$$

On the other hand, since $h(z) = \frac{z^{n+1}}{(n+1)!} \in H_v^\infty$, from the compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$, it follows that $\phi \in \mathcal{B}_{w,0}$. Then there exists a $\rho \in (r, 1)$ such that

$$w(z)|\phi'(z)| < \varepsilon \inf_{t\in[0,r]} v(t)(1-|t|^2)^{n+1} \quad \text{if} \quad \rho < |z| < 1, \tag{11}$$

Therefore, when $\rho < |z| < 1$ and $r < |\phi(z)| < 1$, we have that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} < \varepsilon. \tag{12}$$

If $\rho < |z| < 1$ and $|\phi(z)| \le r$, combine with (11), we have that

$$\frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^{n+1}} \le \frac{w(z)|\phi'(z)|}{\inf\limits_{t\in[0,r]} v(t)(1-|t|^2)^{n+1}} < \varepsilon. \tag{13}$$

Inequalities (12) and (13) imply (10) holds.

Conversely, assume that (10) holds. Then (3) holds, which along with (4) implies that the set $D_\phi^n(\{f \in H_v^\infty : \|f\|_v \le 1\})$ is bounded in $\mathcal{B}_{w,0}$. By Lemma 3 we see that $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_v\le 1} w(z)|f^{(n+1)}(\phi(z))\phi'(z)| = 0.. \tag{14}$$

Taking the supremum in (4) over the unit ball of H_v^∞, then letting $|z| \to 1$, we obtain (14), from which the compactness of $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ follows. □

Noticing the results of Theorem 2 and Theorem 4, we conclude that the boundedness and compactness of the operator $D_\phi^n : H_v^\infty \to \mathcal{B}_{w,0}$ is equivalent. Similarly, for $n = 0$, we obtain necessary and sufficient conditions for the compactness of the operators $C_\phi : H_v^\infty \to \mathcal{B}_w$(*or* $\mathcal{B}_{w,0}$).

Corollary 3. *Suppose that w be a normal weights, v be a radial weight satisfying condition (L1). Then the following statements hold:*

(i) $C_\phi : H_v^\infty \to \mathcal{B}_w$ is compact if and only if

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

(ii) $C_\phi : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0.$$

And for $n = 1$, D_ϕ^n is the operator $C_\phi D$.

Corollary 4. *Suppose that w be a normal weights, v be a radial weight satisfying condition (L1). Then the following statements hold:*

(i) $C_\phi D : H_v^\infty \to \mathcal{B}_w$ is compact if and only if

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

(ii) $C_\phi D : H_v^\infty \to \mathcal{B}_{w,0}$ is compact if and only if

$$\lim_{|z|\to 1} \frac{w(z)|\phi'(z)|}{v(\phi(z))(1-|\phi(z)|^2)^2} = 0.$$

References

[1] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.*, 127, 137-168 (1988).

[2] J. Bonet, P. Domański and M. Lindström, Essential norm and weak compactness of composition opterators on weighted Banach spaces of analytic functions, *Canad. Math. Bull.*, 42(2),139-148 (1999).

[3] K. Avetisyan, S. Stević, Extended Cesàro opterators between different Hardy spaces, *Appl. Math. Comput.*, 207, 346-350 (2009).

[4] K. Madigan, A. Matheson, Compact compositon operators on the Bloch space, *Trans. Amer. Math. Soc.*, 347(7), 2679-2687 (1995).

[5] A. Montes-Rodriguez, Weighted compositon operators on weighted Banach spaces of analytic functions, *J. London Math. Soc.*, 61(3), 872-884 (2000).

[6] W. Lusky, On weighted spaces of harmonic and holomorphic functions, *J. London. Math. Soc.*, 51(2), 309-320 (1995).

[7] W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, *Studia. Math.*, 175(1), 19-45 (2006).

[8] Z. Hu, Extended Cesàro opterators on mixed-norm spaces, *Proc. Amer. Math. Soc.*, 131(7), 2171-2179 (2003).

[9] E. Wolf, Composition followed by differentiation between weighted Banach spaces of holomorphic functions, *Revista Da La Real Academia De Ciencias Exactas, Fisicas Y Naturales. Serie A. Matematics*, 105(2), 315-322 (2011).

[10] Y. Wu, H. Wulan, Products of differentiation and composition operators on the Bloch space, *Collect. Math.*, 63, 93-107 (2012).

[11] C.C. Cowen and B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.

[12] J.N. Dai and C.H. Ouyang, Difference of weighted composition optertors on $H_\alpha^\infty(B_N)$, *J. Inequal. Appl.*, 2009, 19p., Article ID 127431,(2009).

[13] S. Stević, Composition opterators between H^∞ and the α-Bloch spaces on the polydisc, *Z. Anal. Anwendungen*, 25(4), 457-466 (2006).

[14] D. García, M. Maestre and P.S. Peris, Composition opterators between weighted spaces of holomorphic functions on Banach spaces, *Ann. Aca. Sci. Fen. Math.*, 29(7) , 81-98 (2004).

[15] Z.H. Zhou, Y.X. Liang and X.T Dong, Differences of weighted composition operators from Hardy space to weighted-type spaces on the unit ball, *Ann. Polon. Math.*, 104(3) , 309-319 (2012).

[16] K.H. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in the Mathematics 226, Springer, New York, 2004.

FUZZY n-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY C^*-ALGEBRAS

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ABSTRACT. Using the fixed point method, we prove the fuzzy version of the Hyers-Ulam stability of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right)+f\left(\frac{a-3c}{3}\right)+f\left(\frac{3a-b+3c}{3}\right)=f(a).$$

1. Introduction and Preliminaries

The stability of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms in 1940. More precisely, he proposed the following problem: Given a group $\mathcal{G}$, a metric group $(\mathcal{G}',d)$ and $\epsilon>0$, does there exist a $\delta>0$ such that if a function $f:\mathcal{G}\to\mathcal{G}'$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in\mathcal{G}$, then there exists a homomorphism $T:\mathcal{G}\to\mathcal{G}'$ such that $d(f(x),T(x))<\epsilon$ for all $x\in\mathcal{G}$? In 1941, Hyers [16] gave a partial solution of the Ulam's problem for the case of approximate additive mappings under the assumption that $\mathcal{G}$ and $\mathcal{G}'$ are Banach spaces. In 1950, Aoki [1] generalized the Hyers' theorem for approximately additive mappings. In 1978, Th. M. Rassias [33] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 9, 11, 12, 13, 14, 19, 30, 31, 34, 35]).

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. ([4, 10]) *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^nx,J^{n+1}x)<\infty,\qquad\forall n\ge n_0$;
(2) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$;*

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(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 27, 28, 32]).

Katsaras [18] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematics have defined fuzzy normed on a vector space from various points of view [15, 21, 23, 24, 25, 29, 37]. In particular, Bag and Samanta [3] following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

We use the definition of fuzzy normed spaces given in [3, 23, 24] to investigate a fuzzy version of the Hyers-Ulam stability of n-Jordan $*$-derivations in induced fuzzy C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a-b+3c}{3}\right) = f(a).$$

Definition 1.2. ([3, 23, 24, 25]) Let $\mathcal{X}$ be a complex vector space. A function $N : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$,

N_1**:** $N(x, t) = 0$ for $t \leq 0$
N_2**:** $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$
N_3**:** $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \in \mathbb{C} - \{0\}$
N_4**:** $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$
N_5**:** $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$
N_6**:** for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair $(\mathcal{X}, N)$ is called a *fuzzy normed vector space*.

Definition 1.3. ([3, 23, 24, 25]) Let $(\mathcal{X}, N)$ be a fuzzy normed vector space.

(1) A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be *convergent* if there exists an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. in this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : \mathcal{X} \to \mathcal{Y}$ between fuzzy normed vector space $\mathcal{X}$, $\mathcal{Y}$ is continuous at point $x_0 \in \mathcal{X}$ if for each sequence $\{x_n\}$ converging to x_0 in $\mathcal{X}$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : \mathcal{X} \to \mathcal{Y}$ is continuous at each $x \in \mathcal{X}$, then $f : \mathcal{X} \to \mathcal{Y}$ is said to be *continuous* on $\mathcal{X}$ (see [2]).

Definition 1.4. Let $\mathcal{X}$ be a $*$-algebra and $(\mathcal{X}, N)$ a fuzzy normed space.

(1) The fuzzy normed space $(\mathcal{X}, N)$ is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t) \qquad \& \qquad N(x^*, t) = N(x, t)$$

(2) A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

Example 1.5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed $*$-algebra. let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t>0,\, x\in\mathcal{X} \\ 0, & t\le 0,\, x\in\mathcal{X}. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on $\mathcal{X}$ and $(\mathcal{X}, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(\mathcal{X}, \|\cdot\|)$ be a C^*-algebra and N a fuzzy norm on $\mathcal{X}$.
(1) The fuzzy normed $*$-algebra $(\mathcal{X}, N)$ is called an induced fuzzy normed $*$-algebra
(2) The fuzzy Banach $*$-algebra $(\mathcal{X}, N)$ is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(\mathcal{X}, N)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (\mathcal{X}, N) \to (\mathcal{X}, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$D(a^n) = D(a)a^{n-1} + aD(a)a^{n-2} + \ldots + a^{n-2}D(a)a + a^{n-1}D(a) \quad \& \quad D(a^*) = D(a)^*$$

for all $a \in \mathcal{X}$.

Throughout this paper, assume that $(\mathcal{X}, N)$ is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let $(\mathcal{Z}, N)$ be a fuzzy normed vector space and let $f : \mathcal{X} \to \mathcal{Z}$ be a mapping such that*

$$N\left(f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x-y+3z}{3}\right), t\right) \ge N\left(f(x), \frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ and all $t > 0$. Then f is additive, i.e., $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \ge N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By N_5 and N_6, $N(f(0), t) = 1$ for all $t > 0$. It follows from N_2 that $f(0) = 0$.

Letting $y = x = 0$ in (2.1), we get

$$N(f(0) + f(-z) + f(z), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that $f(-z) + f(z) = 0$ for all $z \in \mathcal{X}$. So

$$f(-z) = -f(z)$$

for all $z \in \mathcal{X}$.

Letting $x = 0$ and replacing y, z by $3y, -z$, respectively, in (2.1), we get

$$N(f(y) + f(z) + f(-y-z), t) \ge N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that

$$f(y) + f(z) + f(-y-z) = 0 \tag{2.2}$$

for all $y, z \in \mathcal{X}$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in \mathcal{X}$, as desired. □

Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy n-Jordan $*$-derivations on induced fuzzy C^*-algebras.

Theorem 2.2. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function such that there exists an $L < \frac{3}{3^n}$ with*

$$\varphi\left(\frac{x}{3},\frac{y}{3},\frac{z}{3}\right) \le \frac{L}{3}\varphi(x,y,z) \tag{2.3}$$

for all $x,y,z \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping such that

$$\begin{aligned} N\left(\mu f\left(\frac{y-x}{3}\right) + \mu f\left(\frac{x-3z}{3}\right) + \mu f\left(\frac{3x-y+3z}{3}\right) - f(\mu x), t\right) \\ \ge \frac{t}{t+\varphi(x,y,z)}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} N(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) \\ +f(v^*) - f(v)^*, t) \ge \frac{t}{t+\varphi(w,v,0)} \end{aligned} \tag{2.5}$$

for all $x,y,z,w,v \in \mathcal{X}$, all $t>0$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then $D(x) = N-\lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \ge \frac{(1-L)t}{(1-L)t + \varphi(x,2x,0)} \tag{2.6}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Letting $\mu = 1$, $y = 2x$ and $z = 0$ in (2.4), we get

$$N\left(3f\left(\frac{x}{3}\right) - f(x), t\right) \ge \frac{t}{t+\varphi(x,2x,0)} \tag{2.7}$$

for all $x \in \mathcal{X}$.

Consider the set

$$S := \{g : \mathcal{X} \to \mathcal{X}\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\alpha \in \mathbb{R}_+ : N(g(x)-h(x), \alpha t) \ge \frac{t}{t+\varphi(x,2x,0)},\ \forall x \in \mathcal{X}, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see the proof of [22, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 3g\left(\frac{x}{3}\right)$$

for all $x \in \mathcal{X}$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t+\varphi(x,2x,0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Hence

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L}{3}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \varphi\left(\frac{x}{3}, \frac{2x}{3}, 0\right)} \geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \frac{L}{3}\varphi\left(x, 2x, 0\right)} \\ &= \frac{t}{t + \varphi\left(x, 2x, 0\right)} \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that $d(f, Jf) \leq 1$.

By Theorem 1.1, there exists a mapping $D : \mathcal{X} \to \mathcal{X}$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{3}\right) = \frac{1}{3}D(x) \tag{2.8}$$

for all $x \in \mathcal{X}$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.8) such that there exists a $\alpha \in (0, \infty)$ satisfying

$$N(f(x) - D(x), \alpha t) \geq \frac{t}{t + \varphi\left(x, 2x, 0\right)}$$

for all $x \in \mathcal{X}$;

(2) $d(J^k f, D) \to 0$ as $k \to \infty$. This implies the equality

$$N\text{-}\lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right) = D(x)$$

for all $x \in \mathcal{X}$;

(3) $d(f, D) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, D) \leq \frac{1}{1 - L}.$$

This implies that the inequality (2.7) holds.

It follows from (2.3) that

$$\sum_{k=0}^{\infty} 3^k \varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}) < \infty$$

for all $x, y, z \in \mathcal{X}$.

By (2.4),

$$\begin{aligned} N\left(3^k \mu f\left(\frac{y - x}{3^{k+1}}\right) + 3^k \mu f\left(\frac{x - 3z}{3^{k+1}}\right) + 3^k \mu f\left(\frac{3x - y + 3z}{3^{k+1}}\right)\right. \\ \left. -3^k f\left(\frac{\mu x}{3^k}\right), 3^k t\right) \geq \frac{t}{t + \varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})} \end{aligned}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. So

$$N\left(3^k\mu f\left(\frac{y-x}{3^{k+1}}\right) + 3^k\mu f\left(\frac{x-3z}{3^{k+1}}\right) + 3^k\mu f\left(\frac{3x-y+3z}{3^{k+1}}\right)\right.$$
$$\left. -3^k f\left(\frac{\mu x}{3^k}\right), t\right) \geq \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})} = \frac{t}{t + 3^k\varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\varphi(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k})} = 1$ for all $x, y, z \in \mathcal{X}$ and all $t > 0$,

$$N\left(\mu D\left(\frac{y-x}{3}\right) + \mu D\left(\frac{x-3z}{3}\right) + \mu D\left(\frac{3x-y+3z}{3}\right) - D(\mu x), t\right) = 1$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Thus

$$\mu D\left(\frac{y-x}{3}\right) + \mu D\left(\frac{x-3z}{3}\right) + \mu D\left(\frac{3x-y+3z}{3}\right) = D(\mu x) \tag{2.9}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Letting $x = y = z = 0$ in (2.9), we get $D(0) = 0$. Let $\mu = 1$ and $x = 0$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that D is additive. Letting $y = 2x$ and $z = 0$ in (2.9), we get

$$\mu D(x) = 3\mu D\left(\frac{x}{3}\right) = D(\mu x)$$

for all $x \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. By [26, Theorem 2.1], the mapping $D : \mathcal{X} \to \mathcal{X}$ is $\mathbb{C}$-linear.

By (2.5) and letting $v = 0$ in (2.5), we have

$$N\left(3^{nk}f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk}f\left(\frac{w}{3^k}\right)w^{n-1} - 3^{nk}wf\left(\frac{w}{3^k}\right)w^{n-2} - \cdots - 3^{nk}w^{n-2}f\left(\frac{w}{3^k}\right)w\right.$$
$$\left. -3^{nk}w^{n-1}f\left(\frac{w}{3^k}\right), 3^{nk}t\right) \geq \frac{t}{t + \varphi(\frac{w}{3^k}, 0, 0)}$$

for all $w \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^{nk}f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk}f\left(\frac{w}{3^k}\right)w^{n-1} - 3^{nk}wf\left(\frac{w}{3^k}\right)w^{n-2} - \cdots - 3^{nk}w^{n-2}f\left(\frac{w}{3^k}\right)w\right.$$
$$\left. -3^{nk}w^{n-1}f\left(\frac{w}{3^k}\right), t\right) \geq \frac{\frac{t}{3^{nk}}}{\frac{t}{3^{nk}} + \varphi(\frac{w}{3^k}, 0, 0)} = \frac{t}{t + (3^{n-1}L)^k\varphi(w, 0, 0)}$$

for all $w \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+(3^{n-1}L)^k\varphi(w,0,0)} = 1$ for all $w \in \mathcal{X}$ and all $t > 0$,

$$N(D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} \cdots w^{n-2}D(w)w - w^{n-1}D(w), t) = 1$$

for all $w \in \mathcal{X}$ and all $t > 0$. Thus

$$D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} \cdots w^{n-2}D(w)w - w^{n-1}D(w) = 0$$

for all $w \in \mathcal{X}$.

By (2.5) and letting $w = 0$ in (2.5), we have

$$N\left(3^k f\left(\frac{v^*}{3^k}\right) - 3^k f\left(\frac{v}{3^k}\right)^*, 3^k t\right) \geq \frac{t}{t + \varphi(0, \frac{v}{3^k}, 0)}$$

for all $v \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^k f\left(\frac{v^*}{3^k}\right) - 3^k f\left(\frac{v}{3^k}\right)^*, t\right) \geq \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \varphi(0, \frac{v}{3^k}, 0)} = \frac{t}{t + 3^k\varphi(0, \frac{v}{3^k}, 0)}$$

for all $v \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\varphi(0,\frac{v}{3^k},0)} = 1$ for all $v \in \mathcal{X}$ and all $t > 0$,

$$N(D(v^*) - D(v)^*, t) = 1$$

for all $x \in \mathcal{X}$ and all $t > 0$. Thus $D(v^*) - D(v)^* = 0$ for all $v \in \mathcal{X}$.

Therefore, the mapping $D : \mathcal{X} \to \mathcal{X}$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > n$. Let $\mathcal{X}$ be a normed vector space with norm $\|\cdot\|$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$N\left(\mu f\left(\frac{y-x}{3}\right) + \mu f\left(\frac{x-3z}{3}\right) + \mu f\left(\frac{3x-y+3z}{3}\right) - f(\mu x), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \tag{2.10}$$

$$N(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w) + f(v^*) - f(v)^*, t) \geq \frac{t}{t + \theta(\|w\|^p + \|v\|^p)} \tag{2.11}$$

for all $x, y, z, w, v \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then $D(x) = N - \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \geq \frac{(3^p - 3)t}{(3^p - 3)t + 3^p\theta\|x\|^p}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

Theorem 2.4. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 3L\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$$

for all $x, y, z \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying (2.4) and (2.5). Then $D(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that

$$N(f(x) - D(x), t) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, 0, 0)} \tag{2.12}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{3} g(3x)$$

for all $x \in \mathcal{X}$.

It follows from (2.7) that

$$N\left(f(x) - \frac{1}{3}f(3x), \frac{1}{3}t\right) \geq \frac{t}{t+\varphi(3x,0,0)} \geq \frac{t}{t+3L\varphi(x,0,0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f, D) \leq \frac{L}{1-L},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a positive real number with $p < 1$. Let $\mathcal{X}$ be a normed vector space with norm $\|\cdot\|$. Let $f : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying* (2.10) *and* (2.11). *Then $D(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan $*$-derivation $D : \mathcal{X} \to \mathcal{X}$ such that*

$$N(f(x) - D(x), t) \geq \frac{(3-3^p)t}{(3-3^p)t + 3^p\theta\|x\|^p}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{p-1}$. □

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References

1. Aoki, T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.
2. Bag, T., Samanta, S.K., *Fuzzy bounded linear operators*, Fuzzy Set and Systems **151** (2005) 513–547.
3. Bag, T., Samanta, S.K., *Finite dimensional fuzzy normed linear space*, J. Fuzzy Math. **11** (2003) 687–705.
4. Cădariu, L., V. Radu, V., *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
5. Cădariu, L., Radu, V., *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004) 43–52.
6. Cădariu, L., Radu, V., *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications **2008**, Art. ID 749392 (2008).
7. I. Chang, *Stability of higher ring derivations in fuzzy Banach algebras,* J. Computat. Anal. Appl. **14** (2012), 1059–1066.
8. Cheng, S.C., Mordeson J.M., *Fuzzy linear operators and fuuzy normed linear spaces*, Cacutta Math. Soc. **86** (1994) 429–436.
9. I. Cho, D. Kang, H. Koh, *Stability problems of cubic mappings with the fixed point alternative,* J. Computat. Anal. Appl. **14** (2012), 132–142.
10. Diaz, J., Margolis, B., *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968) 305–309.
11. M. Eshaghi Gordji, M. Bavand Savadkouhi, M. Bidkham, *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **12** (2010), 454–462.
12. M. Eshaghi Gordji, A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 724–729.

13. M. Eshaghi Gordji, R. Farokhzad Rostami, S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras,* J. Computat. Anal. Appl. **13** (2011), 734–742.
14. M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi, M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras,* J. Computat. Anal. Appl. **12** (2010), 463–470.
15. Felbin, C., *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992) 239–248.
16. Hyers, D.H., *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941) 222–224.
17. Isac, G., Rassias, Th.M., *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996) 219–228.
18. Katsaras, A.K., *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984) 143–154.
19. H.A. Kenary, J. Lee, C. Park, *Non-Archimedean stability of an AQQ-functional equation,* J. Computat. Anal. Appl. **14** (2012), 211–227.
20. Kramosil, I., Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975) 326–334.
21. Krishna, S.V., Sarma, K.K.M., *Separation of fuzzy normed linear spaces* Fuzzy Sets and Systems **63** (1994) 207–217.
22. Miheţ, D., Radu, V., *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008) 567–572.
23. Mirmostafaee, A.K., Mirzavaziri, M., Moslehian M.S., *Fuzzy stability of the Jensen functional equation* Fuzzy Sets and Systems **159** (2008) 730–738.
24. Mirmostafaee, A.K., Moslehian, M.S., *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008) 720–729.
25. Mirmostafaee, A.K., Moslehian, M.S., *Fuzzy approximately cubic mappings*, Inform. Sca. **178** (2008) 3791–3798.
26. Park, C., *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005) 79–97.
27. Park, C., *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).
28. Park, C., *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications **2008**, Art. ID 493751 (2008).
29. Park, C., *Fuzzy stability of additive functional inequalities with the fixed point alternative*, J. Inequal. Appl. **2009**, Art. ID 410576 (2009).
30. C. Park, Y. Cho, H.A. Kenary, *Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **14** (2012), 526–535.
31. C. Park, S. Jang, R. Saadati, *Fuzzy approximate of homomorphisms,* J. Computat. Anal. Appl. **14** (2012), 833–841.
32. Radu, V., *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003) 91–96.
33. Rassias, Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.
34. S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 1097–1105.
35. S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Computat. Anal. Appl. **13** (2011), 1106–1114.
36. Ulam, S.M., *Problems in Modern Mathematics*, Chapter VI. Science ed. Wily, New York, 1940.
37. Xiao, J.Z., Zhu, X.H., *Fuzzy normed spaces of opeators and its completeness*, Fuzzy Sets and Systems **133** (2003) 389–399.

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Hyers-Ulam stability of a Tribonacci functional equation in 2-normed spaces

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the Tribonacci functional equation

$$f(x) = f(x-1) + f(x-2) + f(x-3)$$

in 2-Banach spaces.

Keywords: Hyers-Ulam stability, 2-Banach space, Fibonacci functional equation, Tribonacci functional equation.

1. Introduction and preliminaries

The concept of 2-normed spaces was first introduced by S. Gähler [8]. Let X be a complex vector space of a dimension greater than one. Suppose that $\|\cdot,\cdot\|$ is a real valued mapping on $X \times X$ satisfying the following conditions

N1: $\|b,a\| = \|a,b\|$
N2: $\|a,b\| = \mathbf{0} \Leftrightarrow$ *a and b are linearly dependent*
N3: $\|\alpha a, b\| = |\alpha|\ \|a,b\|$
N4: $\|a+\tilde{a}, b\| \ \leq\ \|a,b\| + \|\tilde{a},b\|$

for all $a, b \in X$ and $\alpha \in \mathbb{C}$. Then $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space. Some of the basic properties of 2-norms are that they are non-negative and $\|a, b+\alpha a\| = \|a,b\|$ for all $a, b \in X$ and $\alpha \in \mathbb{C}$. As an example of a 2-normed space, we may take an inner product space $(X, <\cdot,\cdot>)$, and define the standard 2-norm on X by

$$\|a,b\| = \begin{vmatrix} <a,a> & <a,b> \\ <b,a> & <b,b> \end{vmatrix}.$$

A sequence $\{x_n\}$ in a 2-normed space $(X; \|\cdot,\cdot\|)$ is said to converge to some $x \in X$ in the 2-norm if $\|x - x_n, u\| \to 0$ as $n \to \infty$ for all $u \in X$. A sequence $\{x_n\}$ in a 2-normed

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space $(X, \|\cdot,\cdot\|)$ is said to be Cauchy with respect to the 2-norm if

$$\lim_{n,m\to\infty} \|x_n - x_m, u\| = 0$$

for all $u \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

Throughout this paper, we denote by T_n the nth Tribonacci number for $n \in \mathbb{N}$. In particular, we define $T_0 = 0$, $T_1 = T_2 = 1$ and $T_n = T_{n-1}+T_{n-2}+T_{n-3}$ for $n \geq 3$. Similar application of Pascal's triangle in the Fibonacci numbers can be applied to calculate the Tribonacci numbers.

											1
					1						1
				1		1					2
			1		3		1				4
		1		5		5		1			7
	1		7		13	†	7		1		13
1		9		25		25		9		1	⋮

(a) Numbers in the n^{th} row are the sum of three neighbours: $25 = 13 + 5 + 7$.

(b) Sums of shallow diagonals giving Tribonacci numbers: 4= 1+3.

Let X be 2-Banach space. A function $f : R \to X$ is called a Tribonacci function if it satisfies

$$f(x) = f(x-1) + f(x-2) + f(x-3). \tag{1.1}$$

The stability of functional equations originated from a question of Ulam [15] in 1940. In the next year, Hyers [9] proved the problem for the Cauchy functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14]).

Recently, Bidkham and et al. [7] investigated the solution and the Hyers-Ulam stability of (1.1) in normed spaces.

In this paper, we establish the Hyers-Ulam stability of (1.1) in 2-normed spaces.

We denote the roots of the equation $x^3 - x^2 - x - 1 = 0$ By α, β and γ. β and γ are complex, $|\beta| = |\gamma|$ and α is greater than one. We have

$$\alpha + \beta + \gamma = 1\ ,\ \alpha\beta + \alpha\gamma + \beta\gamma = -1,\ \alpha\beta\gamma = 1. \tag{1.2}$$

2. **Main result**

As we shall see in the following theorem, the general solution of the Tribonacci functional equation is strongly related to the Tribonacci numbers T_n.

Theorem 2.1. ([7]) *Let X be a real vector space. A function $f : R \to X$ is a Tribonacci function if and only if there exists a function $g : [-2, 2] \to X$ such that*

$$f(x) = T_{[x]+2}\, g(x - [x]) + T'_{[x]}\, g(x - [x] - 1) + T_{[x]+1}\, g(x - [x] - 2),$$

where $T'_{[x]} = T_{[x]+3} - T_{[x]+2}$ for all $x \in \mathbb{R}$.

In the following theorem, we prove the Hyers-Ulam stability of the Tribonacci functional equation (1.1) in 2-Banach spaces. We try to prove this theorem under condition

$$\|f(x) - [f(x-1) + f(x-2) + f(x-3)], z\| \ \leq \epsilon$$

for all $x \in \mathbb{R}$ and $z \in X$, but this condition is very heavy and often inaccessible. In the following we offer a condition to obtain a best result.

Theorem 2.2. Let $(X, \|\cdot,\cdot\|)$ be a real 2-Banach space. If a function $f : \mathbb{R} \to X$ satisfies the inequality

$$\|f(x), f(x-1) + f(x-2) + f(x-3)\| \ \leq \epsilon$$

for all $x \in \mathbb{R}$ and some $\epsilon > 0$, then there exists a Tribonacci function $G : \mathbb{R} \to X$ such that

$$\|f(x), G(x)\| \leq \frac{1}{|\alpha^2(\beta-\gamma) + \beta^2(\gamma-\alpha) + \gamma^2(\alpha-\beta)|}[\frac{2(1+|\beta|)+|\beta|^2}{1-|\beta|^2}]\epsilon$$

for all $x \in \mathbb{R}$.

Proof. By (1.2), it follows from (1.1) that

$$\|f(x), (\alpha+\beta+\gamma)f(x-1) - (\alpha\beta+\alpha\gamma+\beta\gamma)f(x-2) + \alpha\beta\gamma\ f(x-3)\| \ \leq \epsilon$$

for all $x \in \mathbb{R}$. If we replace x by $x-r$ and $x+r$ in the last inequality, then we have

$$\begin{aligned}
&\|f(x-r), \alpha[f(x-r-1) - \gamma f(x-r-2)]\\
&+\beta\ [f(x-r-1) - (\alpha+\gamma)f(x-r-2) + \alpha\gamma f(x-r-3)] + \gamma f(x-r-1)\| \leq \epsilon,\\
&\|f(x-r), \alpha[f(x-r-1) - \beta f(x-r-2)]\\
&+\gamma\ [f(x-r-1) - (\alpha+\beta)f(x-r-2) + \alpha\beta f(x-r-3)] + \beta f(x-r-1)\| \leq \epsilon,\\
&\|f(x+r), \alpha[f(x+r-1) - \gamma f(x+r-2)]\\
&+\beta\ [f(x+r-1) - (\alpha+\gamma)f(x+r-2) + \alpha\gamma f(x+r-3)] + \gamma f(x+r-1)\| \leq \epsilon
\end{aligned}$$

for all $x \in \mathbb{R}$ and all $r \in \mathbb{Z}$. Hence we have

$$\begin{aligned}
&\|f(x-r), \beta^r\alpha[f(x-r-1) - \gamma f(x-r-2)] + \beta^{r+1}\ [f(x-r-1)\\
&\qquad - (\alpha+\gamma)f(x-r-2) + \alpha\gamma f(x-r-3)] + \beta^r\gamma f(x-r-1)\| \ \leq |\beta^r|\epsilon, \qquad (2.1)\\
&\|f(x-r), \gamma^r\beta[f(x-r-1) - \alpha f(x-r-2)] + \gamma^{r+1}\ [f(x-r-1)\\
&\qquad - (\alpha+\beta)f(x-r-2) + \alpha\beta f(x-r-3)] + \gamma^r\alpha f(x-r-1)\| \ \leq |\gamma^r|\epsilon, \qquad (2.2)\\
&\|f(x+r), \alpha^{-r}\beta[f(x+r-1) - \gamma f(x-r-2)] + \alpha^{-r+1}\ [f(x+r-1)\\
&\qquad - (\beta+\gamma)f(x+r-2) + \beta\gamma f(x+r-3)]\alpha^{-r}\beta f(x-r-1)\| \ \leq |\alpha^{-r}|\epsilon \qquad (2.3)
\end{aligned}$$

for all $x \in \mathbb{R}$ and all $r \in \mathbb{Z}$. Then we have

$$\begin{aligned}
&\|f(x), \alpha[f(x-1)-\gamma f(x-2)]+\gamma f(x-1)+\beta^n[f(x-n)-(\alpha+\gamma)f(x-n-1)+\alpha\gamma f(x-n-2)]\|\\
&\leq \sum_{r=0}^{n=1} \|f(x-r), \beta^r\alpha[f(x-r-1)-\gamma f(x-r-2)]+\beta^{r+1}[f(x-r-1)-(\alpha+\gamma)f(x-r-2)\\
&\qquad + \alpha\gamma f(x-r-3)] + \beta^r\gamma f(x-r-1)\| \ \leq \ \sum_{r=0}^{n-1}|\beta^r|\epsilon, \qquad (2.4)
\end{aligned}$$

$$\|f(x), \beta[f(x-1)-\alpha f(x-2)]+\alpha f(x-1)+\gamma^n[f(x-n)-(\alpha+\beta)f(x-n-1)+\alpha\beta f(x-n-2)]\|$$
$$\leq \sum_{r=0}^{n=1} \|f(x-r), \gamma^r\beta[f(x-r-1)-\alpha f(x-r-2)]+\gamma^{r+1}[f(x-r-1)-(\alpha+\beta)f(x-r-2)$$
$$+ \alpha\beta f(x-r-3)] + \gamma^r\alpha f(x-r-1)\| \leq \sum_{r=0}^{n-1} |\gamma^r|\epsilon, \tag{2.5}$$

$$\|f(x), \gamma[f(x-1)-\beta f(x-2)]+\gamma f(x-1)+\alpha^{-n}[f(x-n)-(\beta+\gamma)f(x-n-1)+\beta\gamma f(x-n-2)]\|$$
$$\leq \sum_{r=0}^{n=1} \|f(x+r), \alpha^{-r}\beta[f(x+r-1)-\gamma f(x-r-2)]+\alpha^{-r+1}\ [f(x+r-1)-(\beta+\gamma)f(x+r-2)$$
$$+ \beta\gamma f(x+r-3)] + \alpha^{-r}\beta f(x-r-1)\| \leq \sum_{r=0}^{n-1} |\alpha^{-r}|\epsilon \tag{2.6}$$

for all $x \in \mathbb{R}$ and all $r \in \mathbb{Z}$.

By (2.1), (2.2) and (2.3), we obtain that

$$\{\beta^n[f(x-r-1)-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)]\},$$
$$\{\gamma^n[f(x-r-1)-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)]\},$$
$$\{\alpha^{-n}[f(x+r-1)-(\beta+\gamma)f(x+r-2)+\beta\gamma f(x+r-3)]\}$$

are Cauchy sequences for any fixed $x \in \mathbb{R}$. Hence we can define the functions $G_1 : \mathbb{R} \to X$, $G_2 : \mathbb{R} \to X$ and $G_3 : \mathbb{R} \to X$ by

$$G_1 = \lim_{n\to\infty} \beta^n[f(x-r-1)-(\alpha+\gamma)f(x-r-2)+\alpha\gamma f(x-r-3)],$$
$$G_2 = \lim_{n\to\infty} \gamma^n[f(x-r-1)-(\alpha+\beta)f(x-r-2)+\alpha\beta f(x-r-3)],$$
$$G_3 = \lim_{n\to\infty} \alpha^{-n}\ [f(x+r-1)-(\beta+\gamma)f(x+r-2)+\beta\gamma f(x+r-3)]$$

for all $x \in \mathbb{R}$ and all $r \in \mathbb{Z}$. Using the above definition of G_1, G_2 and G_3, we show that there are Tribonacci functions

$$G_1(x-1) + G_1(x-2) + G_1(x-3)$$
$$= \beta^{-1} \lim_{n\to\infty} \beta^{n+1}[f(x-(n+1))-(\alpha+\gamma)f(x-(n+1)-1)+\alpha\gamma f(x-(n+1)-2)]$$
$$+\beta^{-2} \lim_{n\to\infty} \beta^{n+2}[f(x-(n+2))-(\alpha+\gamma)f(x-(n+2)-1)+\alpha\gamma f(x-(n+2)-2)]$$
$$+\beta^{-3} \lim_{n\to\infty} \beta^{n+3}[f(x-(n+3))-(\alpha+\gamma)f(x-(n+3)-1)+\alpha\gamma f(x-(n+3)-2)]$$
$$= \beta^{-1}G_1(x) + \beta^{-2}G_1(x) + \beta^{-3}G_1(x) = G_1(x),$$

$$G_2(x-1) + G_2(x-2) + G_2(x-3)$$
$$= \gamma^{-1} \lim_{n\to\infty} \gamma^{n+1}[f(x-(n+1))-(\alpha+\beta)f(x-(n+1)-1)+\alpha\beta f(x-(n+1)-2)]$$

$$+\gamma^{-2}\lim_{n\to\infty}\gamma^{n+2}[f(x-(n+2))-(\alpha+\beta)f(x-(n+2)-1)+\alpha\beta f(x-(n+2)-2)]$$

$$+\gamma^{-3}\lim_{n\to\infty}\gamma^{n+3}[f(x-(n+3))-(\alpha+\beta)f(x-(n+3)-1)+\alpha\beta f(x-(n+3)-2)]$$

$$=\gamma^{-1}G_2(x)+\gamma^{-2}G_2(x)+\gamma^{-3}G_2(x)=G_2(x),$$

$$G_3(x-1)+G_3(x-2)+G_3(x-3)$$

$$=\alpha^{-1}\lim_{n\to\infty}\alpha^{-n+1}\ [f(x-(n+1))-(\beta+\gamma)f(x-(n+1)-1)+\beta\gamma f(x-(n+1)-2)]$$

$$+\alpha^{-2}\lim_{n\to\infty}\alpha^{-n+2}\ [f(x-(n+2))-(\beta+\gamma)f(x-(n+2)-1)+\beta\gamma f(x-(n+2)-2)]$$

$$+\alpha^{-3}\lim_{n\to\infty}\alpha^{-n+3}\ [f(x-(n+3))-(\beta+\gamma)f(x-(n+3)-1)+\beta\gamma f(x-(n+3)-2)]$$

$$=\alpha^{-1}G_3(x)+\alpha^{-2}G_3(x)+\alpha^{-3}G_3(x)=G_3(x)$$

for all $x\in\mathbb{R}$. It follows from (2.4), (2.5) and (2.6) that

$$\|f(x),(\alpha+\gamma)f(x-1)-\alpha\gamma f(x-2)+G_2(x)\| \ \leq\ \frac{1}{1-|\beta|}\epsilon, \tag{2.7}$$

$$\|f(x),(\alpha+\beta)f(x-1)-\alpha\beta f(x-2)+G_2(x)\| \ \leq\ \frac{1}{1-|\gamma|}\epsilon=\frac{1}{1-|\beta|}\epsilon, \tag{2.8}$$

$$\|f(x),(\beta+\gamma)f(x-1)-\beta\gamma f(x-2)+G_3(x)\| \ \leq\ \frac{|\alpha^{-1}|}{1-|\alpha^{-1}|}\epsilon=\frac{|\beta^2|}{1-|\beta^2|}\epsilon \tag{2.9}$$

for all $x\in\mathbb{R}$. Now, put $\Delta=\alpha^2(\beta-\gamma)+\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)$, and define

$$G(x):=\frac{\beta^2(\gamma-\alpha)}{\Delta}G_1(x)+\frac{\gamma^2(\alpha-\beta)}{\Delta}G_2(x)+\frac{\alpha^2(\beta-\gamma)}{\Delta}G_3(x)$$

for all $x\in\mathbb{R}$. By (2.7), (2.8) and (2.9), we have

$$\begin{aligned}
&\|f(x),G(x)\|\\
&=\|f(x),\frac{\beta^2(\gamma-\alpha)}{\Delta}G_1(x)+\frac{\gamma^2(\alpha-\beta)}{\Delta}G_2(x)+\frac{\alpha^2(\beta-\gamma)}{\Delta}G_3(x)\|\\
&\leq\frac{1}{|\Delta|}[\|f(x),\beta^2(\gamma^2-\alpha^2)f(x-1)-\beta^2(\gamma-\alpha)\alpha\gamma f(x-2)+\beta^2(\gamma-\alpha)G_1\|\\
&\quad+\|f(x),\gamma^2(\alpha^2-\beta^2)f(x-1)-\gamma^2(\alpha-\beta)\alpha\beta f(x-2)+\gamma^2(\alpha-\beta)G_2\|\\
&\quad+\|f(x),\alpha^2(\beta^2-\gamma^2)f(x-1)-\alpha^2(\beta-\gamma)\beta\gamma f(x-2)+\alpha^2(\beta-\gamma)G_3\|]\\
&\leq\frac{1}{|\Delta|}[\frac{2}{1-|\beta|}+\frac{|\beta|^2}{1-|\beta|^2}]\epsilon\\
&\leq\frac{1}{|\Delta|}[\frac{2(1+|\beta|)+|\beta|^2}{1-|\beta|^2}]\epsilon
\end{aligned}$$

for all $x\in\mathbb{R}$.

On the other hand, it is easy to show that G is a Tribonacci function and this completes the proof. □

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References

[1] I. Chang, *Stability of higher ring derivations in fuzzy Banach algebras,* J. Computat. Anal. Appl. **14** (2012), 1059–1066.
[2] I. Cho, D. Kang, H. Koh, *Stability problems of cubic mappings with the fixed point alternative,* J. Computat. Anal. Appl. **14** (2012), 132–142.
[3] M. Eshaghi Gordji, M. Bavand Savadkouhi, M. Bidkham, *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **12** (2010), 454–462.
[4] M. Eshaghi Gordji, A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 724–729.
[5] M. Eshaghi Gordji, R. Farokhzad Rostami, S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras,* J. Computat. Anal. Appl. **13** (2011), 734–742.
[6] M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi, M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras,* J. Computat. Anal. Appl. **12** (2010), 463–470.
[7] M. Eshaghi Gordji, M. Naderi and W. Y. Lee, *Solution and stability of Tribonacci functional equation,* Discrete Dynamics in Nature and Society **2012**, Article ID 207356, 11 pages (2012).
[8] S. Gähler, 2-metriche Raume und ihre topologische struktur, Math. Nachr. **26** (1963), 115–148.
[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 221–224.
[10] H.A. Kenary, J. Lee, C. Park, *Non-Archimedean stability of an AQQ-functional equation,* J. Computat. Anal. Appl. **14** (2012), 211–227.
[11] C. Park, Y. Cho, H.A. Kenary, *Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **14** (2012), 526–535.
[12] C. Park, S. Jang, R. Saadati, *Fuzzy approximate of homomorphisms,* J. Computat. Anal. Appl. **14** (2012), 833–841.
[13] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 1097–1105.
[14] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Computat. Anal. Appl. **13** (2011), 1106–1114.
[15] S.M. Ulam, *Problems in Modern Mathematics,* Chapter VI, Science ed., Wiley, New York, 1940.

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An identity of the twisted q-Euler polynomials with weak weight α associated with the p-adic q-integrals on $\mathbb{Z}_p$

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Abstract : In [7], we studied the twisted q-Euler numbers and polynomials with weak weight α. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the twisted q-Euler polynomials with weak weight α.

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1. Introduction

The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics(see [1-12]). Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-6])} .$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1-5]} . \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.2}$$

Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$.

In [7], we defined the twisted q-Euler numbers and polynomials with weak weight α and investigate their properties. For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, and $\zeta \in T_p$, the twisted q-Euler polynomials $\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x)$ with weak weight α are defined by

$$\widetilde{F}^{(\alpha)}_{q,\zeta}(x,t) = \sum_{n=0}^{\infty} \widetilde{E}^{(\alpha)}_{n,q,\zeta}(x) \frac{t^n}{n!} = \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1} e^{xt}. \tag{1.3}$$

The twisted q-Euler numbers $\widetilde{E}_{n,q,\zeta}^{(\alpha)}$ with weak weight α are defined by the generating function:

$$\widetilde{F}_{q,\zeta}^{(\alpha)}(t)=\sum_{n=0}^{\infty}\widetilde{E}_{n,q,\zeta}^{(\alpha)}\frac{t^n}{n!}=\frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t+1}. \tag{1.4}$$

The following elementary properties of the q-Euler numbers $\widetilde{E}_{n,q,\zeta}^{(\alpha)}$ and polynomials $\widetilde{E}_{n,q,\zeta}^{(\alpha)}(x)$ with weak weight α are readily derived form (1.1), (1.2), (1.3) and (1.4) (see, for details, [7]). We, therefore, choose to omit details involved.

Theorem 1(Witt formula). For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p<1$, and $\zeta \in T_p$, we have

$$\widetilde{E}_{n,q,\zeta}^{(\alpha)}=\int_{\mathbb{Z}_p}\zeta^x x^n d\mu_{-q^\alpha}(x), \quad \widetilde{E}_{n,q,\zeta}^{(\alpha)}(x)=\int_{\mathbb{Z}_p}\zeta^y (x+y)^n d\mu_{-q^\alpha}(y).$$

Theorem 2. For any positive integer n, we have

$$\widetilde{E}_{n,q,\zeta}^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}\widetilde{E}_{k,q,\zeta}^{(\alpha)}x^{n-k}.$$

In this paper, by using the symmetry of p-adic q-integral on $\mathbb{Z}_p$, we obtain the recurrence identities the twisted q-Euler polynomials with weak weight α.

2. The alternating sums of powers of consecutive q-integers

Let q be a complex number with $|q|<1$ and ζ be the p^N-th root of unity. By using (1.3), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty}\widetilde{E}_{n,q,\zeta}^{(\alpha)}\frac{t^n}{n!}=\frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t+1}=[2]_{q^\alpha}\sum_{n=0}^{\infty}(-1)^n\zeta^n q^{\alpha n}e^{nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty}(-1)^n\zeta^n q^{\alpha n}e^{(n+k)t}+\sum_{n=0}^{\infty}(-1)^{n-k}\zeta^{n-k}q^{\alpha(n-k)}e^{nt}=\sum_{n=0}^{k-1}(-1)^{n-k}\zeta^{n-k}q^{\alpha(n-k)}e^{nt}.$$

Thus, we have

$$\begin{aligned}&-[2]_{q^\alpha}\sum_{n=0}^{\infty}(-1)^n\zeta^n q^{\alpha n}e^{(n+k)t}+[2]_{q^\alpha}(-1)^{-k}\zeta^{-k}q^{-\alpha k}\sum_{n=0}^{\infty}(-1)^n\zeta^n q^{\alpha n}e^{nt}\\&=[2]_{q^\alpha}(-1)^{-k}\zeta^{-k}q^{-\alpha k}\sum_{n=0}^{k-1}(-1)^n\zeta^n q^{\alpha n}e^{nt}.\end{aligned} \tag{2.1}$$

By using (1.3)and (1.4), and (2.1), we obtain

$$-\sum_{j=0}^{\infty}\widetilde{E}_{j,q,\zeta}^{(\alpha)}(k)\frac{t^j}{j!}+(-1)^{-k}\zeta^{-k}q^{-\alpha k}\sum_{j=0}^{\infty}\widetilde{E}_{j,q,\zeta}^{(\alpha)}\frac{t^j}{j!}=[2]_{q^\alpha}\sum_{j=0}^{\infty}\left((-1)^{-k}\zeta^{-k}q^{-\alpha k}\sum_{n=0}^{k-1}(-1)^n\zeta^n q^{\alpha n}n^j\right)\frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1}(-1)^n\zeta^n q^{\alpha n}n^j=\frac{(-1)^{k+1}\zeta^k q^{\alpha k}\widetilde{E}_{j,q,\zeta}^{(\alpha)}(k)+\widetilde{E}_{j,q,\zeta}^{(\alpha)}}{[2]_{q^\alpha}}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$\widetilde{T}^{(\alpha)}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1}(-1)^n \zeta^n q^{\alpha n} n^j = \frac{(-1)^{k+1}\zeta^k q^{\alpha k}\widetilde{E}^{(\alpha)}_{j,q,\zeta}(k) + \widetilde{E}^{(\alpha)}_{j,q,\zeta}}{[2]_{q^\alpha}}.$$

Remark 4. For $\zeta = 1$, we have

$$\lim_{q\to 1}\widetilde{T}^{(\alpha)}_{j,q,\zeta}(k-1) = \sum_{n=0}^{k-1}(-1)^n n^j = \frac{(-1)^{k+1}E_j(k) + E_j}{2},$$

where $E_j(x)$ and E_j denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We obtain recurrence identities the q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1}(-1)^{n-1-l} q^l g(l),$$

where $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1}(-1)^{n-1-l} q^l g(l) \text{ (cf. [1-5])}. \tag{2.2}$$

It will be more convenient to write (2.2) as the equivalent integral form

$$q^{\alpha n}\int_{\mathbb{Z}_p} g(x+n)d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-q^\alpha}(x) = [2]_{q^\alpha}\sum_{k=0}^{n-1}(-1)^k q^{\alpha k} g(k). \tag{2.3}$$

Substituting $g(x) = \zeta^x e^{xt}$ into the above, we obtain

$$\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) = [2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j \zeta^j q^{\alpha j} e^{jt}. \tag{2.4}$$

After some elementary calculations, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) &= \frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1},\\ \int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) &= e^{nt}\frac{[2]_{q^\alpha}}{\zeta q^\alpha e^t + 1}.\end{aligned} \tag{2.5}$$

By using (2.4) and (2.5), we have

$$\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p} \zeta^x e^{(x+n)t}d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} \zeta^x e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]_{q^\alpha}(1+\zeta^n q^{\alpha n} e^{nt})}{\zeta q^\alpha e^t + 1}.$$

From the above, we get

$$\frac{[2]_{q^\alpha}(1+\zeta^n q^{\alpha n} e^{nt})}{\zeta q^\alpha e^t + 1} = \frac{[2]_{q^\alpha}\int_{\mathbb{Z}_p}\zeta^x e^{xt}d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p}\zeta^{nx} q^{\alpha(n-1)x} e^{ntx}d\mu_{-q^\alpha}(x)}. \tag{2.6}$$

By substituting Taylor series of e^{xt} into (2.4), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\zeta^n q^{\alpha n}\int_{\mathbb{Z}_p}\zeta^x (x+n)^m d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)\right)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left([2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j \zeta^j q^{\alpha j} j^m\right)\frac{t^m}{m!}.\end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\zeta^n q^{\alpha n}\sum_{k=0}^{m}\binom{m}{k}n^{m-k}\int_{\mathbb{Z}_p}\zeta^x x^k d\mu_{-q^\alpha}(x)+\int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)=[2]_{q^\alpha}\sum_{j=0}^{n-1}(-1)^j\zeta^j q^{\alpha j}j^m.$$

By using Theorem 3, we have

$$\zeta^n q^{\alpha n}\sum_{k=0}^{m}\binom{m}{k}n^{m-k}\int_{\mathbb{Z}_p}\zeta^x x^k d\mu_{-q^\alpha}(x)+\int_{\mathbb{Z}_p}\zeta^x x^m d\mu_{-q^\alpha}(x)=[2]_{q^\alpha}\widetilde{T}^{(\alpha)}_{m,q,\zeta}(n-1). \tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we have

$$\frac{\int_{\mathbb{Z}_p}\zeta^x e^{xt}d\mu_{-q^\alpha}(x)}{\int_{\mathbb{Z}_p}\zeta^{nx}q^{\alpha(n-1)x}e^{ntx}d\mu_{-q^\alpha}(x)}=\sum_{m=0}^{\infty}\left(\widetilde{T}^{(\alpha)}_{m,q,\zeta}(n-1)\right)\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By (2.5), Theorem 5, and after some elementary calculations, we obtain the following theorem.

Theorem 6. Let w_1 and w_2 be odd positive integers. Then we have

$$\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x}e^{w_2xt}d\mu_{-q^{w_2\alpha}}(x)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2tx}d\mu_{-q^\alpha}(x)}=\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\left(\widetilde{T}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w-1)w_2^m\right)\frac{t^m}{m!}. \tag{2.8}$$

By (1.1), we obtain

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\zeta^{w_1x_1+w_2x_2}e^{(w_1x_1+w_2x_2+w_1w_2x)t}d\mu_{-q^{w_1\alpha}}(x_1)d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\\&=\frac{e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{w_1x_1t}d\mu_{-q^{w_1\alpha}}(x_1)\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{w_2x_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}.\end{aligned} \tag{2.9}$$

By using (2.8) and (2.9), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{(w_1x_1+w_1w_2x)t}d\mu_{-q^{w_1\alpha}}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}\widetilde{E}^{(\alpha)}_{m,q^{w_1},\zeta^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\widetilde{T}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.10}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.11}$$

By using the symmetry in (2.10), we obtain

$$\begin{aligned}a&=\left(\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{(w_2x_2+w_1w_2x)t}d\mu_{-q^{w_2\alpha}}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^\alpha}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}\widetilde{E}^{(\alpha)}_{m,q^{w_2},\zeta^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^\alpha}}\sum_{m=0}^{\infty}\widetilde{T}^{(\alpha)}_{m,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we obtain

$$a=\sum_{m=0}^{\infty}\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_2},\zeta^{w_2}}(w_1x)w_2^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \tag{2.12}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.11) and (2.12), we arrive at the following theorem.

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&[2]_{q^{w_2\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1)w_2^{m-j}\\&=[2]_{q^{w_1\alpha}}\sum_{j=0}^{m}\binom{m}{j}\widetilde{E}^{(\alpha)}_{j,q^{w_2},\zeta^{w_2}}(w_1x)w_2^j\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)w_1^{m-j},\end{aligned}$$

where $\widetilde{E}^{(\alpha)}_{k,q,\zeta}(x)$ and $\widetilde{T}^{(\alpha)}_{m,q,\zeta}(k)$ denote the twisted q-Euler polynomials with weak weight α and the q-analogue of alternating sums of powers of consecutive integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&[2]_{q^{w_1\alpha}}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j}x^{j-k}\widetilde{E}^{(\alpha)}_{k,q,\zeta^{w_2}}\widetilde{T}^{(\alpha)}_{m-j,q^{w_1},\zeta^{w_1}}(w_2-1)\\&=[2]_{q^{w_2\alpha}}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k}x^{j-k}\widetilde{E}^{(\alpha)}_{k,q,\zeta^{w_1}}\widetilde{T}^{(\alpha)}_{m-j,q^{w_2},\zeta^{w_2}}(w_1-1).\end{aligned}$$

By using (2.9), we have

$$\begin{aligned}a&=\left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^{\alpha}}(x)}\right)\\&=\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2\alpha j}\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{\left(x_1+w_2x+j\frac{w_2}{w_1}\right)(w_1t)}d\mu_{-q^{w_1\alpha}}(x_1)\\&=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_2\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.\end{aligned}\tag{2.13}$$

By using the symmetry property in (2.13), we also have

$$\begin{aligned}a&=\left(e^{w_1w_2xt}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{x_2w_2t}d\mu_{-q^{w_2\alpha}}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}\zeta^{w_1x_1}e^{x_1w_1t}d\mu_{-q^{w_1\alpha}}(x_1)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}q^{\alpha(w_1w_2-1)x}e^{w_1w_2xt}d\mu_{-q^{\alpha}}(x)}\right)\\&=\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1\alpha j}\int_{\mathbb{Z}_p}\zeta^{w_2x_2}e^{\left(x_2+w_1x+j\frac{w_1}{w_2}\right)(w_2t)}d\mu_{-q^{w_2\alpha}}(x_2)\\&=\sum_{n=0}^{\infty}\left(\frac{[2]_{q^{w_1\alpha}}}{[2]_{q^{\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}q^{w_1\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.\end{aligned}\tag{2.14}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.13) and (2.14), we have the following theorem.

Theorem 9. Let w_1 and w_2 be odd positive integers. Then we have

$$\begin{aligned}&[2]_{q^{w_2\alpha}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}q^{w_2\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_1},\zeta^{w_1}}\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n\\ =&[2]_{q^{w_1\alpha}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}q^{w_1\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{2.15}$$

Remark 10. Let w_1 and w_2 be odd positive integers. If $q\to 1$ and $\zeta=1$, we have

$$\sum_{j=0}^{w_1-1}(-1)^jE_n\left(w_2x+j\frac{w_2}{w_1}\right)w_1^n=\sum_{j=0}^{w_2-1}(-1)^jE_n\left(w_1x+j\frac{w_1}{w_2}\right)w_2^n.$$

Substituting $w_1=1$ into (2.15), we arrive at the following corollary.

Corollary 11. Let w_2 be odd positive integer. Then we obtain

$$\widetilde{E}^{(\alpha)}_{n,q,\zeta}(x)=\frac{[2]_{q^\alpha}}{[2]_{q^{w_2\alpha}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^jq^{\alpha j}\widetilde{E}^{(\alpha)}_{n,q^{w_2},\zeta^{w_2}}\left(\frac{x+j}{w_2}\right)w_2^n.$$

REFERENCES

1. T. Kim, q-Volkenborn integration, *Russ. J. Math. Phys.*, **9**(2002), 288-299.

2. T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals, *J. Nonlinear Math. Phys.*, **14**(2007), 15-27.

3. T. Kim, An identity of the symmetry for the Frobenius-Euler polynomials associated with the Fermionic p-adic invariant q-integral on $\mathbb{Z}_p$, *Rocky Mountain Journal of Mathematics,* **41**, (2011), 239-247.

4. S-H. Rim, T. Kim and C.S. Ryoo, On the alternating sums of powers of consecutive q-integers, *Bull. Korean Math. Soc.*, **43**(2006), 611-617.

5. T. Kim, J. Choi, Y-.H. Kim, C. S. Ryoo, A Note on the weighted p-adic q-Euler measure on $\mathbb{Z}_p$, *Advan. Stud. Contemp. Math.*, **21**(2011), 35-40.

6. C. S. Ryoo, T. Kim, L.-C. Jang, Some relationships between the analogs of Euler numbers and polynomials, *J. Inequal.Appl.*, **2007**(2007), Art ID 86052, 22pp.

7. C. S. Ryoo, A note on the twisted q-Euler numbers and polynomials with weak weight α, *Adv. Studied Theor. Phys.*, **6**(2012), 1109-1116.

8. C. S. Ryoo, A note on the weighted q-Euler numbers and polynomials, *Advan. Stud. Contemp. Math.*, **21**(2011), 47-54.

9. C.S. Ryoo, A numerical computation of the structure of the roots of the second kind q-Euler polynomials, *Journal of Computational Analysis and Applications*, **14**(2012), pp. 321-327.

10. C.S. Ryoo, Calculating zeros of the second kind Euler polynomials, *Journal of Computational Analysis and Applications*, **12**(2010), 828-833.

11. Y. Simsek, Theorem on twisted L-function and twisted Bernoulli numbers, *Advan. Stud. Contemp. Math.*, **12**(2006), 237-246.

12. Y. Simsek, O. Yurekli, V. Kurt, On interpolation functions of the twisted generalized Frobenius-Euler numbers, *Adv. Stud. Contemp. Math.*, **14**(2007), 49-68.

Two-Level Hierarchical Basis Preconditioner for Elliptic Equations with Jump Coefficients*

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Abstract. This paper provides a proof of robustness of the two-level hierarchical basis preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. As a result, we prove that the convergence rate of the conjugate gradient method with two-level preconditioner is uniform with respect to large jumps and mesh sizes.

Key words. Jump Coefficients, Conjugate Gradient, Effective Condition Number, Two-Level Hierarchical Basis

AMS subject classifications. 65N30, 65N55, 65F10

1 Introduction

In this paper, we will discuss the two-level hierarchical basis preconditioned conjugate gradient methods for the linear finite element approximation of the second order elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (\omega \nabla u) = f & in \quad \Omega \\ u = g_D & on \quad \Gamma_D \\ -\omega \dfrac{\partial u}{\partial n} = g_N & on \quad \Gamma_N \end{cases} \tag{1.1}$$

where $\Omega \in R^d (d = 1, 2 \text{ or } 3)$ is a polygonal or polyhedral domain with

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Dirichlet boundary Γ_D and Neumann boundary Γ_N. The coefficient $\omega = \omega(x)$ is a positive and piecewise constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral regions $\Omega_m(m = 1, \cdots, M)$ satisfying $\cup_{m=1}^M \overline{\Omega}_m = \overline{\Omega}$ with

$$\omega_m = \omega|_{\Omega_m}, m = 1, \cdots, M$$

where each $\omega_m > 0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subdomain.

We assume that the subdomain $\Omega_m : m = 1, \cdots, M$ are given and fixed but may possibly have complicated geometry. We are concerned with the robustness of the preconditioned conjugate gradient method in regard to both the meshsize and jump coefficients. This model problem is relevant to many applications, such as groundwater flow [1, 15], fluid pressure prediction [19], electromagnetics [13], semiconductor modeling [9], electrical power network modeling [14] and fuel cell modeling [20, 21], where the coefficients have large discontinuities across interfaces between subdomains with different material properties.

The goal of the current paper is to provide proof of the robustness of the two-level hierarchical basis preconditioner (Two-Level-PCG).

The rest of the paper is organized as follows. To the paper is comprehensive and self-contained, we refer directly to parts of contents in [23] and [25].(Section 2 in the paper). In Section 2, we introduce some basic notation, the PCG algorithm and some theoretical foundations. In Section 3, we introduce the two-level hierarchical basis method and preconditioner. In Section 4, we analyze the eigenvalue distribution of the two-level preconditioned system and prove the convergence rate of the PCG algorithm. Section 5 is the conclusions. Following [22], short notation $x \lesssim y$ means $x \leq Cy$; and $x \eqsim y$ means $cx \leq y \leq Cx$.

2 Preliminaries

2.1 Notation

We introduce the bilinear form

$$a(u, v) = \sum_{m=1}^{M} \omega_m (\nabla u, \nabla v)_{L^2(\Omega_m)}, \quad \forall u, v \in H_D^1(\Omega),$$

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where $H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$, and introduce the H^1-norm and seminorm with respect to any subregion Ω_m by

$$|u|_{1,\Omega_m} = \|\nabla u\|_{0,\Omega_m}, \quad \|u\|_{1,\Omega_m} = (\|u\|^2_{0,\Omega_m} + |u|^2_{1,\Omega_m})^{\frac{1}{2}}.$$

Thus,

$$a(u,u) = \sum_{m=1}^{M} \omega_m |u|^2_{1,\Omega_m} := |u|^2_{1,\omega}.$$

We also need the weighted L^2-inner product

$$(u,v)_{0,\omega} = \sum_{m=1}^{M} \omega_m (u,v)_{L^2(\Omega_m)}$$

and the weighted L^2- and H^1-norms

$$\|u\|_{0,\omega} = (u,u)^{\frac{1}{2}}_{0,\omega}, \quad \|u\|_{1,\omega} = (\|u\|^2_{0,\omega} + |u|^2_{1,\omega})^{\frac{1}{2}}.$$

For any subset $O \subset \Omega$, let $|u|_{1,\omega,O}$ and $\|u\|_{0,\omega,O}$ be the restrictions of $|u|_{1,\omega}$ and $\|u\|_{0,\omega}$ on the subset O, respectively.

For the distribution of the coefficients, we introduce the index set

$$I = \{m : meas(\partial\Omega_m \cap \Gamma_D) = 0\}$$

where meas$(\cdot)$ is the $d-1$ measure. In other words, I is the index set of all subregions which do not touch the Dirichlet boundary. We assume that the cardinality of I is m_0. We shall emphasize that m_0 is a constant which depends only on the distribution of the coefficients.

2.2 The Discrete Systems

Given a quasi-uniform triangulation $\mathcal{T}_h$ with the meshsize h, let

$$\mathcal{V}_h = \{v \in H^1_D(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_h\}$$

be the piecewise linear finite element space, where $\mathcal{P}_1$ denotes the set of linear polynomials. The finite element approximation of (1.1) is the function $u \in \mathcal{V}_h$, such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \quad \forall v \in \mathcal{V}_h.$$

We define a linear symmetric positive definite operator $A : \mathcal{V}_h \to \mathcal{V}_h$ by

$$(Au, v)_{0,\omega} = a(u, v).$$

The related inner product and the induced energy norm are denoted by

$$(\cdot, \cdot)_A := a(\cdot, \cdot), \quad \| \cdot \|_A := \sqrt{a(\cdot, \cdot)}.$$

Then we have the following operator equation,

$$Au = F. \tag{2.1}$$

2.3 Preconditioned Conjugate Gradient (PCG) Methods

The well known conjugate gradient method is the basis of all the preconditioning techniques to be studied in this paper. The PCG methods can be viewed as a conjugate gradient method applied to the preconditioned system

$$BAu = BF.$$

Here, B is an SPD operator, known as a preconditioner of A. Note that BA is symmetric with respect to the inner product $(\cdot, \cdot)_{B^{-1}}$ (or $(\cdot, \cdot)_A$). For the implementation of the PCG algorithm, we refer to the monographs [2, 17, 18].

Let $u_k, k = 0, 1, \cdots$, be the solution sequence of the PCG algorithm. It is well known that

$$\|u - u_k\|_A \leq 2 \left(\frac{\sqrt{k(BA)} - 1}{\sqrt{k(BA)} + 1} \right)^k \|u - u_0\|_A, \tag{2.2}$$

which implies that the PCG method generally converges faster with a smaller condition number $k(BA)$.

Even though the estimate given in (2.2) is sufficient for many applications, in general it is not sharp. One way to improve the estimate is to look at the eigenvalue distribution of BA(see [2, 12] for more details). More specifically, suppose that we can divide $\sigma(BA)$, the spectrum of BA, into two sets, $\sigma_0(BA)$ and $\sigma_1(BA)$, where σ_0 consists of all "bad" eigenvalues and the remaining eigenvalues in σ_1 are bounded above and below, then we have the following theorem.

Theorem 2.1 *Suppose that $\sigma(BA) = \sigma_0(BA) \cup \sigma_1(BA)$ such that there are m elements in $\sigma_0(BA)$ and $\lambda \in [a, b]$ for each $\lambda \in \sigma_1(BA)$. Then*

$$\|u - u_k\|_A \leq 2K \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m} \|u - u_0\|_A, \tag{2.3}$$

where

$$K = \max_{\lambda \in \sigma_1(BA)} \prod_{\mu \in \sigma_0(BA)} \left| 1 - \frac{\lambda}{\mu} \right|.$$

If there are only m small eigenvalues in σ_0, say

$$0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_m \ll \lambda_{m+1} \leq \cdots \leq \lambda_n,$$

then

$$K = \prod_{i=1}^{m} \left| 1 - \frac{\lambda_n}{\lambda_i} \right| \leq \left(\frac{\lambda_n}{\lambda_1} - 1 \right)^m = (k(BA) - 1)^m.$$

In this case, the convergence rate estimate (2.3) becomes

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \leq 2(k(BA) - 1)^m \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m}. \qquad (2.4)$$

Based on (2.4), given a tolerance $0 < \epsilon < 1$, the number of iterations of the PCG algorithm needed for $\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$ is given by

$$k \geq m + \left(log\left(\frac{2}{\epsilon}\right) + mlog(k(BA) - 1) \right) / log\left(\frac{\sqrt{b/a} + 1}{\sqrt{b/a} - 1} \right). \qquad (2.5)$$

Observing the convergent estimate (2.4), if there are only a few small eigenvalues of BA in $\sigma_0(BA)$, then the convergent rate of the PCG methods will be dominated by the factor $\frac{\sqrt{b/a}+1}{\sqrt{b/a}-1}$, i.e., by b/a where $b = \lambda_n(BA)$ and $a = \lambda_{m+1}(BA)$. We define this quantity as the "effective condition number".

Definition. ([23]) Let $\mathcal{V}$ be a Hilbert space. The m-th effective condition number of an operator $A : \mathcal{V} \to \mathcal{V}$ is defined by

$$k_{m+1}(A) = \frac{\lambda_{max}(A)}{\lambda_{m+1}(A)}$$

where $\lambda_{m+1}(A)$ is the (m+1)-th minimal eigenvalue of A.

To estimate the effective condition number, we need to estimate $\lambda_{m+1}(A)$. A fundamental tool is the following *Courant-Fisher min-max theorem.*

Theorem 2.2 *The eigenvalues of a SPD operator $A : \mathcal{V} \to \mathcal{V}$ are characterized by the relation*

$$\lambda_{m+1}(A) = \min_{S, dim(S) = n-m.} \max_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}. \qquad (2.6)$$

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Especially, for any subspace $\mathcal{V}_0 \subset \mathcal{V}$ with $dim(\mathcal{V}_0) = n - m$, the following estimation of $\lambda_{m+1}(A)$ holds:

$$\lambda_{m+1}(A) \geq \min_{0 \neq x \in \mathcal{V}_0} \frac{(Ax, x)}{(x, x)}. \qquad (2.7)$$

3 Two-Level Hierarchical Basis Preconditioner

The classical two-level hierarchical basis method was proposed and developed by Axelsson, Bank, Dupont, and Yserentant [3, 4, 5, 6, 24]. As usual, we assume that V is decomposed as a direct sum

$$V = SV_s \oplus PV_c. \qquad (3.1)$$

for some components V_s and V_c isomorphic to $\mathbb{R}^{n_s}$ and $\mathbb{R}^{n_c}$ respectively, with $n = n_s + n_c$. A typical and simple example to keep in mind is $S = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $P = \begin{pmatrix} W \\ I \end{pmatrix}$ for some W such that the square matrix (S, P) is unit upper triangular, and hence invertible.

3.1 Some notation

Two ingredients (the space decomposition (3.1) and the smoother M) are important in the two-level hierarchical basis method. Various restrictions of M and A to the subspaces mentioned before will be needed. We first define the exact coarse grid matrix A_c and its hierarchical complement A_s as follows

$$A_c = P^T AP, \quad A_s = S^T AS.$$

Later we will see, in the case of a two level hierarchical basis preconditioner, one needs M to be well-defined only on the first component SV_s. In that case, we refer to M as M_s. Then

$$M_s = S^T MS.$$

In order to define the hierarchical basis preconditioner, we also need two symmetrized version of the smoother M:

$$\widetilde{M} = M^T(M^T + M - A)^{-1}M, \qquad (3.2)$$

$$\overline{M} = M(M^T + M - A)^{-1}M^T. \qquad (3.3)$$

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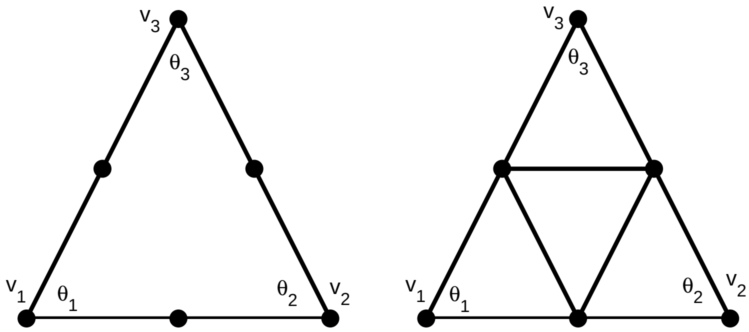

Figure 1: Quadratic element(left) and piecewise linear element(right).

If we assume that $A = D - L - L^T$ where D, L, L^T are the diagonal, lower triangle, and upper triangle part of A, and let $M = D - L^T$, then

$$\widetilde{M_s}^{-1} = (D - L^T)_s^{-1} D_s (D - L)_s^{-1}, \tag{3.4}$$

where $D_s = S^T DS$, $(D-L^T)_s = S^T(D-L^T)S$, and $(D-L)_s = S^T(D-L)S$.

3.2 The Element Stiffness Matrix for The Hierarchical Basis

In this subsection, we consider the stiffness matrix for the hierarchical basis in each element. Following Braess [7], and Bank [6], simply we let $\omega = 1$ in (1.1) and let t be a triangle with vertices v_i, edges e_i, and angles θ_i, $1 \leq i \leq 3$. Here, we consider two kinds of different hierarchical basis: the quadratic element and piecewise linear element.

For the space of continuous quadratic finite elements (illustrated on the left in Figure 1), we let $\phi_i, 1 \leq i \leq 3$ denote the linear basis functions for element t. Then on element t, the subspace PV_c will be the span of $\langle\phi_i\rangle_{i=1}^3$.

And the subspace SV_s is composed of the quadratic bump functions $\langle\psi_i\rangle_{i=1}^3$, where $\psi_i = 4\phi_j\phi_k$, and (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

For the space of continuous piecewise linear polynomials on a refined mesh (illustrated on the right in Figure 1), let PV_c be defined as the quadratic finite elements above. But the subspace SV_s contains the continuous piecewise polynomials on the fine grid that are zero at the vertices of t. Then on element t, the subspace $SV_s = \langle\widehat{\phi_i}\rangle_{i=1}^3$, where $\widehat{\phi_i}$ is the standard nodal piecewise linear basis functions associated with the midpoint of edge e_i of t.

Following [6] and [7], we can establish the relation

$$L_i = cot\theta_i = -2|t|\nabla\phi_j \cdot \nabla\phi_k, \tag{3.5}$$

where $|t|$ is measure of element t, it is about $h^d, d = 1, 2, 3$.

Then the element stiffness matrix for the quadratic hierarchical basis can be shown to be

$$A_Q^t = \begin{pmatrix} * & * \\ * & A_s^t \end{pmatrix}, \tag{3.6}$$

where A_s^t is the restriction of A_s on the element t, and

$$A_s^t = \frac{4}{3}\begin{pmatrix} L_1+L_2+L_3 & -L_3 & -L_2 \\ -L_3 & L_1+L_2+L_3 & -L_1 \\ -L_2 & -L_1 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.7}$$

The diagonal of A_s^t is

$$D_s^t = \frac{4}{3}\begin{pmatrix} L_1+L_2+L_3 & 0 & 0 \\ 0 & L_1+L_2+L_3 & 0 \\ 0 & 0 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.8}$$

The element stiffness matrix for the piecewise linear hierarchical basis is given by

$$A_L^t = \begin{pmatrix} * & * \\ * & A_s^t \end{pmatrix}. \tag{3.9}$$

In this case,

$$A_s^t = \begin{pmatrix} L_1+L_2+L_3 & -L_3 & -L_2 \\ -L_3 & L_1+L_2+L_3 & -L_1 \\ -L_2 & -L_1 & L_1+L_2+L_3 \end{pmatrix}, \tag{3.10}$$

and

$$D_s^t = \begin{pmatrix} L_1+L_2+L_3 & 0 & 0 \\ 0 & L_1+L_2+L_3 & 0 \\ 0 & 0 & L_1+L_2+L_3 \end{pmatrix}. \tag{3.11}$$

3.3 The Two-Level Hierarchical Basis Preconditioner

In this subsection, we can define the two-level hierarchical basis preconditioner using some notations in above subsections. Let

$$\widehat{B}_{TL} = \begin{pmatrix} I & 0 \\ P^T ASM_s^{-1} & I \end{pmatrix} \begin{pmatrix} \overline{M}_s & 0 \\ 0 & A_c \end{pmatrix} \begin{pmatrix} I & M_s^{-T} S^T AP \\ 0 & I \end{pmatrix}. \quad (3.12)$$

Then, the two-level hierarchical basis preconditioner is defined by

$$B_{TL}^{-1} = \begin{pmatrix} S, P \end{pmatrix} \widehat{B}_{TL}^{-1} \begin{pmatrix} S, P \end{pmatrix}^T. \quad (3.13)$$

4 The Condition Number Analysis of $B_{TL}^{-1}A$

Following [11], for the two-level hierarchical basis preconditioner B_{TL}^{-1}, we have following estimate.

Lemma 4.1 *Assume that $(M_s + M_s^T - A_s)$ is S.P.D, for any $v \in V$, the following bounds hold:*

$$\frac{1}{K} v^T B_{TL} v \le v^T A v \le v^T B_{TL} v, \quad K \lesssim \sup_w \frac{1}{\lambda(\widetilde{M}_s^{-1} A_s)}. \quad (4.1)$$

If $\widetilde{M}_s^{-1}$ is given by (3.4), then we have the following relationship between the symmetric Gauss-Seidel preconditioner and the Jacobi preconditioner.

Lemma 4.2 *For any $v \in V$, we have*

$$\frac{1}{4} v^T D_s v \le v^T \widetilde{M}_s v \le v^T D_s v. \quad (4.2)$$

Proof:

$$\widetilde{M}_s = (D-L)_s D_s^{-1} (D-L^T)_s.$$

Following the Schwarz inequality we can prove the second inequality. Then, we prove the first inequality.

Using the fact that D_s and A_s are S.P.D, then for any $v \in V$ we have

$$((D-L)_s v, v)_A = \frac{1}{2}((A_s + D_s)v, v)_A \ge \frac{1}{2}(D_s v, v)_A.$$

Taking $v = (D-L^T)_s^{-1} w$, we have for all $w \in V$:

$$\frac{1}{2}(D_s(D-L)_s^{-T} w, (D-L)_s^{-T} w)_A \le ((D-L)_s^{-1} w, w)_A = (D_s(D-L)_s^{-T} w, D_s^{-1} w)_A.$$

On the other hand,

$$\frac{1}{2}(\widetilde{M}_s^{-1}w, w)_A \leq (\widetilde{M}_s^{-1}w, w)_A^{1/2}(D_s^{-1}w, w)_A^{1/2}.$$

Consequently,

$$(\widetilde{M}_s^{-1}w, w)_A \leq 4(D_s^{-1}w, w)_A,$$

and

$$(\widetilde{M}_s v, v)_A \geq \frac{1}{4}(D_s v, v)_A.$$

The proof of Lemma 4.2 can be found in [26].

Following lemma provides the eigenvalue estimate of Jacobi preconditioner.

Lemma 4.3 *For any $v \in V$, we have*

$$v^T D_s v \eqsim h^{-2}\|v\|_{0,\omega}^2.$$

Proof: Note that on each element, we have

$$\sum_{i=1}^{3} L_i = -2h^d(\nabla\phi_1\nabla\phi_2 + \nabla\phi_1\nabla\phi_3 + \nabla\phi_2\nabla\phi_3) \eqsim h^{d-2}.$$

Consequently, following [23] we have

$$v^T D_s v \eqsim h^{d-2}(v, v)_{l^2,\omega} \eqsim h^{-2}\|v\|_{0,\omega}^2.$$

This completes the proof.

In order to research the effective condition number for the Jacobi preconditioner, we need to define the space

$$\widetilde{\mathcal{V}}_h = \left\{ v \in \mathcal{V}_h : \int_{\Omega_m} v = 0, m \in I \right\}.$$

On this space, the following Poincare-Friedrichs inequality holds:

$$\|v\|_{0,\omega} \lesssim \|\nabla v\|_{0,\omega}, \quad \forall v \in \widetilde{\mathcal{V}}_h. \tag{4.3}$$

Then, we have following important lemma.

Lemma 4.4 *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, then we have*

$$h^2 J(\omega)^{-1} v^T D_s v \lesssim v^T A_s v, \quad \forall v \in \mathcal{R}^n, \tag{4.4}$$

and

$$h^2 v^T D_s v \lesssim v^T A_s v, \quad \forall v \in \widetilde{\mathcal{V}}_h. \tag{4.5}$$

where $J(\omega) = \frac{\max_m \omega_m}{\min_m \omega_m}$.

Proof: In fact, we have

$$a(v,v) \geq \min_m\{\omega_m\}|v|^2_{1,\Omega} \gtrsim \min_m\{\omega_m\}\|v\|^2_{0,\Omega} \geq \frac{\min_m\{\omega_m\}}{\max_m\{\omega_m\}}h^{-2}(h^2)\|v\|^2_{0,\Omega}.$$

Applying Lemma 4.3 and inequality (4.3), we also have

$$v^T D_s v \lesssim \sum_{\tau\in\mathcal{T}_h} h^{-2}\|v\|^2_{0,\omega,\tau} = h^{-2}\sum_{m=1}^{M}\omega_m\|v\|^2_{0,\Omega_m} \lesssim h^{-2}|v|^2_{1,\omega} = h^{-2}v^T A_s v.$$

This completes the proof. Followed by Lemmas 4.1-4.4, we have the following results regarding the condition number of $B_{TL}^{-1}A$.

Theorem 4.1 *For the hierarchical basis preconditioner B_{TL}^{-1} defined by (3.13), the condition number and m_0-th effective condition number satisfies:*

$$k(B_{TL}^{-1}A) \leq J(\omega)h^{-2}, \quad k_{m_0+1}(B_{TL}^{-1}A) \leq h^{-2}.$$

Theorem 4.2 *For the hierarchical basis preconditioned conjugate gradient methods, we have the following convergence rate*

$$\frac{\|u-u_k\|_A}{\|u-u_0\|_A} \leq 2(C_1J(\omega)h^{-2}-1)^{m_0}\left(1-\frac{2}{1+C_2h^{-1}}\right)^{k-m_0}, k \geq m_0. \quad (4.6)$$

5 Conclusions

In this paper, we provided a proof of robustness of the two-level hierarchical basis preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. We discussed the eigenvalue distribution of the Two-Level-preconditioner and found that only a few small eigenvalues infected by the large jump.

References

[1] R.E. Alcouffe, A. Brandt, J.E. Dendy, Jr., and J.W. Painter, The multigrid method for the diffusion equation with strongly discontinuous coefficients, *SIAM J. Sci. Stat. Comput.* **2(4)** (1981), 430-454.

[2] O. Axelsson, Iterative solution methods, *Cambridge University Press, Cambridge*, 1994.

[3] O. Axelsson and I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximation, *Mathematics of Computations.* **40** (1983), 219-242.

[4] R. E. Bank and T. Dupont, Analysis of a two-level scheme for solving finite element equations, *Report CNA-159, Center for Numerical Analysis, The University of Texas at Austin, Austin, TX*, 1980.

[5] R. E. Bank, T. Dupont, and H. Yserentant, The hierarchical basis multigrid method, *Numerische Mathematik.* **52** (1988), 427-458.

[6] R. E. Bank, Hierarchical bases and the finite element method, *vol. 5 of Acta Numerica, Cambridge University Press, Cambridge.* (1996), 1-43.

[7] D. Braess, The contraction number of a multigrid method for solving Poisson equation, *Numerische Mathematik.* **37** (1981), 387-404.

[8] T.F. Chan and W. Wan, Robust multigrid methods for nonsmooth coefficient elliptic linear systems, *J. Comput. Appl. Math*, **123** (2000), 323-352.

[9] R.K. Coomer, and I.G. Graham, Massively parallel mehtods for semiconductor device modelling, *Computing*, **56(1)** (1996), 1-27.

[10] M. Dryja, M.V. Sarkis, and O.B. Widlund, Multilevel schwarz methods for elliptic problems with discontinous coefficients in three dimensions, *Numerische Mathematik*, **72(3)** (1996), 313-348.

[11] R. D. Falgout, P. S. Vassilevski, and L. T. Zikatanov, On two-grid convergence estimates, *Numer. Linear Algebra Appl.* **12(5-6)** (2005), 471 C 494.

[12] I.G. Graham and M.J. Hagger, Unstructured additive schwarz conjugate gradient method for elliptic problems with discontinuous coefficients, *SIAM J. SCI. Comput*, **20** (1999), 2041-2066.

[13] B. Heise, and M. Kuhn, Parallel solvers for linear and nonlinear exterior magnetic field problems based upon coupled FE/BE formulations, *Computing*, **56(3)** (1996), 237-258. International GAMM-Workshop on Multi-level Methods (Meisdorf, 1994).

[14] V. Howle, and S. Vavasis, Preconditioning complex-symmetric layered systems arising in electric power modeling, *Preconditioning complex-symmetric layered systems arising in electric power modeling. In T.*

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A. Manteuffel and S. F. McCormick, editors, Fifth Copper Mountain Conference on Iterative Methods, Copper Mountain, Colorado, 1998.

[15] C. E. Kees, C. T. Miller, E. W. Jenkins, and C. T. Kelley, Versatile two-level Schwarz preconditioners for multiphase flow, *Comput. Geosci*, **7(2)**, (2003), 91-114.

[16] J. Mandel, and M. Brezina, Balancing domain decomposition for problems with large jumps in coefficients, *Math. Comp*, **65(216)**, (1996), 1387-1401.

[17] P. Oswald, On the robustness of the BPX-preconditioner with respect to jumps in the coefficients, *Math. Comp*, **68** (1999), 633-650.

[18] Y. Saad, Iterative methods for sparse linear systems, *SIAM, Philadelphia*,**2nd**, 2003.

[19] C. Vuik, A. Segal, and J. A. Meijerink, An efficient preconditioned cg method for the solution of a class of layered problems with extreme contrasts in the coefficients, *Journal of Computational Physics*, **152(1)** (1999),385-403.

[20] C. Wang, Fundamental models for fuel cell engineering, *an invited review article for Chemical Reviews*, **104** (2004),4727-4766.

[21] Z. Wang, C. Wang, and K. Chen, Two phase flow and transport in the air cathode of proton exchange membrane fuel cells, *J.Power Sources*, **94** (2001),40-50.

[22] J. Xu, Iterative methods by space decomposition and subspace correction, *SIAM Review*, **34** (1992), 581-613.

[23] J. Xu and Y. Zhu, Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients, *M3AS*,**18(1)** (2008), 77-106.

[24] H. Yserentant, On the multilevel splitting of finite element spaces, *Numerische Mathematik*. **49** (1986), 379-412.

[25] Y. Zhu, Domain decomposition preconditioners for elliptic equations with jump coefficients, *Numer. Linear Algebra Appl*, **15** (2008), 271-289.

[26] L. Zikatanov, Two-sided bounds on the convergence rate of two-level methods, *Numer. Linear Algebra Appl*, **15** (2008), 439-454.

A New Fourth-Order Explicit Finite Difference Method for the Solution of Parabolic Partial Differential Equation with Nonlocal Boundary Conditions

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Abstract

In this paper, a new fourth-order explicit finite difference method is proposed for solving linear and nonhomogeneous parabolic partial differential equation with nonlocal boundary conditions. The advantage of the explicit finite difference methods is easier to implement than the implicit methods. Moreover, the explicit method need lesser CPU time than the implicit schemes. Numerical results show that the proposed method is very accurate and effective.

Key words: Finite difference method, Fourth-order explicit method, Nonlocal boundary conditions, nonhomogeneous parabolic partial differential equation.

1 Introduction

Many physical phenomena can be modelled by parabolic partial differential equations which involve integral terms in the boundary conditions. These boundary conditions are called nonlocal boundary conditions. One-dimensional parabolic equation with nonlocal boundary conditions have important applications in electro-chemistry, porous media flow, thermo-elasticity, heat conduction and several others. The existence, uniqueness and theoretical aspects of these equations have been studied by [17, 20, 35]. Generally, it is difficult to find the analytical solution of parabolic partial differential equations with nonlocal boundary conditions.

Approximate and numerical techniques for obtaining approximate solution of these equations have been developed by many researchers [5, 6, 7, 8, 9, 11, 19, 23, 27, 29, 28]. Some standard numerical methods have been used for the solution of one dimensional diffusion equation with nonlocal boundary conditions such as finite difference method, finite element method, adomian decomposition method (ADM), Chebyshev spectral collocation method, reducing kernel space method and method of lines [1, 13, 21, 24, 25, 26, 30].

In this paper a method based on explicit finite difference method is introduced and applied to obtain the numerical solution of the following parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \qquad 0 \le x \le 1 \quad , \quad 0 \le t \le T, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{1.2}$$

and subject to the boundary conditions

$$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_1(t), \qquad 0 < t \le T, \tag{1.3}$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t), \qquad 0 < t \le T, \tag{1.4}$$

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where $q(x,t)$, $f(x)$, $g_1(t)$, $g_2(t)$, $\phi(x,t)$ and $\psi(x,t)$ are known functions.

Many authors applied various type of finite difference methods to obtain the numerical solution of equations (1.1)–(1.4). Dehghan [15, 16] applied the forward Euler, backward Euler, BTCS (backward in time and centered in space) schemes, Crandall's implicit formula, FTCS (forward in time and centered in space) method for the heat equation. The nonlocal boundary conditions have been approximated by Trapezoidal rule and fourth-order Simpson composite formula. Zhou et al. [36], Mu and Du [25] introduced an efficient technique based on reproducing kernel space to solve the partial differential equations with nonlocal boundary conditions. The BTCS and explicit Crandall's formula have been developed by Martin-Vaquero and Vigo-Aguiar [32, 31] to solve the above mentioned equations .

The aim of this paper is to describe an efficient technique based on explicit finite difference method to find out the numerical solution of parabolic equation with nonlocal boundary conditions. The new method is of fourth order and it is compared with BTCS, Crank-Nicolson and Crandall's formula. The basic idea of this approach is to write $q(x,t)$ as a linear combination of q_{i-1}^{n+1}, q_i^{n+1}, q_{i+1}^{n+1}, q_{i-1}^n, q_i^n and q_{i+1}^n. The objective of this technique is to improve the results obtained by many researchers in some papers [3, 10, 16, 26, 31, 32, 36]. We considered that coefficient q_{i+1}^{n+1}, q_{i-1}^{n+1} and q_{i-1}^n, q_{i+1}^n are not equal. The nonlocal boundary conditions are solved by higher order Integration rules.

This paper is organized as follows: In section 2, the new fourth-order explicit technique is presented, the composite Simpson rule and sixth-order formula for the nonlocal boundary conditions are also introduced. Numerical results are presented in section 3. Finally conclusion is given in section 4.

2 Explicit Finite Difference Method

The domain $[0,1]\times[0,T]$ is divided into an $M\times N$ mesh with a spatial size of $h=1/M$ and temporal size $k=T/N$. The grid points (x_i,t_n) are defined by

$$\begin{aligned} x_i &= ih, \qquad & i &= 0,1,...,M,\\ t_n &= nk, \qquad & n &= 0,1,...,N, \end{aligned}$$

where M and N are integers. The notation u_i^n, q_i^n, ϕ_i^n, ψ_i^n, g_1^n and g_2^n represents, respectively, the finite difference approximations of $u(x_i,t_n)$, $q(x_i,t_n)$, $\phi(x_i,t_n)$, $\psi(x_i,t_n)$, $g_1(t_n)$ and $g_2(t_n)$. The FTCS (forward in time and centrad in space) finite difference scheme for the heat equation (1.1) can be written as

$$u_i^{n+1} = ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n + kq_i^n, \tag{2.1}$$

for $i=1,...,M-1$, $n=0,1,...,N-1$ and $r=\frac{k}{h^2}$. The stability condition for this method is proved in [16]:

$$r \le \frac{1}{2}.$$

The local truncation error of this method can be written as [31, 33]:

$$\tau = (u_t - u_{xx} - q) + \frac{6ru_{tt} - u_{xxxx}}{12}h^2 + \frac{60r^2u_{ttt} - u_{xxxxxx}}{360}h^4 + O(h^6). \tag{2.2}$$

Now it can be verified that

$$u_{tt} = u_{xxxx} + q_{xx} + q_t. \tag{2.3}$$

By substituting (2.3) into (2.2) the truncation error can be obtained as

$$\tau = \frac{6rq_t + 6rq_{xx} + (6r-1)u_{xxxx}}{12}h^2 + \frac{60r^2u_{ttt} - u_{xxxxxx}}{360}h^4 + O(h^6). \tag{2.4}$$

It is clear that (2.4) is second order. Now, we write q_i^n using a linear combination of q_{i-1}^{n+1}, q_i^{n+1}, q_{i+1}^{n+1}, q_{i-1}^n, q_i^n and q_{i+1}^n, then we have

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + a_1q_{i-1}^{n+1} + a_2q_i^{n+1} + a_3q_{i+1}^{n+1} + a_4q_{i-1}^n + a_5q_i^n + a_6q_{i+1}^n. \tag{2.5}$$

By using the Taylor's expansion in (2.5), we can obtain

$$\begin{aligned}\left(\sum_{i=1}^{6} a_i - 1\right) + h(a_3 - a_1 - a_4 + a_6)q_x + rh^2(a_1 + a_2 + a_3)q_t + \frac{h^2}{2}(a_1 + a_3 + \\ a_4 + a_6)q_{xx} - \frac{h^3}{6}(a_6 - a_4)q_{xxx} + rh^3(a_3 - a_1)q_{xt} + A(u(x,t), q)h^4 + O(h^6) = 0.\end{aligned} \tag{2.6}$$

where

$$A(u(x,t), q) = \frac{-15q_{tt} + 10u_{tt} - 15q_{xxxx} - 6u_{xxxxxx} + (60 - 180a_5)q_{xxt}}{2160}. \tag{2.7}$$

By substituting (2.4), (2.6) and (2.7) into (2.5), we get the following system of linear equations

$$\begin{aligned}&a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1,\\ &a_3 - a_1 - a_4 + a_6 = 0,\\ &a_1 + a_2 + a_3 = \frac{1}{2},\\ &a_6 - a_4 = 0,\\ &a_3 - a_1 = 0,\\ &a_1 + a_3 + a_4 + a_6 = r,\end{aligned} \tag{2.8}$$

and $r = 1/6$. By selecting $a_5 = m$ thus equation (2.5) can be written as

$$\begin{aligned}u_i^{n+1} = ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n + \frac{k}{12}[(6m-2)(q_{i-1}^{n+1} + q_{i+1}^{n+1})\\ + (10 - 12m)q_i^{n+1} - (6m-3)(q_{i-1}^n + q_{i+1}^n) + 12mq_i^n].\end{aligned} \tag{2.9}$$

It should be noted that this technique is fourth order accurate when $r = 1/6$. By considering (2.9), we can consider many values for m such that the solution of these equation become converges to the exact solution. Our goal in this paper is to improve the results obtained in the literature. To find the optimal value of m, we can apply the following algorithm.

1. Step 1: We consider, $\quad m_1 = \dfrac{a_1}{b_1}, \quad m_2 = \dfrac{a_2}{b_2}$
2. Step 2: We calculate, E_{m_1} and E_{m_2}
3. Step 3: If $E_{m_1} < E_{m_2}$ then, $m_2 = \dfrac{a_1 + a_2}{b_1 + b_2}, \quad$ else $m_1 = \dfrac{a_1 + a_2}{b_1 + b_2}$
4. Step 4: If $|E_{m_i}| < l$ then $m = m_i$ is optimal $i = 1$ or 2, else we repeat Step 1.

Equation (2.9) has $M-1$ linear equations and $M+1$ unknowns. Thus two more equations are needed. The integral in the boundary conditions can be approximated by composite Simpson rule and sixth order formula.

2.1 Composite Simpson formula

The Simpson composite formula for solving the nonlocal boundary conditions (1.3) and (1.4) can be written as [16, 31]:

$$\begin{aligned}u_0^{n+1} = \int_0^1 \phi(x, t^{n+1})u(x, t^{n+1})dx + g_1^{n+1} = \frac{h}{3}\left(\phi_0^{n+1}u_0^{n+1}\right.\\ \left. + 4\sum_{i=1}^{M/2} \phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2\sum_{i=1}^{M/2-1} \phi_{2i}^{n+1}u_{2i}^{n+1} + \phi_M^{n+1}u_M^{n+1}\right) + g_1^{n+1} + O(h^4),\end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
u_M^{n+1} = \int_0^1 \psi(x,t^{n+1})u(x,t^{n+1})dx + g_2^{n+1} = \frac{h}{3}(\psi_0^{n+1}u_0^{n+1} \\
+ 4\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1} + \psi_M^{n+1}u_M^{n+1}) + g_2^{n+1} + O(h^4).
\end{aligned}
\tag{2.11}
$$

Thus

$$
(h\phi_0^{n+1}-3)u_0^{n+1} + 4h\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2h\sum_{i=1}^{M/2-1}\phi_{2i}^{n+1}u_{2i}^{n+1} + h\phi_M^{n+1}u_M^{n+1} = -3g_1^{n+1}, \tag{2.12}
$$

$$
h\psi_0^{n+1}u_0^{n+1} + 4h\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 2h\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1} + (h\psi_M^{n+1}-3)u_M^{n+1} = -3g_1^{n+1}, \tag{2.13}
$$

Combining (2.12) and (2.13) with (2.9) gives $(M+1)\times(M+1)$ linear system of equations. We can obtain

$$
u_0^{n+1} = \frac{F_1(\Phi,U)(h\psi_M^{n+1}-3) - hF_2(\Psi,U)\phi_M^{n+1}}{J(\Phi,\Psi,U)},
$$

$$
u_M^{n+1} = \frac{F_2(\Psi,U)(h\phi_0^{n+1}-3) - hF_1(\Phi,U)\psi_0^{n+1}}{J(\Phi,\Psi,U)},
$$

where

$$
F_1(\Phi,U) = -4h\left(\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1}\right) - 2h\left(\sum_{i=1}^{M/2-1}\phi_{2i}^{n+1}u_{2i}^{n+1}\right) - 3g_1^{n+1},
$$

$$
F_2(\Psi,U) = -4h\left(\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1}\right) - 2h\left(\sum_{i=1}^{M/2-1}\psi_{2i}^{n+1}u_{2i}^{n+1}\right) - 3g_2^{n+1},
$$

and

$$
J(\Phi,\Psi,U) = (h\phi_0^{n+1}-3)(h\psi_M^{n+1}-3) - h^2\phi_M^{n+1}\psi_0^{n+1} \neq 0.
$$

2) Sixth-order formula

The sixth-order integration formula can be used to approximate numerically the integral present in the boundary conditions (1.3) and (1.4). We can write the sixth-order formula as [31]:

$$
\begin{aligned}
u_0^{n+1} = u(0,t_{n+1}) &= \int_0^1 \phi(x,t_{n+1})u(x,t_{n+1})dx + g_1(t_{n+1}) \\
&= \frac{2h}{45}[7\phi_0^{n+1}u_0^{n+1} + 32\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 12\sum_{i=0}^{M/4-1}\phi_{4i+2}^{n+1}u_{4i+2}^{n+1} \\
&+ 14\sum_{i=0}^{M/4-2}\phi_{4i+4}^{n+1}u_{4i+4}^{n+1} + 7\phi_M^{n+1}u_M^{n+1}] + g_1^{n+1} + O(h^6),
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
u_M^{n+1} = u(1,t_{n+1}) &= \int_0^1 \psi(x,t_{n+1})u(x,t_{n+1})dx + g_1(t_{n+1}) \\
&= \frac{2h}{45}[7\psi_0^{n+1}u_0^{n+1} + 32\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} + 12\sum_{i=0}^{M/4-1}\psi_{4i+2}^{n+1}u_{4i+2}^{n+1} \\
&+ 14\sum_{i=0}^{M/4-2}\psi_{4i+4}^{n+1}u_{4i+4}^{n+1} + 7\psi_M^{n+1}u_M^{n+1}] + g_1^{n+1} + O(h^6),
\end{aligned}
\tag{2.15}
$$

where M should be a multiple of 4. So

$$(14h\phi_0^{n+1} - 45)u_0^{n+1} + 14h\phi_M^{n+1}u_M^{n+1} = F_1(\Phi, U), \tag{2.16}$$

$$14h\psi_0^{n+1}u_0^{n+1} + (14h\psi_M^{n+1} - 45)u_M^{n+1} = F_2(\Psi, U), \tag{2.17}$$

where

$$F_1(\Phi, U) = -64h\sum_{i=1}^{M/2}\phi_{2i-1}^{n+1}u_{2i-1}^{n+1} - 24h\sum_{i=0}^{M/4-1}\phi_{4i+2}^{n+1}u_{4i+2}^{n+1} - 28h\sum_{i=0}^{M/4-2}\phi_{4i+4}^{n+1}u_{4i+4}^{n+1} - 45g_1^{n+1},$$

$$F_2(\Psi, U) = -64h\sum_{i=1}^{M/2}\psi_{2i-1}^{n+1}u_{2i-1}^{n+1} - 24h\sum_{i=0}^{M/4-1}\psi_{4i+2}^{n+1}u_{4i+2}^{n+1} - 28h\sum_{i=0}^{M/4-2}\psi_{4i+4}^{n+1}u_{4i+4}^{n+1} - 45g_1^{n+1},$$

Combining (2.16), (2.17) with (2.9) gives $(M+1)\times(M+1)$ linear system of equations. We can obtain

$$u_0^{n+1} = \frac{F_1(\Phi, U)(14h\psi_M^{n+1} - 45) - 14hF_2(\Psi, U)\phi_M^{n+1}}{J(\Phi, \Psi, U)},$$

$$u_M^{n+1} = \frac{F_2(\Psi, U)(14h\phi_0^{n+1} - 45) - 14hF_1(\Phi, U)\psi_0^{n+1}}{J(\Phi, \Psi, U)},$$

where

$$J(\Phi, \Psi, U) = 196h^2(\phi_0^{n+1}\psi_M^{n+1} - \phi_M^{n+1}\psi_0^{n+1}) - 630h(\phi_0^{n+1} + \psi_M^{n+1}) + 2025.$$

It should be noted that the system (2.8) does not have unique solution. Thus it can help us to obtain the optimal value of m whilst it was not considered in [31]. To check the accuracy of present method, we compared our results with the results obtained in [31]. It should also be noted that the explicit finite difference methods are easier to implement than the implicit schemes or Crank-Nicolson method, because in explicit schemes there is only one unknown is involved in the finite difference formula. Moreover, implicit finite difference schemes require the solution of a large number of simultaneous linear algebraic equations at each steps resulting in an extensive amount of CPU time utilized compared to explicit finite difference methods for the same values of s and h.

3 Illustrative Examples

In this section, the new explicit finite difference method (NFTCS) applied to linear and nonhomogeneous parabolic partial differential equation (1.1) with nonlocal boundary conditions (1.3)–(1.4). The results show that the described method is very accurate, capable and powerful. The numerical results indicate that the approximate solution convergence to the exact solution as h tends to zero. The Simpson formula and sixth-order formula are used to approximate the integral in the examples. The MATHEMATICA software is used to find the approximate solution and CPU time. For describing the error, we define relative error E_R and the absolute error E_A as follows:

$$E_R(u(x,t)) = \frac{|u(ih, jk)_{\text{approx}} - u(ih, jk)_{\text{exact}}|}{|u(ih, jk)_{\text{exact}}|}$$

and

$$E_A(u(x,t)) = |u(ih, jk)_{\text{approx}} - u(ih, jk)_{\text{exact}}|$$

where $u(ih, jk)_{\text{approx}}$ is the approximate solution and $u(ih, jk)_{\text{exact}}$ is the exact solution.

Example 1:

We consider the nonhomogeneous parabolic partial differential equation [16, 31, 32, 33, 36]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \qquad 0 \le x \le 1 \quad , \quad 0 \le t \le T, \tag{3.1}$$

with the following initial and boundary conditions

$$
\begin{aligned}
&f(x) = x^2, && && 0 < x < 1,\\
&\phi(x,t) = x, && 0 < x < 1, && 0 < t < 1,\\
&\psi(x,t) = x, && 0 < x < 1, && 0 < t < 1,\\
&g_1(t) = \frac{-1}{4(t+1)^2}, && && 0 < t < 1,\\
&g_2(t) = \frac{3}{4(t+1)^2}, && && 0 < t < 1,\\
&q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3}, && 0 \le t \le 1, && 0 < x < 1.
\end{aligned}
$$

It can be verified that the exact solution is

$$u(x,t) = \left(\frac{x}{t+1}\right)^2.$$

By applying the algorithm which is introduced in section 2, we take $m = 0.427487$ in (2.9) and obtained results are shown in tables 1 and 2 for various values of x and h. The Simpson composite rule and sixth-order formula are used to approximate the nonlocal boundary conditions (1.3) and (1.4). It can be seen that the errors are very small. In the last row in each table, we have obtained the CPU time consumed in the implementation of NFTCS for various step size h and x at $t = 1$. As expected, the CPU time increases as the step size h decrease.

Table 1: Absolute error NFTCS at $t = 1$ by using the Simpson formula.

x/h	0.25	0.125	0.0625	0.03125
0	1.97727×10^{-9}	2.73501×10^{-13}	4.75005×10^{-14}	1.46390×10^{-14}
0.25	8.07057×10^{-8}	4.87357×10^{-9}	3.03945×10^{-10}	1.89884×10^{-11}
0.5	3.92653×10^{-8}	2.26709×10^{-9}	1.40977×10^{-10}	8.80097×10^{-12}
0.75	3.27403×10^{-8}	2.21648×10^{-9}	1.39182×10^{-10}	8.70845×10^{-12}
1	1.97727×10^{-9}	2.73392×10^{-13}	4.75009×10^{-14}	1.43112×10^{-14}
CPU	0.078	2.248	76	1006.45

Table 2: Absolute error NFTCS at $t = 1$ by using the sixth-order formula.

x/h	0.25	0.125	0.0625	0.03125
0	2.32131×10^{-10}	7.89537×10^{-13}	5.05059×10^{-14}	3.47797×10^{-14}
0.25	7.77293×10^{-8}	4.86301×10^{-9}	3.03905×10^{-10}	1.89883×10^{-11}
0.5	3.60167×10^{-8}	2.25557×10^{-9}	1.40933×10^{-10}	8.80078×10^{-11}
0.75	3.57167×10^{-8}	2.22705×10^{-9}	1.39223×10^{-10}	8.70862×10^{-12}
1	2.32131×10^{-10}	7.89538×10^{-13}	5.05068×10^{-14}	3.47777×10^{-14}
CPU	0.078	1.391	53.687	807.840

The relative error E_R for $u(0.5, 1)$ is obtained for different step size h and compared the results obtain by [31] (scheme FTCS4). We used the algorithm introduced in section 2 with optimal value of m. The Simpson formula (NFTCS4) and sixth-order formula (NFTCS6) have been used for approximating the integrals in the nonlocal boundary conditions.

Table 3: Relative error E_R for $u(0.5, 1)$ at various spatial length.

h	FTCS4 [31]	NFTCS4	NFTCS6
0.25	0.000127462	6.28724×10^{-7}	5.76746×10^{-7}
0.125	7.96658×10^{-6}	3.63034×10^{-8}	3.61190×10^{-8}
0.0625	4.97912×10^{-7}	2.25753×10^{-9}	2.25683×10^{-9}
0.03125	3.11195×10^{-8}	1.40816×10^{-10}	1.40812×10^{-10}

In table 4, we compared the relative error E_R at $x = 0.5$ and $t = 1$ by using NFTCS described in this paper and FTCS, explicit Crandall's formula (ECF) and implicit Crandall's formula (ICF) obtained in [31]. It is observed that the our results are better than the obtained results in [31].

Table 4: Comparison between relative error E_R in [31] and our solution for $u(0.5, 1)$.

h	FTCS [31]	ECF [31]	ICF [31]	Our method
1/8	7.96658×10^{-6}	6.19972×10^{-6}	1.32985×10^{-5}	3.63034×10^{-8}
1/16	4.97912×10^{-7}	3.87463×10^{-7}	8.31040×10^{-7}	2.25753×10^{-9}
1/32	3.11195×10^{-8}	2.42163×10^{-8}	5.19395×10^{-8}	4.37617×10^{-11}

In Figure 1, we show the absolute error E_A using the NFTCS at $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in the boundary conditions. Similarly, the graph of absolute error NFTCS at $x = 0.0625$ and $x = 1$ with $h = 1/16$ is shown in Figure 2.

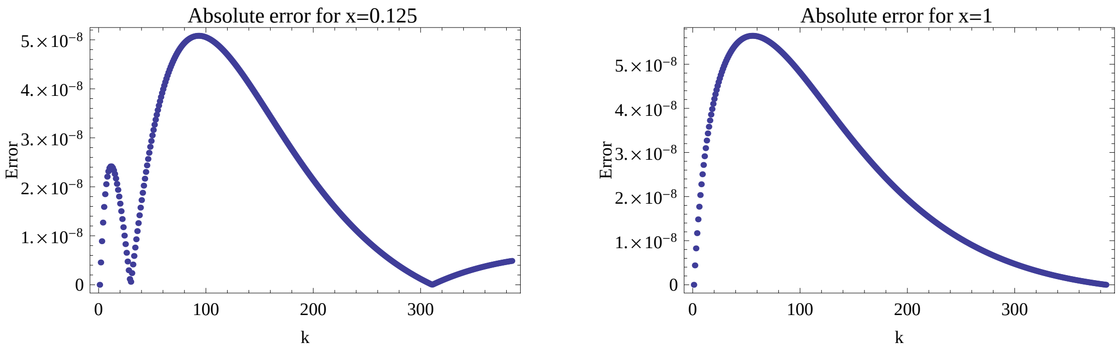

Figure 1: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in boundary conditions.

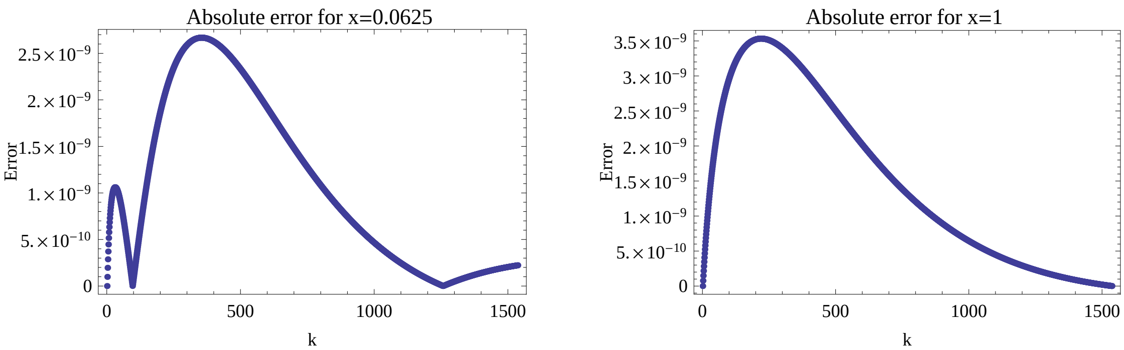

Figure 2: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.0625$ and $x = 1$ with $h = 1/16$, while the Simpson formula is used for approximating the integrals in boundary conditions.

The consumed CPU time of three numerical schemes FTCS [31], FTCS4, and FTCS6 is obtained and the graph of CPU time is shown in Figure 3. It is clear from the Figure 3 that with the same step size h, our method consumed less CPU time than other two numerical schemes.

Example 2:

To check the performance of the explicit finite difference scheme described in section 2, we take the

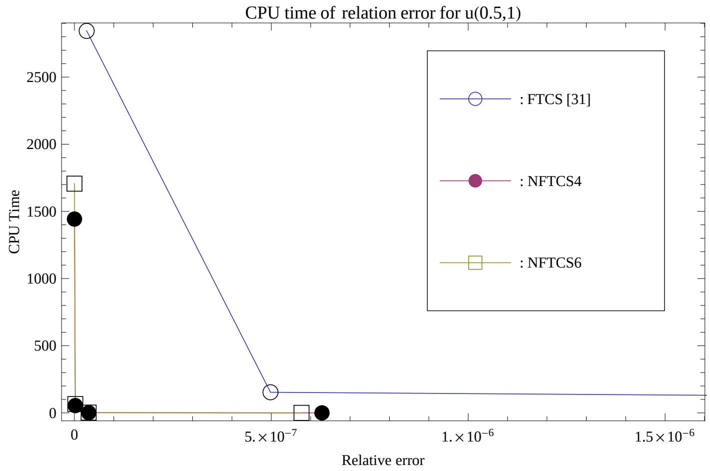

Figure 3: The CPU time spent to find the relative error E_R for $u(0.5, 1)$.

heat equation (1.1) with the initial condition and the boundary conditions as follows [12, 24, 18]:

$$\begin{aligned}
&f(x) = x^2 - x + \frac{\delta}{6(1+\delta)}, && && 0 < x < 1,\\
&\phi(x,t) = -\delta, && 0 < x < 1, && 0 \le t \le 1,\\
&\psi(x,t) = -\delta, && 0 < x < 1, && 0 \le t \le 1,\\
&g_1(t) = 0, && && 0 < x < 1,\\
&g_2(t) = 0, && && 0 < x < 1,\\
&q(x,t) = -\left(x^2 - x + \frac{\delta}{6(1+\delta)} + 2\right)e^{-t}, && 0 < x < 1, && 0 \le t \le 1,
\end{aligned}$$

with $\delta = 0.0144$. It can be verified that the exact solution is

$$u(x,t) = \left(x^2 - x + \frac{\delta}{6(1+\delta)}\right)e^{-t}.$$

We take $m = 0.5166667$, in equation (2.9) and the obtained results are shown in table 5 and table 6. The Simpson composite rule and sixth-order formula are used to approximate the nonlocal boundary conditions (1.3) and (1.4). It can be seen that the errors are very small with different value of x and t. In the last row in each table, we observed that the consumed CPU time increase with the decrease of step size h .

Table 5: Absolute error NFTCS at $t = 1$ by using the Simpson formula.

x/h	0.25	0.125	0.0625	0.03125
0	7.11516×10^{-12}	5.71649×10^{-13}	3.62120×10^{-14}	2.23064×10^{-15}
0.25	1.37993×10^{-9}	8.67588×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
0.5	2.54796×10^{-9}	1.58515×10^{-10}	9.90530×10^{-12}	6.22738×10^{-13}
0.75	1.37993×10^{-9}	8.67587×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
1	7.11516×10^{-12}	5.71649×10^{-13}	3.62121×10^{-14}	2.23064×10^{-15}
CPU	0.093	1.407	53.578	1670.780

Table 6: Absolute error NFTCS at $t = 1$ by using the Sixth-order formula.

x/h	0.25	0.125	0.0625	0.03125
0	9.18984×10^{-12}	5.79563×10^{-13}	3.62435×10^{-14}	2.23118×10^{-15}
0.25	1.37764×10^{-9}	8.67501×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
0.5	2.55032×10^{-9}	1.58524×10^{-10}	9.90531×10^{-12}	6.22738×10^{-13}
0.75	1.37764×10^{-9}	8.67501×10^{-11}	5.42373×10^{-12}	3.36300×10^{-13}
1	9.18984×10^{-12}	5.79563×10^{-13}	3.62435×10^{-14}	2.23117×10^{-15}
CPU	0.079	1.406	52.985	1706.580

In table 7, we present a comparison between the numerical solution of this problem by using new explicit finite difference method and those obtained by the method described in [31]. we observed that the relative error E_R of present method is better than the method described in [31]. Further more the Simpson formula (NFTCS4) and sixth-order formula (NFTCS6) have been used for approximating the integrals in the nonlocal boundary conditions.

Table 7: Relative error E_R for $u(0.5, 1)$ at various spatial length.

h	Method in [31]	NFTCS4	NFTCS6
0.25	3.12655×10^{-5}	2.79690×10^{-8}	2.79949×10^{-8}
0.125	1.95407×10^{-6}	1.74002×10^{-9}	1.74012×10^{-9}
0.0625	1.22130×10^{-7}	1.08731×10^{-10}	1.08731×10^{-10}
0.03125	7.63314×10^{-9}	6.83580×10^{-12}	6.83580×10^{-12}

In Figure 4, we show the absolute error E_A using the NFTCS at $x = 0.125$ and $x = 1$ with step size $h = 1/8$ when the Simpson formula is used for approximating the integrals in the boundary conditions. Similarly, we show the absolute error NFTCS at $x = 0.0625$ and $x = 1$ with step size $h = 1/16$ in Figure 5.

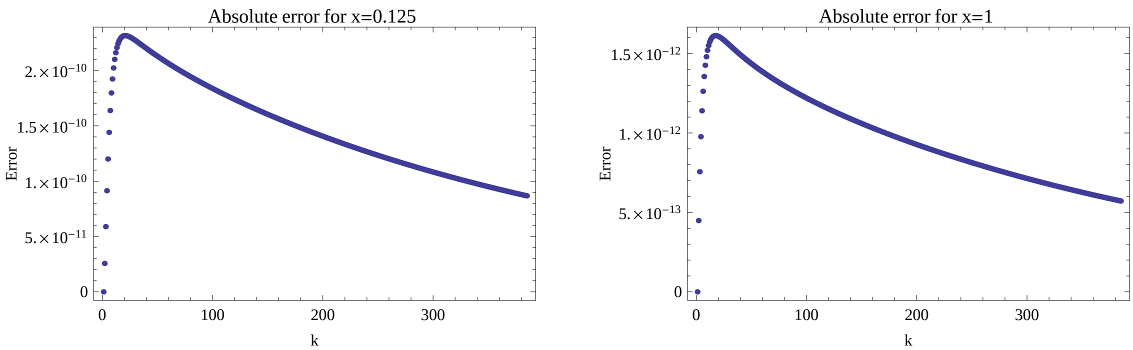

Figure 4: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.125$ and $x = 1$ with $h = 1/8$, while the Simpson formula is used for approximating the integrals in boundary conditions.

The consumed CPU time to obtain the numerical solution of the present method with different algorithm to to solve the non local boundary conditions are shown Figure 6. It is clear that for the all small value of step size h, the consumed CPU time in applying the our methods and consumed CPU time by the algorithm proposed in [31] are almost same.

4 Conclusion

Several approaches have been developed for obtaining the numerical solution of heat equation with non-local boundary conditions. A new fourth-order explicit finite difference method has been applied to

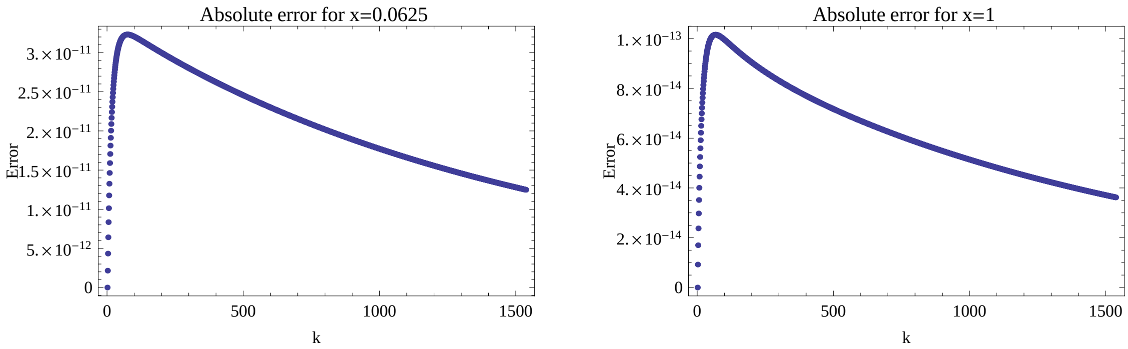

Figure 5: The absolute error between solution obtained by using NFTCS and the exact solution for $x = 0.0625$ and $x = 1$ with $h = 1/16$, while the Simpson formula is used for approximating the integrals in boundary conditions.

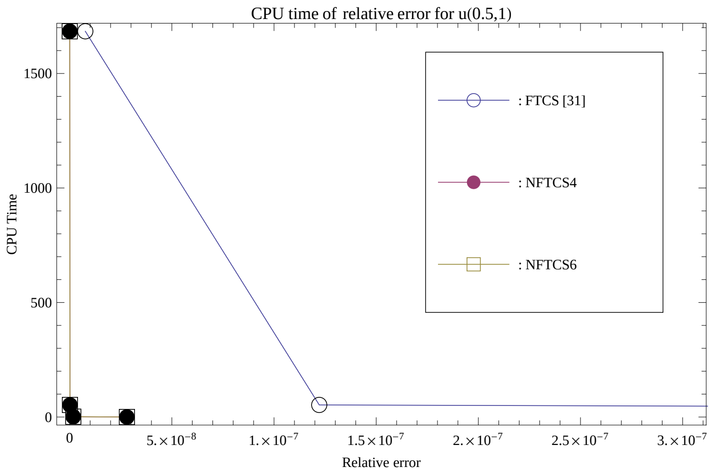

Figure 6: The CPU time spent to find the relation error E_R for $u(0.5, 1)$ in table 7.

obtain numerical solution of one dimensional linear and non-homogeneous parabolic partial differential equation with nonlocal boundary conditions in this paper. The present method is also capable for solving parabolic type partial differential equations with non-local boundary conditions.

References

[1] A. Whye-Teong, Numerical solution of a non-classical parabolic problem: An integro-differential approach, *Appl. Math. Comput.*, **175** (2006), 969–979.

[2] N. Borovykh, Stability in the numerical solution of the heat equation with nonlocal boundary conditions, *Appl. Numer. Math.*, **42** (2002), 17–27.

[3] A. Boutayeb and A. Chetouani, A numerical comparison of different methods applied to the solution of problems with non-local boundary conditions, *Appl. Math. Sci.*, **1(44)** (2007), 2173–2185.

[4] A. Boutayeb and A. Chetouani, Global extrapolation of numerical methods for solving a parabolic problem with nonlocal boundary conditions, *Int. J. Comput. Math.*, **80** (2003), 789–797.

[5] B. Cahlon, D.M. Kulkarni and P. Shi, Stepwise stability for the heat equation with a nonlocal constraint, *SIAM J. Numer. Anal.* **32** (1995), 571–593.

[6] J. R. Cannon, The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.,* **21** (1963) 155–160.

[7] J. R. Cannon and H. M. Yin, On a class of non-classic parabolic problems, *J. Differen. Equat.,* **79** (1989), 266–288.

[8] V. Capasso, K. Kunisch, A reaction-diffusion system arising in modelling man-environment diseases, *Quart. Appl. Math.,* **46** (1988), 431–449.

[9] Y. Choi and K. Chan, A parabolic equation with nonlocal boundary conditions arising from electrochemistry. *Non. Anal. Theor. Math. Appl.,* **18** (1992), 317–327.

[10] D.S. Daoud, Determination of the source parameter in a heat equation with a non-local boundary conditions, *J. Comput. Appl. Math.,* **221** (2008), 261–272.

[11] W.A. Day, Parabolic equations and thermodynamics, *Quart. Appl. Math.,* **50** (1992), 523–533.

[12] W.A. Day, Existence of a property of solutions of the heat equation subject to linear thermoelasticity and other theories, *Quart. Appl. Math.,* **40** (1982), 319–330.

[13] M. Dehghan, The use of Adomian decomposition method for solving the one dimensional parabolic equation with non-local boundary specification, *Int. J. Comput. Math.,* **81** (2004) 25–34.

[14] M. Dehghan, Numerical solution of a parabolic equation with non-local boundary specifications, *Appl. Math. Comput.,* **145** (2003), 185–194.

[15] M. Dehghan, The one-dimensional heat equation subject to a boundary integral specification, *Chaos Solit. Fract.,* **32** (2007), 661–675.

[16] M. Dehghan, Efficient techniques for the second-order parabolic equation subject to nonlocal specifications, *Appl. Numer. Math.,* **52** (2005), 39–62.

[17] K. Deng, Comparison Principle for some nonlocal problems, *Quart. Appl. Math.,* **50** (1992), 517–522.

[18] G. Ekolin, Finite difference methods for a non-local boundary value problem for the heat equation. *BIT,* **31** (1991), 245–55.

[19] R. E. Ewing and T. Lin, A class of parameter estimation techniques of fluid flow in porous media, *Adv. Water Resour.,* **14** (1993), 89–97.

[20] A. Friedman, Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions, *Quart. Appl. Math.,* **44** (1986), 401–407.

[21] A. Golbabai and M. Javidi, A numerical solution for non-classical parabolic problem based on Chebyshev spectral collocation method, *Appl. Math. Comput.,* **190** (2007), 179–185.

[22] F. Ivanauskas, T. Mekauskas and M. Sapagovas, Stability of difference schemes for two-dimensional parabolic equations with non-local boundary conditions, *Appl. Math. Comput.,* **215** (2009), 2716–2732.

[23] N.I. Kamynin, A boundary value problem in the theory of the heat conduction with non-classical boundary condition, *USSR Comput. Math. Math. Phys.,* **4** (1964), 33–59.

[24] L. Yunkang, Numerical solution of the heat equation with nonlocal boundary conditions, *J. Comput. Appl. Math.,* **110** (1999), 115–127.

[25] L. Mu and H. Du, The solution of a parabolic differential equation with non-local boundary conditions in the reproducing kernel space, *Appl. Math. Comput.,* **202** (2008), 708–714.

[26] M. A. Rehman and M. S. A. Taj, Fourth-Order method for non-homogeneous heat equation with nonlocal boundary conditions, *Appl. Math. Sci.,* **3(37)** 2009, 1811–1821.

[27] A. Samarskii, Some problems in differential equations theory. *J. Differ. Equat.,* **16** (1980), 1221–1228.

[28] P. Shi, Weak solution to evolution problem with a nonlocal constraint, *SIAM J. Numer. Anal.,* **24** (1993), 46–58.

[29] P. Shi and M. Shillor, Design of contact patterns in one-dimensional thermoelasticity, Theoretical aspects of industrial design. Philadelphia, PA: *Society for Industrial and Applied Mathematics*, 1992.

[30] Z.Z. Sun, A second-order accurate finite difference scheme for a class of nonlocal parabolic equations with natural boundary conditions, *J. Comput. Appl. Math.,* **76** (1996) 137–146.

[31] J. Martn-Vaquero, Two-level fourth-order explicit schemes for diffusion equations subject to boundary integral specifications, *Chao. Solit. Fract.* **42** (2009), 2364–2372.

[32] J. Martin-Vaquero and J. Vigo-Aguiar, A note on efficient techniques for the second-order parabolic equation subject to non-local conditions, *Appl. Numer. Math.,* **59** (2009), 1258–1264.

[33] J. Martin-Vaquero, J. Vigo-Aguiar, On the numerical solution of the heat conduction equations subject to nonlocal conditions, *Appl. Numer. Math.,* **59** (2009), 2507–2514

[34] S. Wang and Y. Lin, A numerical method for the diffusion equation with nonlocal boundary specifications. *Int. J. Eng. Sci.,* **28** (1990), 543–600.

[35] Y.F. Yin, On nonlinear parabolic equations with nonlocal boundary condition, *J. Math. Anal. Appl.,* **185** (1994), 54–60.

[36] Y. Zhoua, M. Cui and Y. Lin, Numerical algorithm for parabolic problems with non-classical conditions, J. Comput. Appl. Math., **230** (2009), 770–780.

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Numerical Solutions of Fourth Order Lidstone Boundary Value Problems Using Discrete Quintic Splines

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Abstract. In this paper, the numerical treatment of a fourth order Lidstone boundary value problem is proposed with the use of a discrete quintic spline based on central differences. It is shown that the method is of order 4 if a parameter takes a specific value, else it is of order 2. A well known numerical example is presented to illustrate our method as well as to compare the performance with other numerical methods proposed in the literature.

Keywords : Discrete quintic spline, central difference, Lidstone boundary value problem, numerical solution, fourth order.

1 Introduction

We consider the fourth order Lidstone boundary value problem

$$\begin{aligned} &y^{(4)}(x) = f(x)y(x) + g(x), \quad a \le x \le b \\ &y(a) = A_1, \quad y(b) = B_1, \quad y''(a) = A_2, \quad y''(b) = B_2 \end{aligned} \tag{1.1}$$

where $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $A_i, B_i, i = 1, 2$ are arbitrary real finite constants.

Lidstone boundary value problems have received a lot of attention in the literature, notably on the existence of positive solutions, see for example [1, 7, 22] and the references cited therein. The fourth order Lidstone boundary value problem (1.1) considered arises from the physical problem of bending a rectangular simply supported beam resting on an elastic foundation [14, 17], here y is the vertical deflection of the plate. The use of polynomial splines in the numerical treatment of (1.1) has gathered substantial interests over the years. Usmani and Warsi [20] have used quintic and sextic splines respectively to develop second and fourth order convergent methods for (1.1). Thereafter, quartic splines

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are employed by Usmani [19] to formulate second order convergent method. Also, during their investigation on fourth order obstacle boundary value problems, Al-Said and Noor [2] and Al-Said et al [3] have respectively used cubic and quartic splines to obtain second order convergent methods for (1.1). Recently, nonpolynomial spline functions have been proposed by Ramadan et al [12] to obtain second and fourth order convergent methods for (1.1), these methods reduced to those of [2, 19, 20] when certain parameters take certain values. A related problem to (1.1) arises from the bending of a long uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges [14, 17], here the boundary conditions are the conjugate type $y(a) - A_1 = y(b) - B_1 = y'(a) - A_2 = y' - B_2 = 0$. For this problem, second order convergent methods based on quintic splines have been established in [13, 16, 18], while fourth order convergent method based on sextic splines has been discussed in [18]. The general observation from all these research is that spline methods usually give better (or comparable) approximation than finite difference methods and shooting type methods.

Motivated by all the above research especially the use of splines in solving (1.1), in this paper we shall employ a *discrete quintic spline* to get a numerical solution of (1.1). Our proposed method is fourth-order convergent when a parameter takes certain value, else it is second-order convergent. Through a well know numerical example, we illustrate that our method outperforms other spline methods for solving (1.1) in the literature [2, 3, 12, 19, 20].

Discrete splines were first introduced by Mangasarian and Schumaker [11] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Subsequent investigations on discrete splines can be found in the work of Schumaker [15], Astor and Duris [4], Lyche [9, 10] and Wong et al [5, 6, 21]. Following [9, 10], the discrete spline we use will involve central differences.

The plan of the paper is as follows. In section 2, we shall derive our method. The matrix form of the method is presented in section 3 and its convergence analysis is performed. In section 4, we present a well known example and compare the performance of our method with other methods in the literature.

2 Numerical Method for (1.1)

Suppose $P : a = x_0 < x_1 < \cdots < x_n = b$ is a uniform mesh of $[a, b]$ with $x_i - x_{i-1} = p,\ 1 \le i \le n$, i.e., the step size $p = \frac{b-a}{n}$.

Let $h \in (0, p]$ be a given constant. We recall the *central difference operator*

D_h applying to a function $F(x)$ gives

$$D_h^{\{0\}}F(x) = F(x); \qquad D_h^{\{1\}}F(x) = \frac{F(x+h)-F(x-h)}{2h};$$
$$D_h^{\{2\}}F(x) = \frac{F(x+h)-2F(x)+F(x-h)}{h^2};$$
$$D_h^{\{3\}}F(x) = \frac{F(x+2h)-2F(x+h)+2F(x-h)-F(x-2h)}{2h^3};$$
$$D_h^{\{4\}}F(x) = \frac{F(x+2h)-4F(x+h)+6F(x)-4F(x-h)+F(x-2h)}{h^4}.$$

We also use the basic polynomials $x^{\{j\}}$ introduced by [10]

$$x^{\{j\}} = x^j, \quad j=0,1,2; \quad x^{\{3\}} = x(x^2-h^2),$$
$$x^{\{4\}} = x^2(x^2-h^2), \quad x^{\{5\}} = x(x^2-h^2)(x^2-4h^2).$$

It is noted that $D_h^{\{1\}}x^{\{j\}} = jx^{\{j-1\}},\ j=0,1,2,3,5$ and $D_h^{\{1\}}x^{\{4\}} = 2x(2x^2+h^2)$.

Definition 1. Let $S(x;h)$ be a piecewise continuous function defined over $[a,b]$ (with mesh P) and $S_i(x)$ be its restriction in $[x_{i-1},x_i],\ 1\le i\le n$ passing through the points (x_{i-1},s_{i-1}) and (x_i,s_i). We say $S(x;h)$ is a *discrete quintic spline* if $S_i(x),\ 1\le i\le n$ is a polynomial of degree 5 or less and

$$(S_{i+1}-S_i)(x_i+jh) = 0, \quad j=-2,-1,0,1,2, \quad 1\le i\le n-1. \tag{2.1}$$

The above definition is in the spirit of *discrete cubic spline* studied in [10]. In fact, in terms of *central differences*, the condition (2.1) has the following equivalent form

$$D_h^{\{j\}}S_i(x_i) = D_h^{\{j\}}S_{i+1}(x_i), \quad j=0,1,2,3,4, \quad 1\le i\le n-1. \tag{2.2}$$

Throughout, we shall use the notations

$$y_i^{(k)} = y^{(k)}(x_i), \quad f_i = f(x_i), \quad g_i = g(x_i), \quad s_i = S_i(x_i),$$
$$M_i = D_h^{\{2\}}S_i(x_i), \quad F_i = D_h^{\{4\}}S_i(x_i), \qquad 0\le i\le n.$$

We propose s_i's to be the numerical solution of (1.1) at the mesh points, i.e.,

$$y_i \cong s_i, \quad 0\le i\le n. \tag{2.3}$$

Discretizing (1.1) and noting the Lidstone boundary conditions, we set

$$\begin{gathered} s_0 = y_0 = A_1, \quad s_n = y_n = B_1, \quad M_0 = y_0'' = A_2, \\ M_n = y_n'' = B_2, \quad F_i = f_is_i + g_i, \quad 0\le i\le n. \end{gathered} \tag{2.4}$$

We shall now obtain an explicit expression of $S_i(x)$ in terms of its central differences. To begin, let the functions $h_i(x)$, $\bar{h}_i(x)$ and $\bar{\bar{h}}_i(x)$ satisfy the following for $0 \le i, j \le n$:

$$\begin{aligned} h_i(x_j) = \delta_{ij}, \quad & D_h^{\{2\}} h_i(x_j) = D_h^{\{4\}} h_i(x_j) = 0, \\ D_h^{\{2\}} \bar{h}_i(x_j) = \delta_{ij}, \quad & \bar{h}_i(x_j) = D_h^{\{4\}} \bar{h}_i(x_j) = 0, \\ D_h^{\{4\}} \bar{\bar{h}}_i(x_j) = \delta_{ij}, \quad & \bar{\bar{h}}_i(x_j) = D_h^{\{2\}} \bar{\bar{h}}_i(x_j) = 0. \end{aligned}$$

By direct computation, we obtain the explicit expressions:

$$\begin{aligned} h_i(x) &= \frac{x - x_{i-1}}{p}, \quad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\ &= \frac{x_{i+1} - x}{p}, \quad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\ &= 0, \qquad \text{otherwise;} \\ \bar{h}_i(x) &= \frac{(x - x_{i-1})^{\{3\}}}{6p} - \frac{(p^2 - h^2)(x - x_{i-1})}{6p}, \\ &\qquad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\ &= \frac{(x_{i+1} - x)^{\{3\}}}{6p} - \frac{(p^2 - h^2)(x_{i+1} - x)}{6p}, \\ &\qquad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\ &= 0, \qquad \text{otherwise;} \\ \bar{\bar{h}}_i(x) &= \frac{(x - x_{i-1})^{\{5\}}}{120p} - \frac{(p^2 - h^2)(x - x_{i-1})^{\{3\}}}{36p} \\ &\quad + \frac{(x - x_{i-1})(p^2 - h^2)(7p^2 + 2h^2)}{360p}, \\ &\qquad x \in [x_{i-1}, x_i], \quad 1 \le i \le n \\ &= \frac{(x_{i+1} - x)^{\{5\}}}{120p} - \frac{(p^2 - h^2)(x_{i+1} - x)^{\{3\}}}{36p} \\ &\quad + \frac{(x_{i+1} - x)(p^2 - h^2)(7p^2 + 2h^2)}{360p}, \\ &\qquad x \in [x_i, x_{i+1}], \quad 0 \le i \le n-1 \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

Clearly, $S_i(x)$, the restriction of $S(x; h)$ in $[x_{i-1}, x_i]$, can be expressed as

$$\begin{aligned} S_i(x) = {} & s_{i-1} h_{i-1}(x) + s_i h_i(x) + M_{i-1} \bar{h}_{i-1}(x) + M_i \bar{h}_i(x) + F_{i-1} \bar{\bar{h}}_{i-1}(x) \\ & + F_i \bar{\bar{h}}_i(x), \qquad x \in [x_{i-1}, x_i], \; 1 \le i \le n. \end{aligned} \tag{2.5}$$

Using (2.5), the 'continuity' requirement $D_h^{\{1\}}S_i(x_i) = D_h^{\{1\}}S_{i+1}(x_i),\ 1 \le i \le n-1$ leads to the equation

$$\begin{aligned}&(p^2-h^2)M_{i-1} + 2(h^2+2p^2)M_i + (p^2-h^2)M_{i+1}\\ &\quad = 6(s_{i-1}-2s_i+s_{i+1}) + \frac{(p^2-h^2)}{60}\left[(2h^2+7p^2)F_{i-1} + 4(4p^2-h^2)F_i\right.\\ &\qquad \left.+(2h^2+7p^2)F_{i+1}\right].\end{aligned} \tag{2.6}$$

Further, the 'continuity' requirement $D_h^{\{3\}}S_i(x_i) = D_h^{\{3\}}S_{i+1}(x_i),\ 1 \le i \le n-1$ yields

$$M_{i-1} - 2M_i + M_{i+1} = \frac{1}{6}\left[(p^2-h^2)F_{i-1} + 2(h^2+2p^2)F_i + (p^2-h^2)F_{i+1}\right]. \tag{2.7}$$

Using (2.6) and (2.7) in a lengthy algebraic procedure, we are able to eliminate M's and get the 'F-equation' as

$$\begin{aligned}&a_1F_{i-2} + a_2F_{i-1} + a_3F_i + a_2F_{i+1} + a_1F_{i+2}\\ &\quad = s_{i-2} - 4s_{i-1} + 6s_i - 4s_{i+1} + s_{i+2}, \qquad 2 \le i \le n-2\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&a_1 = \frac{(p^2-h^2)(p^2-4h^2)}{120}, \quad a_2 = \frac{2(p^2-h^2)(8h^2+13p^2)}{120},\\ &a_3 = \frac{6(4h^4+5h^2p^2+11p^4)}{120}.\end{aligned} \tag{2.9}$$

Upon substituting $F_j = f_js_j + g_j$ into (2.8), we see that (2.8) gives $(n-3)$ equations with $(n-1)$ unknowns $s_i,\ 1 \le i \le n-1$.

In order to solve for the unknown s_i's, we need two more equations which we write as

$$b_1F_0 + b_2F_1 + b_3F_2 + b_4F_3 = p^2M_0 + b_5s_0 + b_6s_1 + b_7s_2 + b_8s_3 \tag{2.10}$$

and

$$c_1F_{n-3} + c_2F_{n-2} + c_3F_{n-1} + c_4F_n = p^2M_n + c_5s_{n-3} + c_6s_{n-2} + c_7s_{n-1} + c_8s_n \tag{2.11}$$

where b_i and $c_i,\ 1 \le i \le 8$ are real numbers. We require the local truncation errors in both (2.10) and (2.11) to be $O(p^8)$ (the reason will be clear when we perform the convergence analysis in section 3). To fulfill this, we carry out Taylor series expansion in (2.10) about x_0 and set the coefficients of $s_0^{(k)},\ 0 \le k \le 7$ to zeros. This yields 8 equations which we can solve to get $b_i,\ 1 \le i \le 8$. Similarly, in (2.11) we expand about x_n and set the coefficients of $s_n^{(k)},\ 0 \le k \le 7$ to zeros, then we solve 8 equations to get $c_i,\ 1 \le i \le 8$. The resulting (2.10) and (2.11) are given as follows

$$\frac{p^4}{360}(28F_0 + 245F_1 + 56F_2 + F_3) - p^2M_0 = -2s_0 + 5s_1 - 4s_2 + s_3, \tag{2.12}$$

$$\frac{p^4}{360}(F_{n-3} + 56F_{n-2} + 245F_{n-1} + 28F_n) - p^2 M_n = s_{n-3} - 4s_{n-2} + 5s_{n-1} - 2s_n. \tag{2.13}$$

Once again, we substitute $F_j = f_j s_j + g_j$ into (2.12) and (2.13) to give two equations in $s_i,\ i = 1, 2, 3, n-3, n-2, n-1$.

Noting (2.4) the values of $s_0,\ s_n,\ M_0$ and M_n are already known, hence we can now solve (2.8), (2.12), (2.13) to obtain the values of $s_i,\ 1 \le i \le n-1$. The solvability of the linear system will be discussed in section 3.

3 Convergence Analysis

In this section, we shall establish the existence of a unique solution for (2.8), (2.12), (2.13) and also conduct a convergence analysis for the method presented in section 2. To begin, we define the norms of a column vector $T = [t_i]$ and a matrix $A = [a_{ij}]$ as follows:

$$\|T\| = \max_i |t_i| \qquad \text{and} \qquad \|A\| = \max_i \sum_j |a_{ij}|.$$

Let $e_i = y_i - s_i,\ 1 \le i \le n-1$ be the errors. Let $Y = [y_i],\ S = [s_i],\ W = [w_i],\ T = [t_i]$ and $E = [e_i]$ be $(n-1)$-dimensional column vectors. The system (2.8), (2.12), (2.13) can be written as

$$AS = W \tag{3.1}$$

where

$$A = A_0 + Q, \qquad Q = BF, \qquad F = \text{diag}(f_i),\ i = 1, 2, \ldots, n-1, \tag{3.2}$$

A_0 and B are $(n-1) \times (n-1)$ five-band symmetric matrices given by

$$A_0 = \begin{pmatrix} 5 & -4 & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix}, \tag{3.3}$$

$$B = \begin{pmatrix} -\frac{245p^4}{360} & -\frac{56p^4}{360} & -\frac{p^4}{360} & & & & \\ -a_2 & -a_3 & -a_2 & -a_1 & & & \\ -a_1 & -a_2 & -a_3 & -a_2 & -a_1 & & \\ & & & \ddots & & & \\ & & -a_1 & -a_2 & -a_3 & -a_2 & -a_1 \\ & & & -a_1 & -a_2 & -a_3 & -a_2 \\ & & & & -\frac{p^4}{360} & -\frac{56p^4}{360} & -\frac{245p^4}{360} \end{pmatrix} \tag{3.4}$$

and for the vector $W = [w_i]$, we have

$$w_i = \begin{cases} 2s_0 - p^2 M_0 + \dfrac{p^4}{360}(28f_0 s_0 + 28g_0 + 245g_1 + 56g_2 + g_3), & i = 1 \\ -s_0 + a_1 f_0 s_0 + a_1 g_0 + a_2 g_1 + a_3 g_2 + a_2 g_3 + a_1 g_4, & i = 2 \\ a_1 g_{i-2} + a_2 g_{i-1} + a_3 g_i + a_2 g_{i+1} + a_1 g_{i+2}, & 3 \le i \le n-3 \\ -s_n + a_1 g_{n-4} + a_2 g_{n-3} + a_3 g_{n-2} + a_2 g_{n-1} + a_1 g_n + a_1 f_n s_n, & \\ \qquad i = n-2 & \\ 2s_n - p^2 M_n + \dfrac{p^4}{360}(g_{n-3} + 56g_{n-2} + 245g_{n-1} + 28g_n + 28f_n s_n), & \\ \qquad i = n-1. & \end{cases} \tag{3.5}$$

From (3.1) we have $A(Y-E) = W$ or

$$AY = W + T \tag{3.6}$$

where

$$T = AE. \tag{3.7}$$

For $2 \le i \le n-2$, the i-th equation of the linear system (3.7) is

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = a_1 y_{i-2}^{(4)} + a_2 y_{i-1}^{(4)} + a_3 y_i^{(4)} + a_2 y_{i+1}^{(4)} + a_1 y_{i+2}^{(4)} + t_i$$

where t_i's are the local truncation errors given by

$$t_i = \frac{p^4(p^2 - 3h^2)}{12} y_i^{(6)} + \frac{p^4(4p^4 - 15p^2h^2 + 8h^4)}{240} y_i^{(8)} + O(p^9). \tag{3.8}$$

For $i = 1, n-1$, the i-th equations of the linear system (3.7) are respectively

$$-2y_0 + 5y_1 - 4y_2 + y_3 = \frac{p^4}{360}\left(28y_0^{(4)} + 245y_1^{(4)} + 56y_2^{(4)} + y_3^{(4)}\right) - p^2 y_0'' + t_1$$

and

$$\begin{aligned} &y_{n-3} - 4y_{n-2} + 5y_{n-1} - 2y_n \\ &\quad = \frac{p^4}{360}\left(y_{n-3}^{(4)} + 56y_{n-2}^{(4)} + 245y_{n-1}^{(4)} + 28y_n^{(4)}\right) - p^2 y_n'' + t_{n-1} \end{aligned}$$

where t_1 and t_{n-1} are the local truncation errors given by

$$t_1 = t_{n-1} = -\frac{241}{60480}p^8 y_i^{(8)} + O(p^9). \tag{3.9}$$

Remark 1. For the special case $h = \frac{p}{\sqrt{3}}$, it is clear from (3.8) that

$$t_i = -\frac{p^8}{2160}y_i^{(8)} + O(p^9), \quad 2 \le i \le n-2.$$

Thus, taking (3.9) into consideration, we have

$$\|T\| = \frac{241}{60480}p^8 L \tag{3.10}$$

where $L = \max_x |y^{(8)}(x)|$.

Lemma 1. [2] The matrix A_0 is invertible and

$$\|A_0^{-1}\| \le \frac{5n^4 + 4n^2}{384} = \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384p^4}. \tag{3.11}$$

Lemma 2. [8] Let D be a square matrix such that $\|D\| < 1$. Then, $(I + D)$ is nonsingular and

$$\|(I + D)^{-1}\| \le \frac{1}{1 - \|D\|}.$$

Our first result guarantees the existence of a unique solution for (2.8), (2.12), (2.13).

Theorem 1. The system (3.1) has a unique solution if

$$\frac{489}{480}K\hat{f} < 1 \tag{3.12}$$

where $K = \frac{5(b-a)^4+4(b-a)^2p^2}{384}$ and $\hat{f} = \max_{1\le i\le n-1}|f_i|$.

Proof. If (3.1) has a unique solution, then it can be written as

$$S = A^{-1}W = (A_0 + Q)^{-1}W = [A_0(I + A_0^{-1}Q)]^{-1}W = (I + A_0^{-1}Q)^{-1}A_0^{-1}W. \tag{3.13}$$

From Lemma 1 the inverse A_0^{-1} exists, hence it remains to show that $(I + A_0^{-1}Q)$ is nonsingular.

From (3.4), a direct computation gives $\|B\| \le \frac{489}{480}p^4$. Since $Q = BF$, we find

$$\|Q\| \le \|B\| \; \|F\| \le \frac{489}{480}p^4\hat{f}. \tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\|A_0^{-1}Q\| \le \|A_0^{-1}\|\ \|Q\| \le \frac{5(b-a)^4+4(b-a)^2p^2}{384p^4}\left(\frac{489}{480}p^4\hat{f}\right) = \frac{489}{480}K\hat{f} < 1 \tag{3.15}$$

where we have used (3.12) in the last inequality. Since $\|A_0^{-1}Q\| < 1$, we conclude from Lemma 2 that $(I+A_0^{-1}Q)$ is nonsingular. Hence, (3.1) has a unique solution given by (3.13). □

The next result gives the order of convergence of our method.

Theorem 2. Suppose $\frac{489}{480}K\hat{f} < 1$. Then,

$$\|E\| \cong O(p^4) \quad \text{if} \quad h = \frac{p}{\sqrt{3}}$$

and $\|E\| \cong O(p^2)$ for other values of $h \in (0, p]$, i.e., the method (3.1) is fourth order convergent if $h = \frac{p}{\sqrt{3}}$ and is second order convergent otherwise.

Proof. First, we consider the special case when $h = \frac{p}{\sqrt{3}}$. From (3.7) we have

$$E = A^{-1}T = (A_0+Q)^{-1}T = (I + A_0^{-1}Q)^{-1}A_0^{-1}T$$

Noting (3.15) we apply Lemma 2, and together with (3.11) and (3.10), we find

$$\begin{aligned}
\|E\| &\le \|(I + A_0^{-1}Q)^{-1}\|\ \|A_0^{-1}\|\ \|T\| \\
&\le \frac{\|A_0^{-1}\|\ \|T\|}{1-\|A_0^{-1}Q\|} \\
&\le \frac{5(b-a)^4+4(b-a)^2p^2}{384p^4}\left(\frac{241}{60480}p^8L\right)\left(\frac{1}{1-\frac{489}{480}K\hat{f}}\right) \\
&= \frac{241KLp^4}{60480\left(1-\frac{489}{480}K\hat{f}\right)} \ \cong\ O(p^4).
\end{aligned}$$

This inequality shows that (3.1) is a fourth order convergence method when $h = \frac{p}{\sqrt{3}}$.

For other values of $h \in (0, p]$, from (3.8) and (3.9) we have $\|T\| \cong O(p^6)$. Using a similar argument as above, we see that (3.1) is second order convergent. □

4 Numerical Example

In this section, we present a numerical example to demonstrate our proposed method as well as to illustrate the comparative performance with some well known numerical methods for solving (1.1).

Consider the Lidstone boundary value problem

$$\begin{aligned} &y^{(4)} + xy = -(8 + 7x + x^3)e^x, \quad 0 \le x \le 1 \\ &y(0) = y(1) = 0, \quad y''(0) = 0, \quad y''(1) = -4e. \end{aligned} \tag{4.1}$$

The analytical solution of (4.1) is

$$y(x) = x(1 - x)e^x.$$

In this example, we have $a = 0,\ b = 1,\ f(x) = -x$ and $g(x) = -(8 + 7x + x^3)e^x$. So $K = \frac{5+4p^2}{384}$ and $\hat{f} < 1$. For any $p \in (0, 1)$, we have $\frac{489}{480}K\hat{f} < 1$ and hence it follows from Theorem 1 that our method gives a unique numerical solution for (4.1).

To compute the numerical solution of (4.1), first we fix the mesh P (and hence the step size p) and choose $h = \frac{p}{\sqrt{3}}$. Then, we solve the system (2.8), (2.12), (2.13) to get s_i, $1 \le i \le n-1$, which approximates y_i.

The maximum absolute errors ($\max_i |y_i - s_i|$) obtained by our method as well as by other methods in the literature are presented in Table 1. From the table we can see that our method is fourth-order convergent when $h = \frac{p}{\sqrt{3}}$. Moreover, a clear comparison shows that our method *outperforms* continuous polynomial spline (cubic, quartic, quintic, sextic) and nonpolynomial spline (quintic) methods.

Table 1: Maximum absolute errors $\max_i |y_i - s_i|$

Methods	$p = 1/8$	$p = 1/16$	$p = 1/32$
Our method	$7.48e-08$	$5.30e-09$	$4.91e-10$
Quintic nonpolynomial spline (4th order)	$2.09e-07$	$7.92e-09$	$1.27e-09$
Sextic spline [20]	$1.26e-06$	$7.87e-08$	$4.91e-09$
Quintic nonpolynomial spline (2nd order) [12]	$9.42e-05$	$6.17e-06$	$3.95e-07$
Quartic spline [19]	$4.24e-04$	$1.08e-04$	$2.70e-05$
Cubic spline [3]	$5.69e-04$	$1.47e-04$	$3.71e-05$
Quintic spline [20]	$8.67e-04$	$2.16e-04$	$5.40e-05$
Quartic spline [2]	$1.62e-03$	$6.39e-04$	$5.88e-05$

A brief description of the methods listed in Table 1:

(i) In [12], second and fourth order convergent methods are derived using a nonpolynomial spline function that has a polynomial part and a trigonometric part. The methods of [2, 19, 20] are special cases of nonpolynomial spline methods when certain parameters take certain values.

(ii) In [20], quintic and sextic splines are employed respectively to establish second and fourth order convergent methods.

(iii) In [19], a second order convergent method is formulated using quartic splines. Here, the consistency relations are obtained at the *midknots*, this approach is different from other spline methods where consistency relations are usually obtained at the uniformly spaced knots.

(iv) In [3], cubic splines are used to develop a second order convergent method.

(v) In [2], a second order convergent method is proposed based on quartic splines.

5 Conclusion

We have developed a numerical method for fourth order Lidstone boundary value problems using discrete quintic splines. The method is shown to be fourth order convergent when the parameter $h = \frac{p}{\sqrt{3}}$, and second order convergent for other values of $h \in (0, p]$. A well known numerical example is presented to demonstrate the outperformance of our method over other continuous spline methods in the literature.

References

[1] R. P. Agarwal and P. J. Y. Wong, Lidstone polynomials and boundary value problems, *Comput. Math. Appl.*, 17, 1397-1421(1989).

[2] E. A. Al-Said and M. A. Noor, Quartic spline method for solving fourth order obstacle boundary value problems, *J. Comput. Appl. Math.*, 143, 107-116(2002).

[3] E. A. Al-Said, M. A. Noor and T. M. Rassias, Cubic splines method for solving fourth-order obstacle problems, *Appl. Math. Comput.*, 174, 180-187(2006).

[4] P. H. Astor and C. S. Duris, Discrete L-splines, *Numer. Math.*, 22, 393-402(1974).

[5] F. Chen and P. J. Y. Wong, Error estimates for discrete spline interpolation: Quintic and biquintic splines, *J. Comput. Appl. Math.*, 236, 3835-3854(2012).

[6] F. Chen and P. J. Y. Wong, On periodic discrete spline interpolation: Quintic and biquintic cases, preprint.

[7] J. M. Davis, J. Henderson and P. J. Y. Wong, General Lidstone problems: Multiplicity and symmetry of solutions, *J. Math. Anal. Appl.*, 251, 527-548(2000).

[8] C. Fröberg, *Numerical Mathematics, Theory and Computer Applications*, Benjamin/Cummings, Reading, MA, 1985.

[9] T. Lyche, *Discrete cubic spline interpolation*, report RRI 5, University of Oslo, 1975.

[10] T. Lyche, Discrete cubic spline interpolation, *BIT*, 16, 281-290(1976).

[11] O. L. Mangasarian and L. L. Schumaker, Discrete splines via mathematical programming, *SIAM J. Control*, 9, 174-183(1971).

[12] M. A. Ramadan, I. F. Lashien and W. K. Zahra, Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem, *Commun. Nonlinear Sci. Numer. Simul.*, 14, 1105-1114(2009).

[13] J. Rashidinia and A. Golbabaee, Convergence of numerical solution of a fourth-order boundary value problem, *Appl. Math. Comput.*, 171, 1296-1305(2005).

[14] E. L. Reiss, A. J. Callegari and D. S. Ahluwalia, *Ordinary Differential Equations with Applications*, Holt, Rinehart & Winston, New York, 1976.

[15] L. L. Schumaker, Constructive aspects of discrete polynomial spline functions, in *Approximation Theory* (G. G. Lorentz, ed.), Academic Press, New York, 1973, pp. 469-476.

[16] S. S. Siddiqi and G. Akram, Quintic spline solutions of fourth order boundary-value problems, *Int. J. Numer. Anal. Model.*, 5, 101-111(2008).

[17] S. Timoshenko and S. Woinowsky-Kreiger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[18] R. A. Usmani, Smooth spline approximations for the solution of a boundary value problem with engineering applications, *J. Comput. Appl. Math.*, 6, 93-98(1980).

[19] R. A. Usmani, The use of quartic splines in the numerical solution of a fourth-order boundary value problem, *J. Comput. Appl. Math.*, 44, 187-199(1992).

[20] R. A. Usmani and S. A. Warsi, Smooth spline solutions for boundary value problems in plate deflection theory, *Comput. Math. Appl.*, 6, 205-211(1980).

[21] P. J. Y. Wong and R. P. Agarwal, Error inequalities for discrete Hermite and spline interpolation, in *Recent Progress in Inequalities* (G. V. Milovanovic, ed.), Kluwer, Dordrecht, 1998, pp. 397-422.

[22] P. J. Y. Wong and R. P. Agarwal, Eigenvalues of Lidstone boundary value problems, *Appl. Math. Comput.*, 104, 15-31(1999).

Periodicity and Global Attractivity of Difference Equation of Higher Order

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In this paper we investigate the dynamics and behavior of the recursive sequence

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}, \quad n = 0, 1, ...$$

where the parameters a ; b; c; d;α ; β and γ are positive real numbers and the initial conditions are positive real numbers.

1 Introduction

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations see for example [[1]-[14]].
The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: In [4] Elabbasy et al. investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$$

Yalnkaya et al. [15], [16] considered the dynamics of the difference equations

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p}, \qquad x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$$

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For some related work see [[15]-[19]].
Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}, \quad n = 0, 1, ... \tag{1}$$

where the parameters a ; b; c; d;α ; β and γ are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, ..., x_{-1}$ and x_0 are positive real numbers where $t = \max\{k, l, s, r\}$.
Here, we recall some basic definitions and some theorems that we need in the sequel.
Let I be some interval of real numbers and let

$$F : I^{k+1} \longrightarrow I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ... \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$

Theorem 1.1. *[9] Assume that $p_i \in R$,$i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$ .Then $\sum_{i=1}^{k} |p_i| \leq 1$, is a sufficient condition for the asymptotic stability of the difference equation*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0 \qquad , n = 0, 1, ...$$

Theorem 1.2. *[10] Let $g : [a, b]^{k+1} \to [a, b]$ be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ... \tag{3}$$

Suppose that g satisfies the following conditions:

1. *For each integer i with $1 \leq i \leq k+1$,the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.*
2. *If If (m, M) is a solution of the system*

$$m = g(m_1, m_2, ..., m_{k+1}) \quad , and \quad M = g(M_1, M_2, ..., M_{k+1})$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

Then there exists exactly one equilibrium $\overline{x}$ of equation (3), and every solution of equation (3) converges to $\overline{x}$.

2 Local Stability of the Equilibrium Point of equation (1)

This section deals with study the local stability character of the equilibrium point of Equation (1) .Equation (1) has equilibrium point and is given by $\overline{x} = a\overline{x} + \frac{b+c+d}{\alpha+\beta+\gamma}$. If $a \leq 1$, then the only positive equilibrium point of equation (1) is given by $\overline{x} = \frac{b+c+d}{(\alpha+\beta+\gamma)(1-a)}$.
$f : (0, \infty)^4 \to (0, \infty)$ be a continuously differentiable function defined by

$$f(u, v, w, t) = au + \frac{bv + cw + dt}{\alpha v + \beta w + \gamma t} \tag{4}$$

Therefore it follows that

$$\frac{\partial f(u, v, w, t)}{\partial u} = a, \quad \frac{\partial f(u, v, w, t)}{\partial v} = \frac{(b\beta - c\alpha)w + (b\gamma - d\alpha)t}{(\alpha v + \beta w + \gamma t)^2}$$

$$\frac{\partial f(u, v, w, t)}{\partial w} = \frac{-(b\beta - c\alpha)v + (c\gamma - d\beta)t}{(\alpha v + \beta w + \gamma t)^2}$$

$$\frac{\partial f(u, v, w, t)}{\partial t} = \frac{-(b\gamma - d\alpha)v - (c\gamma - d\beta)w}{(\alpha v + \beta w + \gamma t)^2}$$

Then we see that

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x}, \overline{x})}{\partial u} = a = -a_3$$

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x}, \overline{x})}{\partial v} = \frac{(b\beta - c\alpha) + (b\gamma - d\alpha)}{(\alpha + \beta + \gamma)^2 \overline{x}} = \frac{(b\beta - c\alpha) + (b\gamma - d\alpha)}{(\alpha + \beta + \gamma)^2 \frac{b+c+d}{(\alpha+\beta+\gamma)(1-a)}}$$

$$= \frac{[(b\beta - c\alpha) + (b\gamma - d\alpha)](1 - a)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_2$$

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x}, \overline{x})}{\partial w} = \frac{[-(b\beta - c\alpha) + (c\gamma - d\beta)](1 - a)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_1$$

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x}, \overline{x})}{\partial t} = \frac{[-(b\gamma - d\alpha) - (c\gamma - d\beta)](1 - a)}{(\alpha + \beta + \gamma)(b + c + d)} = -a_0$$

Then the linearized equation of Equation (1) about $\overline{x}$ is

$$y_{n+1} + a_3 y_n + a_2 y_{n-1} + a_1 y_{n-2} + a_0 y_{n-3} = 0 \tag{5}$$

whose characteristic equation is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{6}$$

Theorem 2.1. *Assume that*

$$(\alpha+\beta+\gamma)(b+c+d) > \max\left\{|2\alpha(c+d)-2b(\beta+\gamma)|, |2\gamma(b+c)-2d(\alpha+\beta)|\right.$$
$$\left., |2\beta(b+d)-2c(\alpha+\gamma)|\right\} \tag{7}$$

Then the positive equilibrium point of Equation (1) *is locally asymptotically stable.*

Proof. It is follows by theorem (1.1) that equation (5) is asymptotically stable if all roots of equation (6) lie in the open disc, $|\lambda| < 1$ that is if

$$|a_3| + |a_2| + |a_1| + |a_0| < 1$$

$$|a| + \left|\frac{[(b\beta-c\alpha)+(b\gamma-d\alpha)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right|$$
$$+\left|\frac{[-(b\beta-c\alpha)+(c\gamma-d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right| + \left|\frac{[-(b\gamma-d\alpha)-(c\gamma-d\beta)](1-a)}{(\alpha+\beta+\gamma)(b+c+d)}\right| < 1$$

and so (after dividing the denominator and numerator by $(1-a)$ gives)

$$|(b\beta-c\alpha)+(b\gamma-d\alpha)| + |-(b\beta-c\alpha)+(c\gamma-d\beta)|$$
$$+|-(b\gamma-d\alpha)-(c\gamma-d\beta)| < (\alpha+\beta+\gamma)(b+c+d) \tag{8}$$

Suppose that
$B_1 = (b\beta-c\alpha)+(b\gamma-d\alpha)$, $\quad B_2 = -(b\beta-c\alpha)+(c\gamma-d\beta)$,
$B_3 = -(b\gamma-d\alpha)-(c\gamma-d\beta)$
We consider the following cases

1. $B_1 > 0$, $B_2 > 0$, and $B_3 > 0$. In this case we see from equation (8) that

$$(b\beta-c\alpha)+(b\gamma-d\alpha)-(b\beta-c\alpha)+(c\gamma-d\beta)-(b\gamma-d\alpha)-(c\gamma-d\beta)$$
$$< (\alpha+\beta+\gamma)(b+c+d)$$

   if and only if $(\alpha+\beta+\gamma)(b+c+d) > 0$ which is always true.

2. $B_1 > 0$, $B_2 > 0$, and $B_3 < 0$. It follows from equation (8) that

$$(b\beta-c\alpha)+(b\gamma-d\alpha)-(b\beta-c\alpha)+(c\gamma-d\beta)+(b\gamma-d\alpha)+(c\gamma-d\beta)$$
$$< (\alpha+\beta+\gamma)(b+c+d)$$

   if and only if $(\alpha+\beta+\gamma)(b+c+d) > 2\gamma(b+c)-2d(\alpha+\beta)$ which is satisfied by Condition (7).

Also, we can prove the other cases. The proof is complete. □

3 Boundedness of Solutions of Equation (1)

Here we study the boundedness nature and persistence of solutions of Equation (1).

Theorem 3.1. *Every solution of Equation* (1) *is bounded and persists if* $a < 1$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}$$

$$= ax_{n-k} + \frac{bx_{n-l}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} + \frac{cx_{n-s}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} + \frac{dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}}$$

Then

$$x_{n+1} \leq ax_{n-k} + \frac{bx_{n-l}}{\alpha x_{n-l}} + \frac{cx_{n-s}}{\beta x_{n-s}} + \frac{dx_{n-r}}{\gamma x_{n-r}} = x_{n-k} + \frac{b}{\alpha} + \frac{c}{\beta} + \frac{d}{\gamma} \quad \text{for all} \quad n \geq 1$$

By using a comparison, we see that

$$\lim_{n\to\infty} \sup x_n \leq \frac{b\beta\gamma + c\alpha\gamma + d\alpha\beta}{\alpha\beta\gamma(1-a)} = M \tag{9}$$

Thus the solution is bounded. Now we wish to show that there exists $m > 0$ such that $x_n \geq m$ for all $n \geq 1$. The transformation $x_n = \frac{1}{y_n}$ will reduce Equation (1) to the equivalent form

$$y_{n+1} = \frac{y_{n-k}(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s})}{a(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s}) + y_{n-k}(by_{n-s}y_{n-r} + cy_{n-l}y_{n-r} + dy_{n-l}y_{n-s})}$$

It follows that

$$y_{n+1} \leq \frac{y_{n-k}(\alpha y_{n-s}y_{n-r} + \beta y_{n-l}y_{n-r} + \gamma y_{n-l}y_{n-s})}{y_{n-k}(by_{n-s}y_{n-r} + cy_{n-l}y_{n-r} + dy_{n-l}y_{n-s})}$$

$$\leq \frac{\alpha y_{n-s}y_{n-r}}{by_{n-s}y_{n-r}} + \frac{\beta y_{n-l}y_{n-r}}{cy_{n-l}y_{n-r}} + \frac{\gamma y_{n-l}y_{n-s}}{dy_{n-l}y_{n-s}} = \frac{\alpha}{b} + \frac{\beta}{c} + \frac{\gamma}{d}$$

$$= \frac{\alpha cd + \beta bd + \gamma bc}{bcd} = H \quad \text{for all} \quad n \geq 1$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{H} = \frac{bcd}{\alpha cd + \beta bd + \gamma bc} = m \quad \text{for all} \quad n \geq 1 \tag{10}$$

From Equations (9) and (10) we see that $m \leq x_n \leq M$ for all $n \geq 1$. Therefore every solution of Equation (1) is bounded and persists. □

Theorem 3.2. *Every solution of Equation* (1) *is unbounded if* $a > 1$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Equation (1). Then from Equation (1) we see that

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-l} + cx_{n-s} + dx_{n-r}}{\alpha x_{n-l} + \beta x_{n-s} + \gamma x_{n-r}} > ax_{n-k} \quad \text{for all} \quad n \geq 1$$

We see that the right hand side can write as follows

$$y_{n+1} = ay_{n-k} \Rightarrow y_{kn+i} = a^n y_{k+i}, \qquad i = 0, 1, ..., k,$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$ Then by using ratio test $\{x_n\}_{n=-t}^{\infty}$ is unbounded from above. □

4 Existence of Periodic Solutions

In this section we study the existence of periodic solutions of equation (1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two and there is clear that there exists a sixteen cases of the indexes s, l, k, r as we see in the following theorem and lemmas.

Theorem 4.1. *Equation* (1) *has positive prime period two solutions if and only if one of the following statements holds*

1. $(b+d-c)(\alpha+\gamma-\beta)(1+a)+4(a\beta(b+d)+c(\alpha+\gamma)) > 0$,$\alpha+\gamma > \beta$,$b+d > c$ *and* l, r*-odd,* k, s*-even.*
2. $(c + d - b)(\beta + \gamma - \alpha)(1 + a) + 4(a\alpha(c + d) + b(\beta + \gamma)) > 0$, $\beta + \gamma > \alpha$, $c + d > b$ *and* k, r*-odd,* l, s*-even.*
3. $(b + c - d)(\alpha + \beta - \gamma)(1 + a) + 4(a\gamma(b + c) + d(\alpha + \beta)) > 0$, $\alpha + \beta > \gamma$, $b + c > d$ *and* k, l*-odd,* r, s*-even.*
4. $(b - c - d)(\alpha - \beta - \gamma)(1 + a) + 4(ab(\beta + \gamma) + \alpha(c + d)) > 0$, $\alpha > \beta + \gamma$, $b > c + d$ *and* l*-odd,* k, s, r*-even.*
5. $(c - b - d)(\beta - \alpha - \gamma)(1 + a) + 4(ac(\alpha + \gamma) + \beta(b + d)) > 0$, $\beta > \alpha + \gamma$, $c > b + d$ *and* k*-odd,* l, s, r*-even.*
6. $(d - b - c)(\gamma - \alpha - \beta)(1 + a) + 4(ad(\beta + \alpha) + \gamma(b + c)) > 0$, $\gamma > \alpha + \beta$, $d > b + c$ *and* r*-odd,* l, k, s*-even.*
7. $(c + d - b)(\alpha - \beta - \gamma) - 4b(\beta + \gamma) > 0$, $a < 1$, $\alpha > \beta + \gamma$, $c + d > b$ *and* k, s, r*-odd,* l*-even*
8. $(b + d - c)(\beta - \alpha - \gamma) - 4c(\alpha + \gamma) > 0$, $a < 1$, $\beta > \alpha + \gamma$, $b + d > c$ *and* l, s, r*-odd,* k*-even.*

9. $(b+c-d)(\gamma-\alpha-\beta)-4d(\alpha+\beta)>0$, $a<1$, $\gamma>\alpha+\beta$, $b+c>d$ *and* *k,s,l-odd, r-even.*
10. $(d-b-c)(\alpha+\beta-\gamma)-4\gamma(b+c)>0$, $a<1$, $\alpha+\beta>\gamma$, $d>b+c$ *and* s,r*-odd,* l,k*-even*
11. $(c-b-d)(\alpha+\gamma-\beta)-4\beta(b+d)>0$, $a<1$, $\alpha+\gamma>\beta$, $c>b+d$ *and* s,k*-odd,* l,r*-even.*
12. $(b-c-d)(\beta+\gamma-\alpha)-4\alpha(c+d)>0$, $a<1$, $\beta+\gamma>\alpha$, $b>c+d$ *and* s,l*-odd,* r,k*-even.*

Proof. We will prove the theorem when Condition (1) is true and the proof of the other cases are similar and so we will be omitted. First suppose that there exists a prime period two solution $..., p, q, p, q, ...$, of equation (1). We will prove that Condition (1) holds.We see from equation (1) that

$$p = aq + \frac{bp+cq+dp}{\alpha p+\beta q+\gamma p} = aq + \frac{ep+cq}{fp+\beta q}$$

where $e=b+d$, $f=\alpha+\gamma$, and

$$q = ap + \frac{bq+cp+dq}{\alpha q+\beta p+\gamma q} = ap + \frac{eq+cp}{fq+\beta p}$$

Then

$$fp^2+\beta pq = afpq + a\beta q^2 + ep + cq, \tag{11}$$

$$fq^2+\beta pq = afpq + a\beta p^2 + eq + cp, \tag{12}$$

Subtracting (11) from (12) gives $f(p^2-q^2) = -a\beta(p^2-q^2)+(e-c)(p-q)$. Since $p\neq q$, it follows that

$$p+q = \frac{e-c}{f+a\beta} \tag{13}$$

Again, adding (11) and (12) yields

$$(f-a\beta)(p^2+q^2)+2(\beta-af)pq = (e+c)(p+q) \tag{14}$$

It follows by (13),and (14) that

$$pq = \frac{(ea\beta+cf)(e-c)}{(f+a\beta)^2(\beta-f)(1+a)} \tag{15}$$

Now it is clear from equation (13) and equation (15) that p and q are the two distinct roots of the quadratic equation

$$(f+a\beta)t^2-(e-c)t+\frac{(ea\beta+cf)(e-c)}{(f+a\beta)(\beta-f)(1+a)} = 0, \tag{16}$$

and so $(e-c)^2 - \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)} > 0$. Therefore Inequality (1) holds.
Conversely , suppose that Inequality (1) is true. We will show that equation (1) has a prime period two solution. Assume that

$$p = \frac{e-c+\zeta}{2(f+a\beta)}, \quad and \quad q = \frac{e-c-\zeta}{2(f+a\beta)}$$

where $\zeta = \sqrt{(e-c)^2 - \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)}}$. We see from Inequality (1) that

$$(e-c)(f-\beta)(1+a) + 4(ea\beta+cf) > 0, e > c, f > \beta,$$

which equivalents to $(e-c)^2 > \frac{4(ea\beta+cf)(e-c)}{(\beta-f)(1+a)}$. Therefore p and q are distinct real numbers. Set $x_{-3} = p, x_{-2} = q, x_{-1} = p$ and $x_0 = q$. We wish to show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$ .It follows from equation (1) that

$$x_1 = aq + \frac{ep+cq}{fp+\beta q} = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \frac{e\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + c\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)}{f\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \beta\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)}$$

Dividing the denominator and numerator by $2(f+a\beta)$ gives

$$x_1 = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big) + \frac{(e-c)[(e+c)+\zeta]}{(f+\beta)(e-c)+(f-\beta)\zeta}$$

Multiplying the denominator and numerator of the right side by
$(f+\beta)(e-c) - (f-\beta)\zeta$ gives

$$x_1 = a\big(\frac{e-c-\zeta}{2(f+a\beta)}\big)$$

$$+\frac{(e-c)\big\{2(e-c)[fc+\beta e - \frac{2(ea\beta+cf)}{1+a}] + 2\zeta(\beta e - cf)\big\}}{4(e-c)\big[\beta f(e-c) + \frac{(\beta-f)(ea\beta+cf)}{(1+a)}\big]}$$

Multiplying the denominator and numerator of the right side by $(1+a)$ we obtain

$$x_1 = \frac{ae-ac-a\zeta+(e-c)(1-a)+\zeta(1+a)}{2(f+a\beta)} = \frac{e-c+\zeta}{2(f+a\beta)} = p.$$

Similarly as before one can easily show that $x_2 = q$. Then it follows by induction that $x_{2n} = q$ and $x_{2n+1} = p$ for all $n \geq -1$.Thus equation (1) has the prime period two solution $..., p, q, p, q, ...$, where p and q are the distinct roots of the quadratic equation (16) and the proof is complete. □

Lemma 4.2. *If l, k, s, r-even. Then there exists a prime period two solutions if and only if $a = -1$.*

Proof. First suppose that there exists a prime period two solution $..., p, q, p, q, ...,$ then we see from equation (1) that when l, k, s, r-even

$$p = aq + \frac{b+c+d}{\alpha+\beta+\gamma}, \tag{17}$$

$$q = ap + \frac{b+c+d}{\alpha+\beta+\gamma} \tag{18}$$

Subtracting equation (17) from equation (18) gives $p - q = -a(p - q)$. Since $p \neq q$, it follows that $a = -1$. Again, adding equation (17) and equation (18) yields $p + q = \frac{b+c+d}{\alpha+\beta+\gamma}$. If we take

$$p = \frac{b+c}{\alpha+\beta+\gamma}, q = \frac{d}{\alpha+\beta+\gamma}, \text{if} \qquad b+c \neq d.$$

Set $x_{-s} = q$, $x_{-l} = p$, $x_{-k} = q$,...,$x_{-2} = q$, $x_{-1} = p$ and $x_0 = q$.We wish to show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$. It follows from equation (1) that $x_1 = aq + \frac{bq+cq+dq}{\alpha q+\beta q+\gamma q} = p$. Similarly as before one can easily show that $x_2 = q$. Then it follows by induction that $x_{2n} = q$ and $x_{2n+1} = p$ for all $n \geq -1$. Thus equation (1) has the prime period two solution and the proof is complete. □

Lemma 4.3. *If l, k, r-odd, s-even. Then there exists a positive prime period two solutions if and only if $a = -1$.*

Lemma 4.4. *If l, k, s, r-odd (or l, k, r-even, s-odd). Then there no prime period two solution.*

5 Global Attractor of the Equilibrium Point of Equation (1)

In this section we investigate the global asymptotic stability of equation (1).

Lemma 5.1. *For any values of the quotient $\frac{b}{\alpha}, \frac{c}{\beta}$ and $\frac{d}{\gamma}$ the function $f(u, v, w, t)$ defined by equation (4) has the monotonicity behavior in its three arguments.*

Proof. The proof follows by some computations and it will be omitted. □

Remark 5.2. It follows from equation (1), when $\frac{b}{\alpha} = \frac{c}{\beta} = \frac{d}{\gamma}$, that $x_{n+1} = ax_{n-k} + \lambda$ for all $n \geq -t$ and for some constant λ.Whenever the quotients $\frac{\alpha}{A}$, $\frac{\beta}{B}$ and $\frac{\gamma}{C}$ are not equal, we get the following result.

Theorem 5.3. *The equilibrium point $\overline{x}$ is a global attractor of equation (1) if one of the following statements holds*

$$(1)\frac{b}{\alpha} \geq \frac{c}{\beta} \geq \frac{d}{\gamma} \quad \text{ and } \quad d \geq b + c \tag{19}$$

$$(2)\frac{b}{\alpha} \geq \frac{d}{\gamma} \geq \frac{c}{\beta} \quad \text{ and } \quad c \geq b + d \tag{20}$$

$$(3)\frac{c}{\beta} \geq \frac{b}{\alpha} \geq \frac{d}{\gamma} \quad and \quad d \geq b+c \tag{21}$$

$$(4)\frac{c}{\beta} \geq \frac{d}{\gamma} \geq \frac{b}{\alpha} \quad and \quad b \geq c+d \tag{22}$$

$$(5)\frac{d}{\gamma} \geq \frac{c}{\beta} \geq \frac{b}{\alpha} \quad and \quad b \geq c+d \tag{23}$$

$$(6)\frac{d}{\gamma} \geq \frac{b}{\alpha} \geq \frac{c}{\beta} \quad and \quad c \geq b+d \tag{24}$$

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of equation (1) and again let f be a function defined by equation (4). We will prove the theorem when case (1) is true and the proof of the other cases are similar and so we will be omitted. Assume that (19) is true, then it is easy from the equations after equation (4) to see that the function $f(u, v, w, t)$ is non-decreasing in u, v and non-increasing in t and it is not clear what is going on with w. So we consider the following two cases:-
Case(1) Assume that the function $f(u, v, w, t)$is non-decreasing in w. Suppose that (m, M) is a solution of the system $M = f(M, M, M, m)$ and $m = g(m, m, m, M)$. Then from equation (1), we see that

$$(\alpha+\beta)(1-a)M^2 + \gamma(1-a)Mm = (b+c)M + dm,$$

$$(\alpha+\beta)(1-a)m^2 + \gamma(1-a)Mm = (b+c)m + dM$$

Subtracting this two equations we obtain

$$(M-m)\big\{(\alpha+\beta)(1-a)(M+m) + (d-b-c)\big\} = 0,$$

under the conditions $d \geq b+c$, $a < 1$, we see that $M = m$. It follows by theorem (1.2) that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete.
Case(2) Assume that the function $f(u, v, w, t)$is non-increasing in w. Suppose that (m, M) is a solution of the system $M = f(M, M, m, m)$ and $m = g(m, m, M, M)$.Then from equation (1), we see that

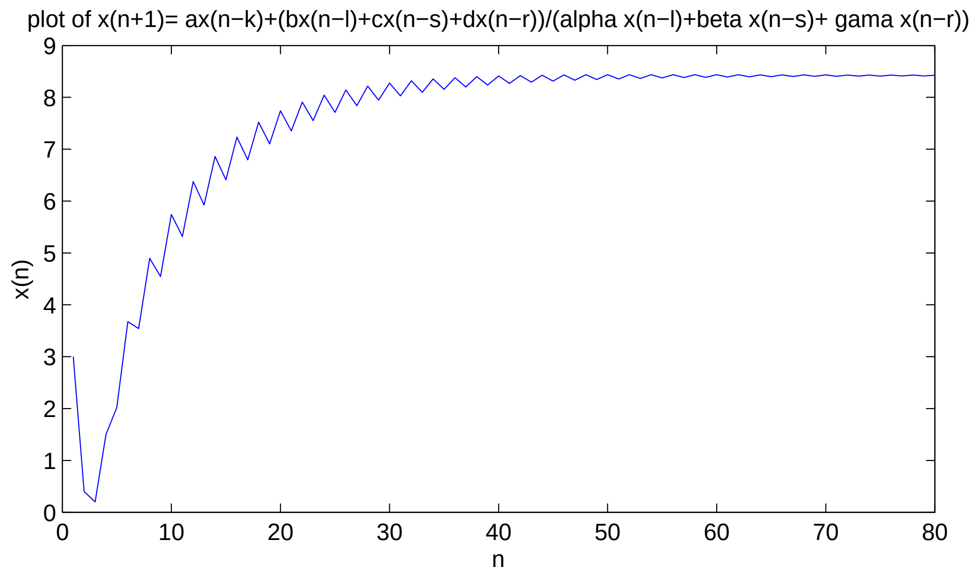
$$M(1-a) = \frac{bM + cm + dm}{\alpha M + \beta m + \gamma m}, \quad m(1-a) = \frac{bm + cM + dM}{\alpha m + \beta M + \gamma M}$$

then under the conditions $d \geq b+c$, $a < 1$, we see that $M = m$. It follows by theorem (1.2) that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete. □

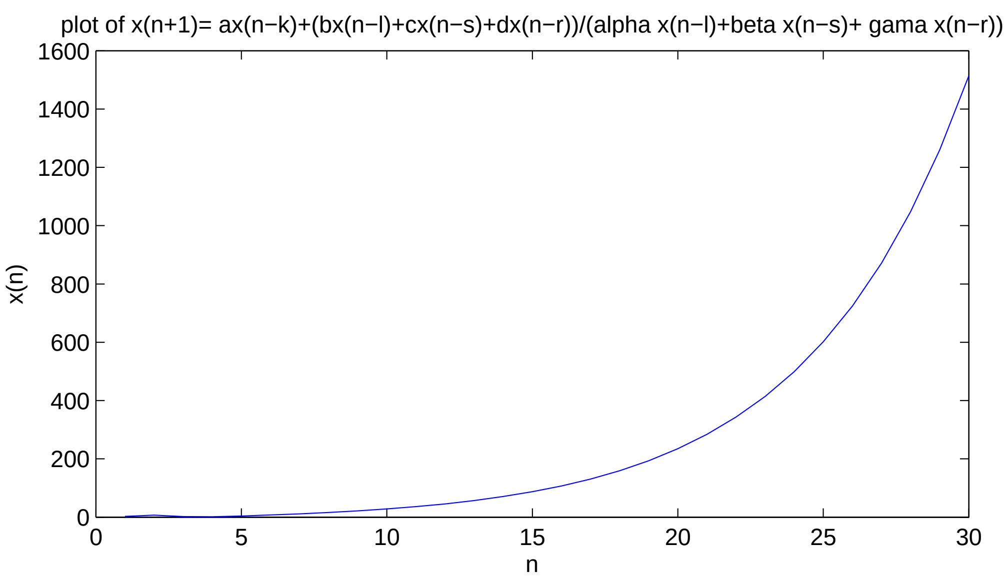
6 Numerical examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to equation (1).

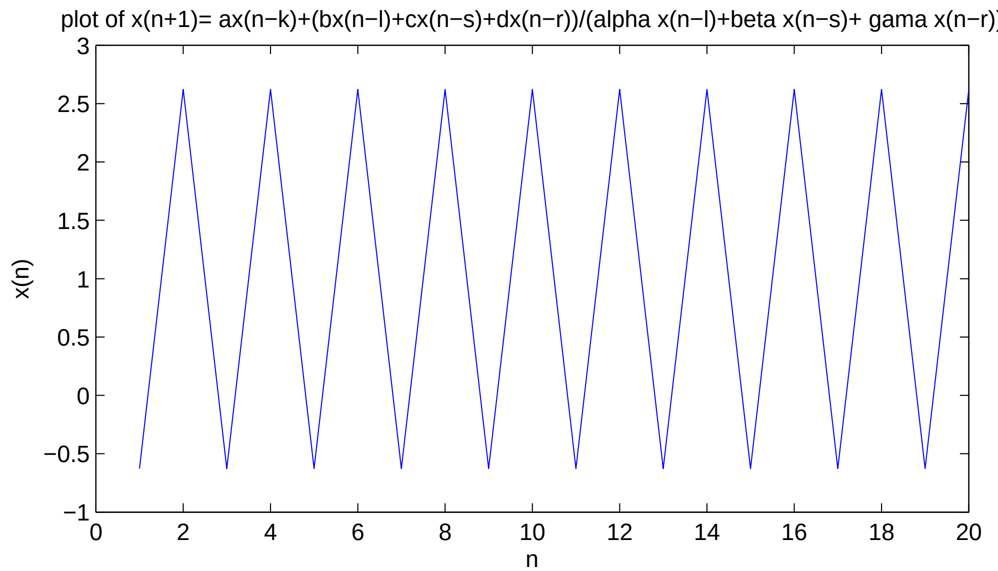
Example 6.1. See Fig.1, since $l = 0, k = 1, s = 2, r = 1, x_{-2} = 3, x_{-1} = 0.4, x_0 = 0.2, a = 0.8, b = 0.4, c = 0.8, d = 2, \alpha = 0.1, \beta = 1, \gamma = 0.8$.



Example 6.2. See Fig.2, since $l = 3, k = 0, s = 1, r = 2, x_{-3} = 3, x_{-2} = 7, x_{-1} = 2, x_0 = 1.5, a = 1.2, b = 3, c = 5, d = 2, \alpha = 1, \beta = 2.1, \gamma = 1.1$.



Example 6.3. See Fig.3, since $l = 1, k = 0, s = 2, r = 3, x_{-3} = x_{-1} = p, x_{-2} = x_0 = q, a = 0.6, b = 7, c = 3, d = 9, \alpha = 3.8, \beta = 0.2, \gamma = 1.2$.



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References

[1] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = (ax_{n-1})/(1 + bx_nx_{n-1})$, Appl. Math. Comp., 156 (2004), 587-590.

[2] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - (bx_n)/(cx_n - dx_{n-1})$, Adv. Differ. Equ., Volume 2006, Article ID 82579, 1–10.

[3] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Advances in Difference Equations 2011, 2011:28.

[4] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5 (2) (2007), 101-113.

[5] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.

[6] E. M. Elsayed, On the solution of some difference equations, European Journal of Pure and Applied Mathematics, 4 (3) (2011), 287-303.

[7] E. M. Elsayed, Solutions of rational difference system of order two, Math. Comp. Mod., 55 (2012), 378–384.

[8] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, Journal of Computational Analysis and Applications, To appear in vol 15 (2013).

[9] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[10] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall / CRC Press, 2001.

[11] M. Saleh and M. Aloqeili, On the difference equation $y_{n+1} = A + ((y_n)/(y_{n-k}))$ with $A < 0$, Appl. Math. Comp., 176 (1) (2006), 359–363.

[12] D. Simsek, C. Cinar and I. Yalcinkaya, On the recursive sequence $x_{n+1} = ((x_{n-3})/(1 + x_{n-1}))$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 475-480.

[13] C. Wang, S. Wang, Z. Wang, H. Gong, R. Wang, Asymptotic stability for a class of nonlinear difference equation, Dis. Dyn. Nat. Soc., Volume 2010, Article ID 791610, 10pages.

[14] C. Wang, Q. Shi, S. Wang. Asymptotic behavior of equilibrium point for a family of rational difference equation. Advances in Difference Equations, Volume 2010, Article ID 505906. 10 pages.

[15] I. Yalnkaya, and C. Cinar, On the dynamics of the difference equation $x_{n+1} = ((ax_{n-k})/(b + cx_n^p))$, Fasciculi Mathematici, 42 (2009), 133-139.

[16] I. Yalnkaya, On the difference equation $x_{n+1} = \alpha+((x_{n-m})/(x_n^k))$, Discrete Dynamics in Nature and Society, Vol. 2008, Article ID 805460, 8 pages, doi: 10.1155/2008/ 805460.

[17] I. Yalnkaya, C. Cinar and M. Atalay, On the solutions of systems of difference equations, Advances in Difference Equations, Vol. 2008, Article ID 143943, 9 pages, doi: 10.1155/2008/143943.

[18] I. Yalnkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Combinatoria, 95 (2010), 151-159.

[19] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - ((bx_n)/(cx_n - dx_{n-k}))$, Comm. Appl. Nonli. Anal., 15 (2008), 47-57.

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Quadratic derivations on non-Archimedean Banach algebras

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Abstract. Let A be an algebra and X be an A-module. A quadratic mapping $D : A \to X$ is called a quadratic derivation if

$$D(ab) = D(a)b^2 + a^2D(b)$$

for all $a_1, a_2 \in A$. We investigate the Hyers-Ulam stability of quadratic derivations from a non-Archimedean Banach algebra A into a non-Archimedean Banach A-module.

1. Introduction

A definition of stability in the case of homomorphisms between metric groups was proposed by a problem by Ulam [32] in 1940. In 1941, Hyers [17] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Rassias [27] for linear mappings by considering an unbounded Cauchy difference (see [3, 4, 8, 10, 18, 19, 22, 25, 29]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [1, 20]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)).$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [31]). Cholewa [6], Czerwik [7] and Grabiec [16] have generalized the results of stability of quadratic mappings. Borelli and Forti [5] generalized the stability result as follows (cf. [23, 24]): Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, where $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property

$$\|f(x) - Q(x)\| \le \Phi(x, x)$$

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for all $x \in G$.

Let $\mathbb{K}$ be a field.

A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is nontrivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

$(NA1)$ $\|x\| = 0$ if and only if $x = 0$;
$(NA2)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
$(NA3)$ the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. It follows from $(NA3)$ that

$$\|x_m - x_\ell\| \leq \max\{\|x_{\jmath+1} - x_\jmath\| : \ell \leq \jmath \leq m - 1\} \qquad (m > \ell).$$

Therefore, a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra A which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. A non-Archimedean Banach space X is a non-Archimedean Banach A-bimodule if X is an A-bimodule which satisfies $\max\{\|xa\|, \|ax\|\} \leq \|a\|\|x\|$ for all $a \in A$, $x \in X$. For more detailed definitions of non-Archimedean Banach algebras, we can refer to [30].

Let A be a normed algebra and let X be a Banach A-module. We say that a mapping $D : A \to X$ is a quadratic derivation if D is a quadratic mapping satisfying

$$D(x_1x_2) = D(x_1)x_2^2 + x_1^2D(x_2) \tag{1.2}$$

for all $x_1, x_2 \in A$.

Recently, the stability of derivations has been investigated by a number of mathematicians including [2, 11, 12, 13, 14, 15, 21, 26, 28] and references therein. More recently, Eshaghi Gordji [9] established the stability of ring derivations on non-Archimedean Banach algebras.

In this paper, we investigate the approximately quadratic derivations on non-Archimedean Banach algebras.

2. Main results

In the following we suppose that A is a non-Archimedean Banach algebra and X is a non-Archimedean Banach A-bimodule. Assume that $|2| \neq 1$.

Theorem 2.1. *Let $f : A \to X$ be a given mapping with $f(0) = 0$ and let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2f(x_2)\| \leq \varphi_1(x_1, x_2), \tag{2.1}$$

$$||f(x + y) + f(x - y) - 2f(x) - 2f(y)|| \leq \varphi_2(x, y) \tag{2.2}$$

for all $x_1, x_2, x, y \in A$. Assume that for each $x \in A$

$$\lim_{n\to\infty} \max\left\{\frac{1}{|2|^{2k}}\frac{\varphi_2(2^kx, 2^kx)}{|2|^2} : 0 \leq k \leq n - 1\right\}$$

denoted by $\Psi(x, x)$, exists. Suppose

$$\lim_{n\to\infty} \frac{\varphi_1(2^nx_1, 2^nx_2)}{|2|^{4n}} = \lim_{n\to\infty} \frac{\varphi_2(2^nx, 2^ny)}{|2|^{2n}} = 0$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \Psi(x, x) \tag{2.3}$$

for all $x \in A$.

Proof. Setting $y = x$ in (2.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi_2(x, x) \tag{2.4}$$

for all $x \in A$, and then dividing by $|2|^2$ in (2.4), we obtain

$$\left\|\frac{f(2x)}{2^2} - f(x)\right\| \leq \frac{\varphi_2(x, x)}{|2|^2} \tag{2.5}$$

for all $x \in A$. Replacing x by $2x$ and then dividing by $|2|^2$ in (2.5), we obtain

$$\left\|\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}\right\| \leq \frac{\varphi_2(2x, 2x)}{|2|^4}. \tag{2.6}$$

Combining (2.5), (2.6) and the strong triangle inequality (NA3) yields

$$\left\|\frac{f(2^2x)}{2^4} - f(x)\right\| \leq \max\left\{\frac{\varphi_2(2x, 2x)}{|2|^4}, \frac{\varphi_2(x, x)}{|2|^2}\right\}. \tag{2.7}$$

Following the same argument, one can prove by induction that

$$\left\|\frac{f(2^nx)}{2^{2n}} - f(x)\right\| \leq \max\left\{\frac{1}{|2|^2}\frac{\varphi_2(2^kx, 2^kx)}{|2|^{2k}} : 0 \leq k \leq n-1\right\}. \tag{2.8}$$

Replacing x by $2^{n-1}x$ and dividing by $|2|^{2(n-1)}$ in (2.5), we find that

$$\left\|\frac{f(2^nx)}{2^{2n}} - \frac{f(2^{n-1}x)}{2^{2(n-1)}}\right\| \leq \frac{\varphi_2(2^{n-1}x, 2^{n-1}x)}{|2|^{2n}}$$

for all positive integers n and all $x \in A$. Hence $\{\frac{f(2^nx)}{2^{2n}}\}$ is a Cauchy sequence. Since X is complete, it follows that $\{\frac{f(2^nx)}{2^{2n}}\}$ is convergent. Set $D(x) = \lim_{n\to\infty}\frac{f(2^nx)}{2^{2n}}$. By taking the limit as $n \to \infty$ in (2.8), we see that $\|D(x) - f(x)\| \leq \Psi(x, x)$ and (2.3) holds for all $x \in A$.

In order to show that D satisfies (1.2), replacing x_1, x_2 by $2^nx_1, 2^nx_2$ in (2.1), and dividing both sides of (2.1) by $|2|^{4n}$, we get

$$\left\|\frac{f(2^nx_1 \cdot 2^nx_2)}{2^{4n}} - \frac{f(2^nx_1)}{2^{4n}} \cdot (2^nx_2)^2 - (2^nx_1)^2.\frac{f(2^nx_2)}{2^{4n}}\right\| \leq \frac{\varphi_1(2^nx_1, 2^nx_n)}{|2|^{4n}}.$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.2).

Replacing x by 2^nx and y by 2^ny in (2.2) and dividing by $|2|^{2n}$, we get

$$\left\|\frac{f(2^nx + 2^ny)}{2^{2n}} + \frac{f(2^nx - 2^ny)}{2^{2n}} - 2\frac{f(2^nx)}{2^{2n}} - 2\frac{f(2^ny)}{2^{2n}}\right\| \leq \frac{\varphi_2(2^nx, 2^ny)}{|2|^{2n}}.$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.1).

Now, suppose that there is another such mapping $D' : A \to X$ satisfying $D'(x+y)+D'(x-y) = 2D'(x)+2D'(y)$ and $\|D'(x) - f(x)\| \leq \Psi(x, x)$. Then for all $x \in A$, we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \lim_{n\to\infty}\frac{1}{|2|^{2n}}\|D(2^nx) - D'(2^nx)\| \\
&\leq \lim_{n\to\infty}\frac{1}{|2|^{2n}}\max\{\|D(2^nx) - f(2^nx)\|, \|D'(2^nx) - f(2^nx)\|\} \\
&\leq \lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{|2|^2}\max\{\frac{\varphi_2(2^jx, 2^jx)}{|2|^{2j}} : n \leq j \leq k+n-1\} = 0.
\end{aligned}$$

It follows that $D(x) = D'(x)$. □

Corollary 2.2. *Let θ_1 and θ_2 be nonnegative real numbers, and let p be a real number such that $p > 4$. Suppose that a mapping $f : A \to X$ satisfies*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2f(x_2)\| \leq \theta_1(\|x_1\|^p + \|x_2\|^p),$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \theta_2(\|x\|^p + \|y\|^p)$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \lim_{n\to\infty} \max\left\{\frac{\theta_2\|x\|^p}{|2|.|2|^{k(2-p)}}\ 0 \leq k \leq n-1\right\}$$

for all $x \in A$.

Proof. Let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that $\varphi_1(x_1,, x_2) = \theta_1(\|x_1\|^p + \|x_2\|^p)$ and $\varphi_2(x, y) = \theta_2(\|x\|^p + \|y\|^p)$ for all $x_1, x_2, x, y \in A$. Then we have

$$\lim_{n\to\infty} \frac{\varphi_2(2^n x, 2^n y)}{|2|^{2n}} = \lim_{n\to\infty} \theta_2 \cdot |2|^{n(p-2)} \cdot (\|x\|^p + \|y\|^p) = 0 \quad (x, y \in A),$$

$$\lim_{n\to\infty} \frac{\varphi_1(2^n x_1, 2^n x_2)}{|2|^{4n}} = \lim_{n\to\infty} \frac{\theta_1|2|^{pn}}{|2|^{4n}}(\|x_1\|^p + \|x_2\|^p) = 0 \quad (x_1, x_2 \in A).$$

Applying Theorem 2.1, we conclude the required result. □

Theorem 2.3. *Let $f : A \to X$ be a mapping and let $\varphi_1 : A \times A \to \mathbb{R}^+$, $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2 f(x_2)\| \leq \varphi_1(x_1, x_2), \tag{2.9}$$

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi_2(x, y) \tag{2.10}$$

for all $x_1, x_2, x, y \in A$. Assume that for each $x \in A$

$$\lim_{n\to\infty} \max\left\{|2|^{2k}\varphi_2\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k \leq n-1\right\}$$

denoted by $\Psi(x, x)$, exists. Suppose

$$\lim_{n\to\infty} |2|^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right) = \lim_{n\to\infty} |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \Psi(x, x) \tag{2.11}$$

for all $x \in A$.

Proof. Setting $y = x$ in (2.10), we obtain

$$\|f(2x) - 4f(x)\| \leq \varphi_2(x, x). \tag{2.12}$$

Replacing x by $\frac{x}{2}$ in (2.12), one obtains

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi_2\left(\frac{x}{2}, \frac{x}{2}\right). \tag{2.13}$$

Again replacing x by $\frac{x}{2}$ in (2.13) and multiplying by $|2|^2$, we obtain that

$$\left\|2^2 f\left(\frac{x}{2}\right) - 2^4 f\left(\frac{x}{2^2}\right)\right\| \leq |2|^2\varphi_2\left(\frac{x}{2^2}, \frac{x}{2^2}\right). \tag{2.14}$$

By using (2.13), (2.14) and strong triangle inequality (NA3), we get

$$\left\|f(x) - 2^4 f\left(\frac{x}{2^2}\right)\right\| \leq \max\left\{\varphi_2\left(\frac{x}{2}, \frac{x}{2}\right), |2|^2\varphi_2\left(\frac{x}{2^2}, \frac{x}{2^2}\right)\right\} \tag{2.15}$$

for $x \in A$.

Next we prove by induction that

$$\left\|f(x) - 2^{2n} f\left(\frac{x}{2^n}\right)\right\| \leq \max\left\{|2|^{2k}\varphi_2\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k \leq n-1\right\}. \tag{2.16}$$

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying by $|2|^{2(n-1)}$ in (2.13), we obtain

$$\left\|2^{2(n-1)} f\left(\frac{x}{2^{n-1}}\right) - 2^{2n} f\left(\frac{x}{2^n}\right)\right\| \leq |2|^{2(n-1)}\varphi_2\left(\frac{x}{2^n}, \frac{x}{2^n}\right) \tag{2.17}$$

for all $x \in A$. Hence $\{2^{2n} f(\frac{x}{2^n})\}$ is a Cauchy sequence. Since X is complete, it follows that $\{2^{2n} f(\frac{x}{2^n})\}$ is convergent. Set $D(x) = \lim_{n\longrightarrow\infty}\{2^{2n} f(\frac{x}{2^n})\}$. By taking the limit as $n \to \infty$ in (2.16), we see that $\|f(x) - D(x)\| \leq \Psi(x, x)$ and (2.11) holds for all $x \in A$.

Replacing x_1, x_2 by $\frac{x_1}{2^n}, \frac{x_2}{2^n}$ in (2.9) and multiplying by $|2|^{4n}$, we get

$$\left\|2^{4n} f\left(\frac{x_1}{2^n} \cdot \frac{x_2}{2^n}\right) - 2^{4n} f\left(\frac{x_1}{2^n}\right)\left(\frac{x_2}{2^n}\right)^2 - 2^{4n}\left(\frac{x_1}{2^n}\right)^2 f\left(\frac{x_2}{2^n}\right)\right\| \leq 2^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right).$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.2).

Replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (2.10) and multiplying by $|2|^{2n}$, we have

$$\left\|2^{2n} f\left(\frac{x}{2^n} + \frac{y}{2^n}\right) + 2^{2n} f\left(\frac{x}{2^n} - \frac{y}{2^n}\right) - 2^{2n} \cdot 2f\left(\frac{x}{2^n}\right) - 2^{2n} \cdot 2f\left(\frac{y}{2^n}\right)\right\| \leq |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right).$$

Taking the limit as $n \to \infty$, we find that D satisfies (1.1).
Now, suppose that there is another such mapping $D' : A \to X$ satisfying $D'(x+y) + D'(x-y) = 2D'(x) + 2D'(y)$ and $\|D'(x) - f(x)\| \leq \Psi(x, x)$. Then for all $x \in A$, we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \lim_{n\to\infty} |2|^{2n} \left\|D\left(\frac{x}{2^n}\right) - D'\left(\frac{y}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} |2|^{2n} \max\left\{\left\|D\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|, \left\|D'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right\} \\
&\leq \lim_{n\to\infty}\lim_{k\to\infty} \max\left\{\varphi_2\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) : n \leq j \leq k+n-1\right\} = 0
\end{aligned}$$

and so $D(x) = D'(x)$ for all $x \in A$. □

Corollary 2.4. *Let θ_1 and θ_2 be nonnegative real numbers, and let p be a positive real number such that $p < 2$. Suppose that a mapping $f : A \to X$ satisfies*

$$\|f(x_1x_2) - f(x_1)x_2^2 - x_1^2 f(x_2)\| \leq \theta_1(\|x_1\|^p + \|x_2\|^p),$$
$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta_2(\|x\|^p + \|y\|^p)$$

for all $x_1, x_2, x, y \in A$. Then there exists a unique quadratic derivation $D : A \to X$ such that

$$\|D(x) - f(x)\| \leq \lim_{n\to\infty} \max\{\theta_2\|x\|^p.|2|^{(k+1)(1-p)}\ 0 \leq k \leq n-1\}$$

for all $x \in A$.

Proof. Let $\varphi_1 : A \times A \to \mathbb{R}^+$ and $\varphi_2 : A \times A \to \mathbb{R}^+$ be functions such that $\varphi_1(x_1, , x_2) = \theta_1(\|x_1\|^p + \|x_2\|^p)$ and $\varphi_2(x, y) = \theta_2(\|x\|^p + \|y\|^p)$ for all $x_1, x_2, x, y \in A$. We have

$$\lim_{n\to\infty} |2|^{2n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n\to\infty}(|2|^{n(2-p)})\theta_2(\|x\|^p + \|y\|^p) = 0 \quad (x, y \in A),$$
$$\lim_{n\to\infty} |2|^{4n}\varphi_1\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}\right) = \lim_{n\to\infty} |2|^{n(4-p)}\theta_1(\|x_1\|^p + \|x_2\|^p) = 0 \quad (x_1, x_2 \in A).$$

Applying Theorem 2.4, we conclude the required result. □

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References

[1] J. Aczel and J. Dhombres, *Functional equations in several variables*, Cambridge Univ. Press, 1989.
[2] M. Bavand Savadkouhi, M. Eshaghi Gordji, J. M. Rassias and N. Ghobadipour, *Approximate ternary Jordan derivations on Banach ternary algebras*, J. Math. Phys. **50** (2009), Article ID 042303.
[3] C. Borelli, *On Hyers-Ulam stability for a class of functional equations*, Aequationes Math. **54** (1997), 74–86.
[4] D.G. Bourgin, *Class of transformations and bordering transformations*, Bull. Amer. Math. Soc. **27** (1951), 223–237.
[5] C. Borelli and G.L. Forti, *On a general Hyers-Ulam stability result*, Internat. J. Math. Math. Sci. **18** (1995), 229–236.
[6] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[7] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[8] A. Ebadian, N. Ghobadipour and M. Eshaghi Gordji, *A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras*, J. Math. Phys. **51** (2010), Article ID 103508.

[9] M. Eshaghi Gordji, *Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras*, Abs. Appl. Anal. **2010**, Article ID 393247 (2010).

[10] M. Eshaghi Gordji and A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 724–729.

[11] M. Eshaghi Gordji, R. Farokhzad Rostami and S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras*, J. Comput. Anal. Appl. **13** (2011), 734–742.

[12] M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi and M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras*, J. Comput. Anal. Appl. **12** (2010), 463–470.

[13] M. Eshaghi Gordji, J.M. Rassias and N. Ghobadipour, *Generalized Hyers-Ulam stability of the generalized (n,k)–derivations*, Abs. Appl. Anal. **2009**, Article ID 437931 (2009).

[14] R. Farokhzad and S.A.R. Hosseinioun, *Perturbations of Jordan higher derivations in Banach ternary algebras: An alternative fixed point approach*, Int. J. Nonlinear Anal. Appl. **1** (2010), No. 1, 42–53.

[15] N. Ghobadipour, A. Ebadian, Th.M. Rassias and M. Eshaghi Gordji, *A perturbation of double derivations on Banach algebras*, Commun. Math. Anal. **11** (2011), 51–60.

[16] A. Grabiec, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen **48** (1996) 217–235.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[18] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel, Berlin, 1998.

[19] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of ψ-additive mappings*, J. Approx. Theory. **72** (1993), 131–137.

[20] Pl. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math. **27** (1995), 368–372.

[21] H. Kim and I. Chang, *stability of the functional equations related to a multiplicative derivation*, J. Appl. Computing. **11** (2003), 413–421.

[22] A. Najati and Th.M. Rassias, *Stability of a mixed functional equation in several variables on Banach modules*, Nonlinear Anal.–TMA **72** (2010), 1755–1767.

[23] C. Park, *Generalized quadratic mappings in several variables*, Nonlinear Anal.–TMA **57** (2004), 713–722.

[24] C. Park, *On the stability of the quadratic mapping in Banach modules*, J. Math. Anal. Appl. **276** (2002), 135–144.

[25] C. Park and Th.M. Rassias, *Hyers-Ulam stability of a generalized Apollonius type quadratic mapping*, J. Math. Anal. Appl. **322** (2006), 371–381.

[26] C. Park and Th.M. Rassias, *Homomorphisms and derivations in proper $JCQ*$-triples*, J. Math. Anal. Appl. **337** (2008), 1404–1414.

[27] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[28] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras*, J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[29] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Nearly ternaty cubic homomorphisms in ternary Fréchet algebras*, J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[30] N. Shilkret, *Non-Archimedian Banach algebras*, Thesis (Ph.D.)-Polytechnic Univ., ProQuest LLC, 1968.

[31] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano. **53** (1983), 113–129.

[32] S.M. Ulam, *A collection of Mathematical problems*, Interscience Publ, New York., 1960.

Soft q-ideals of soft BCI-algebras

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Abstract. The notion of soft q-ideals and q-idealistic soft BCI-algebras are introduced, and several properties of them are investigated. Characterizations of a (fuzzy) q-ideals in BCI-algebras are considered. Relations between fuzzy q-ideals and p-idealistic soft BCI-algebras are discussed.

1. Introduction

D. Molodtsov ([2]) introduced introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical applications. Y. B. Jun ([6]) applied first the notion of soft sets by Moldtsov to the theory of BCK-algebras. Y. B. Jun and C. H. Park ([8]) dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They discussed the algebraic properties of soft sets in BCK/BCI-algebras. In [7], Y. B. Jun, K. J. Lee and J. Zhan introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras, and investigated their properties. Y. S. Hwang and S. S. Ahn ([5]) defined the notion of vague q-ideal of a BCI-algebra and studied several properties of them.

In this paper, we introduced the notion of soft q-ideals and q-idealistic soft BCI-algebras, and investigate several properties of them. We also consider characterizations of a (fuzzy) q-ideals in BCI-algebras and study relations between fuzzy q-ideals and p-idealistic soft BCI-algebras.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)$ $(x * x = 0)$,
(a4) $(\forall x, y \in X)$ $(x * y = 0,\ y * x = 0 \Rightarrow\ x = y)$.

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In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in X)\,(\forall y \in A)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *q-ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A)$.

Note that any q-ideal is an ideal, but the converse is not true in general.

We refer the reader to the book [4] for further information regarding BCI-algebras.

Molodtsov ([2]) defined the soft set in the following way: Let U be an initial set and E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A \subset E$.

Definition 2.1.([2]) A pair $(\mathscr{F}, A)$ is called a *soft set* over U, where $\mathscr{F}$ is a mapping given by

$$\mathscr{F} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $\mathscr{F}(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set $(\mathscr{F}, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [2].

Definition 2.2.([3]) Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cap B$,
(ii) $(\forall e \in C)(\mathscr{H}(e) = \mathscr{F}(e)$ or $\mathscr{G}(e)$, (as both are same sets)).

In this case, we write $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 2.3.([3]) Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cup B$,
(ii) $\forall e \in C$

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \backslash B \\ \mathscr{G}(e) & \text{if } e \in B \backslash A \\ \mathscr{F}(e) \cup \mathscr{G}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 2.4.([3]) If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)AND(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cap \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.5.([3]) If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)OR(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\tilde{\vee}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\tilde{\vee}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cup \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.6.([3]) For two soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over a common universe U, we say that $(\mathscr{F}, A)$ is a *soft subset* of $(\mathscr{G}, B)$, denoted by $(\mathscr{F}, A)\tilde{\subset}(\mathscr{G}, B)$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\epsilon \in A$, $\mathscr{F}(\epsilon)$ and $\mathscr{G}(\epsilon)$ are identical approximations.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh ([11]).

3. Soft q-ideals

In what follows let X and A be a BCI-algebra and a nonempty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of X, that is, R is a subset of $A \times X$ without otherwise specified. A set-valued function $\mathscr{F} : A \to \mathscr{P}(X)$ can be defined as $\mathscr{F}(x) = \{y \in X | (x, y) \in R\}$ for all $x \in A$. The pair $(\mathscr{F}, A)$ is then a soft set over X.

Definition 3.1.([8]) Let S be a subalgebra of X. A subset I of X is called an *ideal* of X related to S (briefly, S-ideal of X), denoted by $I \lhd S$, if it satisfies:
(i) $0 \in I$,
(ii) $(\forall x \in S)(\forall y \in I)(x * y \in I \Rightarrow x \in I)$.

Definition 3.2. Let S be a subalgebra of X. A subset I of X is called a *q-ideal* of X related to S (briefly, S-q-ideal of X), denoted by $I \lhd_q S$, if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x, z \in S)(\forall y \in I)(x * (y * z) \in I \Rightarrow x * z \in I)$.

Example 3.3. Let $X := \{0, 1, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Then $S := \{0, 1, a\}$ is a subalgebra of X and $I := \{0, 1\}$ is an S-q-ideal of X.

Note that every S-q-ideal of X is an S-ideal of X ($\because$ Put $z := 0$ in Definition 3.2(ii)). But the converse is not true in general as seen in the following example.

Example 3.4. Let $X := \{0, 1, 2, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	0	b	a
2	2	1	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Then $S := \{0, a, b\}$ is a subalgebra of X and $\{0\}$ is an S-ideal of X, but not an S-q-ideal of X, since $a * (0 * b) = a * a = 0 \in \{0\}$, $0 \in \{0\}$, and $a * b = b \notin \{0\}$.

Definition 3.5.([6]) Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called a *soft BCI-algebra* over X if $\mathscr{F}(x)$ is a subalgebra of X for all $x \in A$.

Definition 3.6.([8]) Let $(\mathscr{F}, A)$ be a soft set over X. A soft set $(\mathscr{G}, I)$ over X is called a *soft ideal* of $(\mathscr{F}, A)$, denoted by $(\mathscr{G}, I)\tilde{\lhd}(\mathscr{F}, A)$ if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)(\mathscr{G}(x) \lhd \mathscr{F}(x))$.

Definition 3.7. Let $(\mathscr{F}, A)$ be a soft set over X. A soft set $(\mathscr{G}, I)$ over X is called a *soft q-ideal* of $(\mathscr{F}, A)$, denoted by $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$ if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)(\mathscr{G}(x) \lhd_q \mathscr{F}(x))$.

Example 3.8. Consider a BCI-algebra $X = \{0, 1, a, b\}$ which is given in Example 3.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{0, 1, a\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) \in \{0, 1, a\}\}$$

for all $x \in A$. Then $\mathscr{F}(0) = \mathscr{F}(1) = \mathscr{F}(a) = X$, which are subalgebras of X. Hence $(\mathscr{F}, A)$ is a soft BCI-algebra over X. Let $I := \{0, 1\}$ and $\mathscr{G} : I \to (X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $\mathscr{G}(0) = \{0,1\} \lhd_q \mathscr{F}(0)$ and $\mathscr{G}(1) = \{0,1\} \lhd_q \mathscr{F}(1)$. Hence $(\mathscr{G}, I)$ is a soft q-ideal of $(\mathscr{F}, A)$.

Note that every soft q-ideal is a soft ideal. But the converse is not true in general as seen in the following example.

Example 3.9. Let $X := \{0, 1, a, b\}$ be a BCI-algebra ([4]) in which the $*$-operation is given by the following table:

$*$	0	1	a	b
0	0	0	b	a
1	1	0	b	a
a	a	a	0	b
b	b	b	a	0

For $A = \{0,1\}$, define a set-valued function $\mathscr{F}(x) : A \to \mathscr{P}(X)$ by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) = 0\}$$

for all $x \in A$. Then $\mathscr{F}(0) = X$ and $\mathscr{F}(1) = \{0, a, b\}$ are subalgebras of X. Hence $(\mathscr{F}, A)$ is a soft BCI-algebra over X. For $I := \{0\}$, let $\mathscr{G} : I \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $\mathscr{G}(0) = \{0,1\}$. Hence $\mathscr{G}(0) \lhd \mathscr{F}(0)$, but $\mathscr{G}(0) \not\lhd_q \mathscr{F}(0)$ since $a * (0 * b) = 0, 0 \in \{0,1\}$ and $a * b = b \notin \{0,1\}$.

Theorem 3.10. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}_1, I_1)$ and $(\mathscr{G} I_2)$ over X where $I_1 \cap I_2 \neq \emptyset$, we have*

$$(\mathscr{G}_1, I_1) \tilde{\lhd}_q (\mathscr{F}, A), (\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A) \Rightarrow (\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A).$$

Proof. By Definition 2.2, we can write

$$(\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) = (\mathscr{G}, I),$$

where $I = I_1 \cap I_2$ and $\mathscr{G}(x) = \mathscr{G}_1(x)$ or $\mathscr{G}_2(x)$ for all $x \in I$. Obviously, $I \subset A$ and $\mathscr{G} : I \to \mathscr{P}(X)$ is a mapping. Hence $(\mathscr{G}, I)$ is a soft set over X. Since $(\mathscr{G}_1, I_1) \tilde{\lhd}_q (\mathscr{F}, A)$ and $(\mathscr{G}_2, I_2) \tilde{\lhd}_q (\mathscr{F}, A)$, we know that $\mathscr{G}(x) = \mathscr{G}_1(x) \tilde{\lhd}_q \mathscr{F}(x)$ or $\mathscr{G}(x) = \mathscr{G}_2(x) \tilde{\lhd}_q \mathscr{F}(x)$ for all $x \in I$. Hence

$$(\mathscr{G}_1, I_1) \tilde{\cap} (\mathscr{G}_2, I_2) = (\mathscr{G}, I) \tilde{\lhd}_q (\mathscr{F}, A).$$

This completes the proof. $\square$

Corollary 3.11. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}_1, I)$ and $(\mathscr{G}_2, I)$ over X, we have*

$$(\mathscr{G}_1, I) \tilde{\lhd}_q (\mathscr{F}, A), (\mathscr{G}_2, I) \tilde{\lhd}_q (\mathscr{F}, A) \Rightarrow (\mathscr{G}_1, I) \tilde{\cap} (\mathscr{G}_2, I) \tilde{\lhd}_q (\mathscr{F}, A).$$

Proof. Straightforward. □

Theorem 3.12. *Let $(\mathscr{F}, A)$ be a soft BCI-algebra over X. For any soft sets, $(\mathscr{G}, I)$ and $(\mathscr{H}, J)$ over X in where $I \cap J = \emptyset$, we have*

$$(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A), (\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A) \Rightarrow (\mathscr{G}, I)\tilde{\cup}(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A).$$

Proof. Assume that $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$ and $(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A)$. By Definition 2.3, we can write $(\mathscr{G}, I)\tilde{\cup}(\mathscr{H}, J) = (\mathscr{K}, U)$ where $U = I \cup J$ and for every $x \in U$,

$$\mathscr{K}(x) = \begin{cases} \mathscr{G}(x) & \text{if } x \in I\backslash J \\ \mathscr{H}(x) & \text{if } x \in J\backslash I \\ \mathscr{G}(x) \cup \mathscr{H}(x) & \text{if } x \in I \cap J. \end{cases}$$

Since $I \cap J = \emptyset$, either $x \in I\backslash J$ or $x \in J\backslash I$ for all $x \in U$. If $x \in I\backslash J$, then $\mathscr{K}(x) = \mathscr{G}(x) \lhd_q \mathscr{F}(x)$ since $(\mathscr{G}, I)\tilde{\lhd}_q(\mathscr{F}, A)$. If $x \in J\backslash I$, then $\mathscr{K}(x) = \mathscr{H}(x) \lhd_q \mathscr{F}(x)$ since $(\mathscr{H}, J)\tilde{\lhd}_q(\mathscr{F}, A)$. Thus $\mathscr{K}(x) \lhd_q \mathscr{F}(x)$ for all $x \in U$, and so $(\mathscr{G}, I)(\mathscr{H}, J) = (\mathscr{K}, \tilde{\lhd}_q(\mathscr{F}, A)$. □

If I and J are not disjoint in Theorem 3.12, then Theorem 3.12 is not true in general as seen in the following example.

Example 3.13. Consider a BCI-algebra $X = \{0, 1, a, b\}$ which is given in Example 3.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{0, 1, a\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

$$\mathscr{F}(x) = \{0\} \cup \{y \in X | y * (y * x) \in \{0, 1, a\}\}$$

for all $x \in A$. Then $(\mathscr{F}, A)$ is a soft BCI-algebra over X (see Example 3.8). Let $I := \{0, 1\}$ and $\mathscr{G} : I \to (X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | x \leq y\}$$

for all $x \in I$. Then $(\mathscr{G}, I)$ is a soft q-ideal of $(\mathscr{F}, A)$ (see Example 3.8). Let $J := \{0\}$ and $\mathscr{H} : J \to \mathscr{P}(X)$ be defined by

$$\mathscr{H}(x) = \{x, a\}.$$

Then $\mathscr{H}(0) = \{0, a\} \lhd_q \mathscr{F}(0)$. But $\mathscr{G}(0) \cup \mathscr{H}(0) = \{0, 1, a\} \not\lhd_q \mathscr{F}(0)$ since $b * (1 * 0) = a, 1 \in \{0, 1, a\}$ and $b * 0 = b \notin \{0, 1, a\}$.

4. q-idealistic soft BCI-algebras

Definition 4.1.([8]) Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called an *idealistic soft BCI-algebra* over X if $\mathscr{F}(x)$ is an ideal of X for all $x \in A$.

Definition 4.2. Let $(\mathscr{F}, A)$ be a soft set over X. Then $(\mathscr{F}, A)$ is called a *q-idealistic soft BCI-algebra* over X if $\mathscr{F}(x)$ is a q-ideal of X for all $x \in A$.

Example 4.3. Let $X := \{0, a, b, c\}$ be a BCI-algebra ([9]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = X$ and $\mathscr{G} : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{G}(x) = \{0, x\}$$

for all $x \in A$. Then $\mathscr{G}(0) = \{0\}, \mathscr{G}(a) = \{0, a\}$ and $\mathscr{G}(c) = \{0, c\}$, which are ideals of X. Hence $(\mathscr{G}, A)$ is an idealistic soft BCI-algebra over X ([8]). Note that $\mathscr{G}(x)$ is a q-ideal of X for all $x \in A$. Hence $(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X.

For any element x of a BCI-algebra X, we define the order of X, denoted by $o(x)$, as

$$o(x) = \min\{n \in \mathbb{N} | 0 * x^n = 0\}$$

where $0 * x^n = (\cdots((0 * x) * x) * \cdots) * x$ in which x appears n-times.

Example 4.4. Let $X := \{0, a, b, c, de, f, g\}$ be a BCI-algebra ([1]) in which the $*$-operation is given by the following table:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Let $(\mathscr{F}, A)$ be a soft set over X, where $A = \{a, b, c\} \subset X$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined as follows:

$$\mathscr{F}(x) = \{y \in X | o(x) = o(y)\}$$

for all $x \in A$. Then $\mathscr{F}(a) = \mathscr{F}(b) = \mathscr{F}(c) = \{0, a, b, c\}$ is an ideal of X. Hence $(\mathscr{F}, A)$ is an idealistic soft BCI-algebra over X ([6]). If we take $B := \{a, b, d, f\} \subset X$ and define a set-valued function $\mathscr{G} : B \to \mathscr{P}(X)$ by

$$\mathscr{G}(x) = \{0\} \cup \{y \in X | o(x) = o(y)\}$$

for all $x \in B$, then $(\mathscr{G}, B)$ is not a q-idealistic soft BCI-algebra over X. In fact, since $f * (g * e) = d, g \in \{0, d, e, f, g\}$ and $f * e = b \notin \{0, d, e, f, g\}$, $\mathscr{G}(d) = \{0, d, e, f, g\}$ is not a q-ideal of X.

Obviously, every q-idealistic soft BCI-algebra over X is an idealistic soft BCI-algebra over X, but the converse is not true in general as seen in the following example.

Example 4.5. Consider a BCI-algebra $X := Y \times \mathbb{Z}$, where $(Y, *, 0)$ is a BCI-algebra over X and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ integers. Let $\mathscr{F} : X \to \mathscr{P}(X)$ be a set-valued function defined as follows:

$$\mathscr{F}(y, n) = \begin{cases} Y \times \mathbb{N}_0 & \text{if } x \in \mathbb{N}_0 \\ \{(0, 0)\} & \text{otherwise} \end{cases}$$

for all $(y, n) \in X$, where $\mathbb{N}_0$ is the set of all non-negative integers. Then $(\mathscr{F}, X)$ is an idealistic soft BCI-algebra over X([8]). But it is not a q-idealistic soft BCI-algebra over X since $\{(0, 0)\}$ is not a q-ideal of X. In fact, $(0, 3) * ((0, 0) * (0, -3)) = (0, 0) \in \{(0, 0)\}$ and $(0, 3) * (0, -3) = (0, 6) \notin \{(0, 0)\}$.

Proposition 4.6. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be soft sets over X where $B \subseteq A \subseteq X$. If $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X, then so is $(\mathscr{G}, B)$.*

Proof. Straightforward. □

The converse of Proposition 4.6 is not true in general as seen in the following example.

Example 4.7. Consider a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X which is described in Example 4.4. If we take $B := \{a, b, c, d\} \supseteq A = \{a, b, c\}$, then $(\mathscr{F}, B)$ is not a q-idealistic soft BCI-algebra over X since $\mathscr{F}(d) = \{d, e, f, g\}$ is not a q-ideal of X.

Theorem 4.8. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two q-idealistic soft BCI-algebra over X. If $A \cap B \neq \emptyset$, then the intersection $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Using Definition 2.2, we can write $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cap B$ and $\mathscr{H}(x) = \mathscr{F}(x)$ or $\mathscr{G}(x)$ for all $x \in C$. Note that $\mathscr{H} : C \to \mathscr{P}(X)$ is a mapping, and therefore $(\mathscr{H}, C)$ is a soft set over X. Since $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are q-idealistic soft BCI-algebras over X, it follows that $\mathscr{H}(x) = \mathscr{F}(x)$ is a q-ideal of X, or $\mathscr{H}(x) = \mathscr{G}(x)$ is a q-ideal of X for all $x \in C$. Hence $(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. □

Corollary 4.9. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, A)$ be two q-idealistic soft BCI-algebra over X. Then the intersection $(\mathscr{F}, A)\tilde{\cap}(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Straightforward. □

Theorem 4.10. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two q-idealistic soft BCI-algebra over X. If $A \cap B = \emptyset$, then the union $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. Using Definition 2.3, we write $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\mathscr{H}(x) = \begin{cases} \mathscr{F}(x) & \text{if } x \in A \backslash B \\ \mathscr{G}(c) & \text{if } x \in B \backslash A \\ \mathscr{F}(x) \cup \mathscr{G}(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \backslash B$ or $x \in B \backslash A$ for all $x \in C$. If $x \in A \backslash B$, then $\mathscr{H}(x) = \mathscr{F}(x)$ is a q-ideal of X since $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X. If $x \in B \backslash A$, then

$\mathscr{H}(x) = \mathscr{G}(x)$ is a q-ideal of X since $(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. Hence $(\mathscr{F}, A)\tilde{\cup}(\mathscr{G}, A)$ is a q-idealistic soft BCI-algebra over X. □

Theorem 4.11. *If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are q-idealistic soft BCI-algebra over X, then $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X.*

Proof. By Definition 2.4,

$$(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B),$$

where $\mathscr{H}(x, y) = \mathscr{F}(x) \cap \mathscr{G}(y)$ for all $(x, y) \in A \times B$. Since $\mathscr{F}(x)$ and $\mathscr{G}(y)$ are q-ideals of X, the intersection $\mathscr{F}(x) \cap \mathscr{G}(y)$ is also a q-ideal of X. Hence $\mathscr{H}(x, y)$ is a q-ideal of X for all $(x, y) \in A \times B$, and therefore $(\mathscr{F}, A)\tilde{\wedge}(\mathscr{G}, B)$ is a q-idealistic soft BCI-algebra over X. □

Definition 4.12. A q-idealistic BCI-algebra $(\mathscr{F}, A)$ over X is said to be *trivial* (resp., *whole*) if $\mathscr{F}(x) = \{0\}$ (resp., $\mathscr{F}(x) = X$) for all $x \in A$.

Example 4.13. Let $X = \{0, a, b, c\}$ be a BCI-algebra which is given Example 4.3. Let $(\mathscr{F}, A)$ be a soft set over X, where $A := \{a, b, c\} \subset X$, and let $\mathscr{F} : A \to \mathscr{P}(X)$ be a set-valued function defined by

$$\mathscr{F}(x) = \{y \in X | o(x) = o(y)\}$$

for all $x \in X$. Then $\mathscr{F}(a) = \mathscr{F}(b) = \mathscr{F}(c) = X$. It is check that $X \lhd_q X$. Hence $(\mathscr{F}, X \setminus \{0\})$ is a whole q-idealistic soft BCI-algebra over X. Let $\mathscr{G} : \{0\} \to \mathscr{P}(X)$ be a set-valued function defined by $\mathscr{G}(x) = x$ for all $x \in \{0\}$. Then $\mathscr{G}(0) = \{0\}$. It is check that $\{0\} \lhd_q X$. Hence $(\mathscr{G}, \{0\})$ is a trivial q-idealistic soft BCI-algebra over X.

Definition 4.14.([10]) A fuzzy set μ in X is a fuzzy q-ideal of X if it satisfies the following assertions:

(i) $(\forall x \in X)(\mu(0) \geq \mu(x))$,

(ii) $(\forall x, y, z \in X)(\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\})$.

Lemma 4.15. *A fuzzy set μ in X is a fuzzy q-ideal of X if and only if it satisfies:*

$$(\forall t \in [0, 1])(U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t) \text{ is a } q\text{-ideal of } X).$$

Proof. Straightforward. □

Theorem 4.16. *For every fuzzy q-ideal of X, there exists a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X.*

Proof. Let μ be a fuzzy q-ideal of X. Then $U(\mu; t) := \{x \in X | \mu(x) \geq t\}$ is a q-ideal of X for all $t \in Im(\mu)$. If we take $A = Im(\mu)$ and consider a set-valued function $\mathscr{F} : A \to \mathscr{P}(X)$ given by $\mathscr{F}(t) = U(\mu; t)$ for all $t \in A$, then $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X. □

Conversely, the following theorem is straightforward.

Theorem 4.17. *For any fuzzy set μ in X, if a q-idealistic soft BCI-algebra $(\mathscr{F}, A)$ over X is given by $A = Im(\mu)$ and $\mathscr{F}(t) = U(\mu; t)$ for all $t \in A$, then μ is a fuzzy q-ideal of X.*

Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft set over X in which $A = Im(\mu)$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is a set-valued function defined by

(4.1) $(t \in A)(\mathscr{F}(t) = \{x \in X | \mu(x) + t > 1\})$.

Then there exists $t \in A$ such that $\mathscr{F}(t)$ is not a q-ideal of X as seen in the following example.

Example 4.18. For any BCI-algebra X, define a fuzzy set μ in X by $\mu(0) = t_0 < 0.5$ and $\mu(x) = 1 - t_0$ for all $x \neq 0$. Let $A = Im(\mu)$ and $\mathscr{F} : A \to \mathscr{P}(X)$ be a set-valued function given by (4.1). Then $\mathscr{F}(1 - t_0) = X \backslash \{0\}$, which is not a q-ideal of X.

Theorem 4.19. *Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft over X in which $A = [0, 1]$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is given by (4.1). Then the following assertions are equivalent:*

(1) *μ is a fuzzy q-ideal of X,*
(2) *for every $t \in A$ with $\mathscr{F}(t) \neq \emptyset$, $\mathscr{F}(t)$ is a q-ideal of X.*

Proof. Assume that μ is a fuzzy q-ideal of X. Let $t \in A$ be such that $\mathscr{F}(t) \neq \emptyset$. If we select $x \in \mathscr{F}(t)$, then $\mu(0) + t \geq \mu(x) + t > 1$, and so $0 \in \mathscr{F}(t)$. Let $t \in A$ and $x, y, z \in X$ be such that $y \in \mathscr{F}(t)$ and $x * (y * z) \in \mathscr{F}(t)$. Then $\mu(y) + t > 1$ and $\mu(x * (y * z)) + t > 1$. Since μ is a fuzzy q-ideal of X, it follows that

$$\begin{aligned} \mu(x * z) + t &\geq \min\{\mu(x * (y * z)), \mu(y)\} + t \\ &= \min\{\mu(x * (y * z)) + t, \mu(y) + t\} \\ &> 1, \end{aligned}$$

so that $x * z \in \mathscr{F}$. Hence $\mathscr{F}(t)$ is a q-ideal of X with $\mathscr{F}(t) \neq \emptyset$.

Conversely, suppose that (2) is valid. If there exists $a \in X$ such that $\mu(0) < \mu(a)$, then we can select $t_a \in A$ such that $\mu(0) + t_a \leq 1 < \mu(a) + t_a$. It follows that $a \in \mathscr{F}(t_a)$ and $0 \notin \mathscr{F}(t_a)$, which is a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, assume that

$$\mu(a * c) < \min\{\mu(a * (b * c)), \mu(b)\}$$

for some $a, b, c \in X$. Then

$$\mu(a * c) + s_0 \leq 1 < \min\{\mu(a * (b * c)), \mu(b)\} + s_0,$$

for some s_0, which implies $a * (b * c) \in \mathscr{F}(s_0)$ and $b \in \mathscr{F}(s_0)$, but $a * c \in \mathscr{F}(s_0)$. This is a contradiction. Therefore

$$\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\},$$

for all $x, y, z \in X$, and thus μ is a fuzzy q-ideal of X. □

Corollary 4.20. *Let μ be a fuzzy set in X such that $\mu(x) > 0.5$ for some $x \in X$, and let $(\mathscr{F}, A)$ be a soft set over X in which*

$$A := \{t \in Im(\mu) | t > 0.5\}$$

and $\mathscr{F} : A \to \mathscr{P}(X)$ is given by (4.1). If μ is a fuzzy q-ideal of X, then $(\mathscr{F}, A)$ is a q-idealistic soft BCI-algebra over X.

Proof. Straightforward. □

Theorem 4.21. Let μ be a fuzzy set in X and let $(\mathscr{F}, A)$ be a soft set over X in which $A = (0.5, 1]$ and $\mathscr{F} : A \to \mathscr{P}(X)$ is defined by

$$(\forall t \in A)(\mathscr{F}(t) = U(\mu; t)).$$

Then $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$ if and only if the following assertions are valid:

(1) $(\forall x \in X)(\max\{\mu(0), 0.5\} \geq \mu(x))$,
(2) $(\forall x, y, z \in X)(\max\{\mu(x * z), 0.5\} \geq \min\{\mu((x * (y * z)), \mu(y)\})$.

Proof. Assume that $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. If there exists $x_0 \in X$ such that $\max\{\mu(0), 0.5\} < \mu(x_0)$, then we can select $t_0 \in A$ such that $\max\{\mu(0), 0.5\} < t_0 \leq \mu(x_0)$. It follows that $\mu(0) < t_0$, so that $x_0 \in \mathscr{F}(t_0)$ and $0 \notin \mathscr{F}(t_0)$. This is a contradiction, and so (1) is valid. Suppose that there exist $a, b, c \in X$ such that

$$\max\{\mu(a * c), 0.5\} < \min\{\mu(a * (b * c)), \mu(b)\}.$$

Then

$$\max\{\mu(a * c), 0.5\} < u_0 \leq \min\{\mu(a * (b * c)), \mu(b)\}.$$

for some $u_0 \in A$. Thus $a * (b * c) \in \mathscr{F}(u_0)$ and $b \in \mathscr{F}(u_0)$, but $a * c \notin \mathscr{F}(u_0)$. This is a contradiction, and so (2) is valid.

Conversely, suppose that (1) and (2) are valid. Let $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. For any $x \in \mathscr{F}(t)$, we have

$$\max\{\mu(0), 0.5\} \geq \mu(x) \geq t > 0.5$$

and so $\mu(0) \geq t$, i.e., $0 \in \mathscr{F}(t)$. Let $x, y, z \in X$ be such that $y \in \mathscr{F}(t)$ and $x * (y * z) \in \mathscr{F}(t)$. Then $\mu(y) \geq t$ and $\mu(x * (y * z)) \geq t$. It follows from the second condition that

$$\max\{\mu(x * z), 0.5\} \geq \min\{\mu(x * (y * z)), \mu(y)\} \geq t > 0.5,$$

so that $\mu(x * z) \geq t$, i.e., $x * z \in \mathscr{F}(t)$. Therefore $\mathscr{F}(t)$ is a q-ideal of X for all $t \in A$ with $\mathscr{F}(t) \neq \emptyset$. □

References

[1] M. A. Chaudhry, *Weakly positive implicative and weakly implicative BCI-algebras*, Math. Jpn. **35**(1990), 141-151.
[2] D. Molodtsov, *Soft set theory-First results*, Comput. Math. Appl. **37**(1999), 19-31.
[3] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. Appl. **45**(2003), 555-562.
[4] Y. Huang, *BCI-algebras*, Sciences Press, Bejing, 2006.
[5] Y. S. Hwang and S. S. Ahn, *Vague q-ideals in BCI-algebras*, Honam Math. J. **34**(2012), 549-557.
[6] Y. B. Jun, *Soft BCK/BCI-algebras*, Comput. Math. Appl. **56**(2008), 1408-1413.
[7] Y. B. Jun , K. J. Lee and J. Zhan, *Soft p-ideals of soft BCI-algebras*, Comput. Math. Appl. **58**(2009), 2060-2068.
[8] Y. B. Jun and C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci. **178**(2008), 2466-2475.

[9] Y. L. Liu, J. Meng, X. H. Zhang and Z. C. Yue, *q-ideals and a-ideals in BCI-algebras*, Southeast Asian Bull. Math. **24**(2000), 243-253.

[10] S. K. Sung and S. S. Ahn, *Fuzzy q-ideals of BCI-algebras with degrees in the interval* $(0, 1]$, J. Chungcheong Math. Soc. **25**(2012), 241-251.

[11] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8**(1965), 338-353.

Convergence of parallel multisplitting USAOR methods for block $H-$matrices linear systems

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Abstract: In this paper, We present parallel multisplitting blockwise relaxation methods for solving the large sparse blocked linear systems, which come from the discretizations of many discrential equations, and study the convergence of our methods associated with USAOR multisplitting when the coefficient matrices of the blocked linear systems are block H-matrices. A lot of numerical experiments show that our methods are applicable and efficient.

Key words: Block matrix multisplitting; Blockwise relaxation parallel multisplitting method; Convergence; Block H-matrix.,

2000 MR Subject Classification: 65F10, 65F50

1. Introduction

For the linear system

$$Ax = b, \tag{1.1}$$

where A is an $n\times n$ square matrix, and x and b are n-dimensional vectors. O'Leary and White [6]invented the matrix multisplitting method in 1985 for solving parallely the large sparse linear systems on the multiprocessor systems and was further studied by many authors. For example, Neumann and Plemmons [5] developed some more refined convergence results for one of the cases considered in [6], Elsner [7] established the comparison theorems about the asymptotic convergence rate of this case, Frommer and Mayer [8] discussed the successive overrelaxation (SOR) method in the sense of multisplitting, White [9,10] studied the convergence properties of the above matrix multisplitting methods for the symmetric positive definite matrix class, as well as matrix multisplitting methods as preconditioners, respectively, Bai [4] established the convergence theory of a class of asynchronous multisplitting blockwise relaxation methods, Zhang, Huang, et, al. [3] present local relaxed parallel multisplitting method and global relaxed parallel multisplitting method for H-matrices and so on. On the other hand, Since the finite element or the finite difference discretizations of many partial differential equations usually result in the large sparse systems of linear equations of regularly blocked structures, recently, [1,4] further generalized the matrix multisplitting concept of O'Leary and White [6] to a blocked form and proposed a class of parallel matrix multisplitting blockwise relaxation methods. This class of methods, besides enjoying all the advantages of the existing pointwise parallel matrix multisplitting methods discussed in [6,12], possesses better convergence properties and robuster numerical behaviours. Therefore, the parallel matrix multisplitting blockwise relaxation methods for the solution of large and sparse nonsingular blocked linear system have become more and more obvious.

In the following, we recall the mathematical descriptions of the blocked linear system and the BMM introduced in [1,4].

Let $N(\le n)$ and $n_i(\le n)(i=1,2,\dots,N)$ be given positive integers satisfying $\sum_{i=1}^N n_i = n$, and denote

$$V_n(n_1,\dots,n_N)=\{x\in\mathbb{R}^n | x=(x_1^T,\dots,x_N^T)^T, x_i\in\mathbb{R}^{n_i}\},$$

$$\mathbb{L}_n(n_1,\dots,n_N)=\{A\in\mathbb{R}^{n\times n} | A=(A_{ij})_{N\times N}, A_{ij}\in\mathbb{R}^{n_i\times n_j}\},$$

When the context is clear we will simply use $\mathbb{L}_n$ for $\mathbb{L}_n(n_1,\dots,n_N)$ and V_n for $V_n(n_1,\dots,n_N)$. Then, the blocked linear system to be solved can be expressed as the form

$$Ax=b,\quad A\in\mathbb{L}_n,\quad x,b\in V_n \tag{1.2}$$

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where $A \in \mathbb{L}_n$ is nonsingular and $b \in V_n$ are general known coefficient matrix and right-hand vector, respectively, and $x \in V_n$ is the unknown vector.

If blocked matrices $M_k, N_k, E_k \in \mathbb{L}_n (k = 1, 2, \ldots, \alpha)$ satisfy

1. $A = M_k - N_k$, M_k nonsingular, $k = 1, 2, \ldots, \alpha$,
2. $E_k = diag(E_{11}^{(k)}, \ldots, E_{NN}^{(k)})$, $k = 1, 2, \ldots, \alpha$,
3. $\sum_{k=1}^{\alpha} \|E_{ii}^{(k)}\| = 1$, $i = 1, 2, \ldots, N$,

then we call the collection of triples $(M_k, N_k, E_k)(k = 1, 2, \ldots, \alpha)$ is a BMM of the blocked matrix $A \in \mathbb{L}_n$, where $\|\cdot\|$ denotes the consistent matrix norm.

Suppose that we have a multiprocessor with α processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector x^k, then the kth processor only computes those entries of the vector

$$M_k^{-1} N_k x^k + M_k^{-1} b,$$

which correspond to the block diagonal entries $E_{ii}^{(k)}$ of the blocked matrix E_k. The processor then scales these entries so as to be able to deliver the vector

$$E_K (M_k^{-1} N_k x^k + M_k^{-1} b)$$

to the host processor, performing the parallel multisplitting scheme

$$x^{m+1} = \sum_{k=1}^{\alpha} E_K M_k^{-1} N_k x^m + \sum_{k=1}^{\alpha} E_K M_k^{-1} b = H x^m + G b,\ m = 0, 1, 2, \ldots$$

Under reasonable restrictions on the relaxation parameters and the multiple splittings, we establish local parallel multisplitting blockwise relaxation method, global parallel multisplitting blockwise relaxation method and global nonstationary parallel multisplitting blockwise relaxation method for solving the large sparse blocked linear systems and study the convergence of our methods associated with USAOR multisplitting when the coefficient matrices of the blocked linear systems are block H-matrices.

2. Establishments of the methods

Given a positive integer $\alpha (\alpha \le N)$, we separate the number set $\{1, 2, \ldots N\}$ into a nonempty subsets $J_k (k = 1, 2, \ldots, \alpha)$ such that $J_k \subseteq \{1, 2, \ldots, N\}$ and $\bigcup_{k=1}^{\alpha} J_k = \{1, 2, \ldots, N\}$.

Note that there may be overlappings among the subsets $J_1, J_2, \ldots, J_\alpha$. Corresponding to this separation, we introduce matrices

$$D = \mathrm{diag}(A_{11}, \ldots, A_{NN}) \in \mathbb{L}_n,$$
$$L_k = (\mathscr{L}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathscr{L}_{ij}^{(k)} = \begin{cases} L_{ij}^{(k)} & \text{for } i, j \in J_k \text{ and } i > j, \\ 0 & \text{otherwise,} \end{cases}$$
$$U_k = (\mathscr{U}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathscr{U}_{ij}^{(k)} = \begin{cases} U_{ij}^{(k)} & \text{for } i \ne j, \\ 0 & \text{otherwise,} \end{cases}$$
$$E_k = \mathrm{diag}(E_{11}^{(k)}, \ldots, E_{NN}^{(k)}) \in \mathbb{L}_n, \quad E_{ii}^{(k)} = \begin{cases} E_{ii}^{(k)} & \text{for } i \in J_k, \\ 0 & \text{otherwise,} \end{cases} \quad i, j = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, \alpha.$$

Obviously, D is a blocked diagonal matrix, $L_k (k = 1, 2, \ldots, \alpha)$ are blocked strictly lower triangular matrices, $U_k (k = 1, 2, \ldots, \alpha)$ are general blocked matrices, and $E_k (k = 1, 2, \ldots, \alpha)$ are blocked diagonal matrices. If they satisfy

1. D is nonsingular;
2. $A = D - L_k - U_k$, $k = 1, 2, \ldots, \alpha$;
3. $\sum_{k=1}^{\alpha} E_k = I$,

then the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Here, I denotes the identity matrix of order $n \times n$.

We will present *local parallel multisplitting blockwise relaxation USAOR method (LBUSAOR)* and *global parallel multisplitting blockwise relaxation USAOR method (GBUSAOR)*.

Algorithm 2.1. (local parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \dots$ repeat (I) and (II), until convergence.
(I) For $k = 1, 2, \dots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \sum_{k=1}^{\alpha} E_k y_k.$$

Algorithm 2.1 associated with LBUSAOR method can be written as

$$x^{m+1} = H_{LBUSAOR} x^m + G_{LBUSAOR} b, \ \ m = 0, 1, \cdots, \tag{2.1}$$

where

$$\begin{array}{rcl} H_{LBUSAOR} & = & \sum_{k=1}^{\alpha} E_k U_{\omega_2 r_2}(k) L_{\omega_1 r_1}(k), \\ U_{\omega_2 r_2}(k) & = & (D - r_2 U_k)^{-1}\{(1-\omega_2)D + (\omega_2 - r_2)U_k + \omega_2 L_k\}, \\ L_{\omega_1 r_1}(k) & = & (D - r_1 L_k)^{-1}\{(1-\omega_1)D + (\omega_1 - r_1)L_k + \omega_1 U_k\}, \\ G_{LBUSAOR} & = & \sum_{k=1}^{\alpha} E_k (D - r_2 U_k)^{-1}\{(\omega_1 + \omega_2 - \omega_1\omega_2)D + \omega_2(\omega_1 - r_1)L_k \\ & & +\omega_1(\omega_2 - r_2)U_k\}(D - r_1 L_k)^{-1} \end{array} \tag{2.2}$$

By using a suitable positive relaxation parameter β, we will establish global parallel multisplitting blockwise relaxation USAOR method which is based on Algorithm 2.1.

Algorithm 2.2. (global parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \dots$ repeat (I) and (II), until convergence.
(I) For $k = 1, 2, \dots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \beta \sum_{k=1}^{\alpha} E_k y_k + (1-\beta) x^m.$$

Algorithm 2.2 associated with GBUSAOR method can be written as

$$x^{m+1} = H_{GBUSAOR} x^m + \beta G_{LBUSAOR} b, \ \ m = 0, 1, \cdots, \tag{2.3}$$

where $H_{GBUSAOR} = \beta H_{LBUSAOR} + (1-\beta) I$.

In the standard multisplitting method each local approximation is updated exactly once using the same previous iterate x^m. On the other hand, it is possible to update the local approximations more than once, using different iterates computed earlier. In this case, we call this method a nonstationary multisplitting method [15,16,17]. The main idea of the nonstationary method is that at the mth iteration each processor k solves the system $q(m,k)$ times, using in each time the new calculated vector to update the right-hand side; i.e., we have the following algorithm:

Algorithm 2.3. (global nonstationary parallel multisplitting blockwise relaxation method)
Given the initial vector
For $m = 0, 1, 2, \dots$ repeat (I) and (II), until convergence.
(I) For $i = 1, 2, \dots, q(m,k)$, (parallel) solving $y_k^{(i)}$:

$$M_k y_k^{(i)} = N_k y_k^{(i-1)} + b.$$

(II) Computing

$$x^{m+1} = \beta \sum_{k=1}^{\alpha} E_k y_k^{q(m,k)} + (1-\beta) x^m.$$

Algorithm 2.3 associated with GNBUSAOR method can be written as

$$x^{m+1} = H_{GNBUSAOR} x^m + \beta G_{GNBUSAOR} b, \ \ m = 0, 1, \cdots, \tag{2.4}$$

where

$$\begin{array}{rcl} H_{GNBUSAOR} & = & \beta \sum_{k=1}^{\alpha} E_k (P_{\omega r} Q_{\xi\eta})^{q(m,k)} + (1-\beta) I \\ P_{\omega r} & = & (D - r_k U_k)^{-1}\{(1-\omega_k)D + (\omega_k - r_k)U_k + \omega_k L_k\} = M_r^{-1} N_{\omega r}, \\ Q_{\xi\eta} & = & (D - \eta_k L_k)^{-1}\{(1-\xi_k)D + (\xi_k - \eta_k)L_k + \xi_k U_k\} = M_\eta^{-1} N_{\xi\eta}, \\ G_{GNBUSAOR} & = & \beta \sum_{k=1}^{\alpha} E_k \sum_{i=1}^{q(m,k)-1} (M_\eta M_r)^{-1} (N_{\omega r} N_{\xi\eta})^i (M_\eta M_r)^{-1} \omega_k \xi_k. \end{array} \tag{2.5}$$

It follows that when $q(m,k) = 1$, $\omega_k = \omega_2$, $r_k = r_2$, $\xi_k = \omega_1$ and $\eta_k = r_1$ for $1 < k < \alpha$, $m = 0, 1, 2 ....$, Algorithm 2.3 reduces to Algorithm 2.2.

3. Preliminaries

We shall use the following notations and lemmas. A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$. A *Z-matrix* is a nonsingular *M-matrix* if A is nonsingular and if $A^{-1} \geq 0$. Additionally, we denote the spectral radius of A by $\rho(A)$. It is well-known that if $A \geq 0$ and there exists a vector $x > 0$ such that $Ax < \alpha x$, then $\rho(A) < \alpha$. Let

$$\mathbb{L}_{n,I}(n_1, \ldots, n_N) = \{M = (M_{ij}) \in \mathbb{L}_n | M_{ii} \in \mathbb{R}^{n_i \times n_i} \text{ nonsingular}, \ i = 1, \ldots, N\}.$$

We will review the concepts of strictly block diagonally dominant matrix and the block H-matrix. Let $A \in \mathbb{L}_{n,I}$. Then its block comparison matrix $\langle A \rangle$ is defined by

$$\langle A \rangle_{ij} = \begin{cases} \|A_{ij}^{-1}\|^{-1}, & i = j, \\ -\|A_{ij}\|, & i \neq j, \end{cases} \quad i, j = 1, \ldots, N$$

where $\|\cdot\|$ is a consistent matrix norm. If

$$\|A_{ii}^{-1}\|^{-1} > \sum_{i \neq j} \|A_{ij}\|, \ \ j = 1, 2, \ldots, N.$$

Then A is said to be a strictly block diagonally dominant matrix [13], if there exists a positive diagonally matrix X such that AX is a strictly block diagonally dominant matrix, then A is said to be a block H-matrix [14]. Clearly, a strictly block diagonally dominant matrix is certainly a block H-matrix.

Definition 3.1 [1]. Let $M \in \mathbb{L}_n$. We call $[M] = (\|M_{ij}\|) \in \mathbb{R}^{N \times N}$ the block absolute value of the blocked matrix M. The block absolute value $[x] \in \mathbb{R}^N$ of a blocked vector $x \in V_n$ is defined in an analogous way.

These kinds of block absolute values have the following important properties.

Lemma 3.1 [1]. *Let $L, M \in \mathbb{L}_n$, $x, y \in V_n$ and $r \in \mathbb{R}^1$. Then*
1. $|[L] - [M]| \leq [L + M] \leq [L] + [M]$ ($|[x] - [y]| \leq [x + y] \leq [x] + [y]$);
2. $[LM] \leq [L][M]$ ($[xy] \leq [x][y]$);
3. $[rM] \leq |r|[M]$ ($[rx] \leq |r|[x]$);
4. $\rho(M) \leq \rho(|M|) \leq \rho([M])$ (here, $\|\cdot\|$ is either $\|\cdot\|_\infty$ or $\|\cdot\|_1$).

Lemma 3.2 [1]. *Let $M \in \mathbb{L}_{n,I}$ be a strictly block diagonally dominant matrix, then*
1. M is nonsingular;
2. $[(M)^{-1}] \leq \langle M \rangle^{-1}$;
3. $\rho(J(\langle M \rangle)) < 1$.

4. Convergence

For Algorithms 1, 2 and 3, we give convergence theorems for block diagonally dominant matrices and block $H-$matrices.

Theorem 4.1. *Let A be a strictly block diagonally dominant matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k=1,2,\dots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B], \quad k=1,2,\dots,\alpha, \quad 0<\omega_1,\omega_2<\frac{2}{1+\rho}, \quad 0\le r_1\le\omega_1, \quad 0\le r_2\le\omega_2,$$

then LBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho=\rho(J(\langle A\rangle))=\rho(\langle D\rangle^{-1}[B])$.

Proof. By Lemma 3.1, we know $\rho(H_{LBUSAOR}) \le \rho(|H_{LBUSAOR}|) \le \rho([H_{LBUSAOR}])$, and then, the iteration (2) converges for any initial vector $x^0 \in V_N$ if and only if $\rho([H_{LBUSAOR}])<1$. Let $B=L_k+U_k$, by (7), we know that $[B]=[L_k]+[U_k]$, clearly, $D-rL_k$ $(k=1,2,\dots,\alpha)$ are strictly block diagonally dominant matrix for $r>0$, and $\langle D\rangle - r[B]$, $(k=1,2,\dots,\alpha)$ are strictly block diagonally dominant matrix for $0<r<\frac{2}{1+\rho}$ which follow from A is a strictly block diagonally dominant matrix. Since $\langle D\rangle - r[B] \le \langle D\rangle - r[U_k] \le \langle D\rangle$ for $0<r<\frac{2}{1+\rho}$, $k=1,2,\dots,\alpha$, and $\langle A\rangle$ is a strictly diagonally dominant matrix, we have $\langle D\rangle - r[B]$ and $\langle D\rangle$ are strictly diagonally dominant M-matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\dots,\alpha$. Therefore, $\langle D\rangle - r[U_k]$ are strictly diagonally dominant M-matrices, and then $D-rU_k$ are strictly block diagonally dominant matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\dots,\alpha$.

Let $\bar{L_k}=D^{-1}L_k$ and $\bar{U_k}=D^{-1}U_k$, then $I-r\bar{L_k}$ and $I-r\bar{U_k}$ are also strictly block diagonally dominant matrices, for $0<r<\frac{2}{1+\rho}$, $k=1,2,\dots,\alpha$. Thus, by Lemma 3.1 and (7), we have

$$[(I-r_1\bar{L_k})^{-1}] \le (\langle I-r_1\bar{L_k}\rangle)^{-1} = (I-r_1[\bar{L_k}])^{-1},$$
$$[(I-r_2\bar{U_k})^{-1}] \le (\langle I-r_2\bar{U_k}\rangle)^{-1} = (I-r_2[\bar{U_k}])^{-1}.$$

From (3), we have

$$\begin{aligned}
[U_{\omega_2 r_2}(k)] &= [(D-r_2U_k)^{-1}\{(1-\omega_2)D+(\omega_2-r_2)U_k+\omega_2L_k\}] \\
&= [(I-r_2\bar{U_k})^{-1}\{(1-\omega_2)I+(\omega_2-r_2)\bar{U_k}+\omega_2\bar{L_k}\}] \\
&\le (I-r_2[\bar{U_k}])^{-1}\{|1-\omega_2|I+(\omega_2-r_2)[\bar{U_k}]+\omega_2[\bar{L_k}]\} \\
&= I+(I-r_2[\bar{U_k}])^{-1}\{(|1-\omega_2|-1)I+\omega_2([\bar{U_k}]+[\bar{L_k}])\}.
\end{aligned}$$

Since $\bar{L_k}=D^{-1}L_k$ and $\bar{U_k}=D^{-1}U_k$, we have $[\bar{L_k}] \le \langle D\rangle^{-1}[L_k]$ and $[\bar{U_k}] \le \langle D\rangle^{-1}[U_k]$ which follow from Lemma 3.1 and Lemma 3.2, and then

$$[\bar{U_k}]+[\bar{L_k}] \le \langle D\rangle^{-1}[U_k+L_k] = \langle D\rangle^{-1}[B] = J(\langle A\rangle), \ \ k=1,2,\dots,\alpha.$$

Therefore, we have

$$[U_{\omega_2 r_2}(k)] \le I-(I-r_2[\bar{U_k}])^{-1}(I-T(\omega_2)),$$

where $T(\omega_2)=|1-\omega_2|I+\omega_2J(\langle A\rangle)$. Note that $(I-r_2[\bar{U_k}(k)])^{-1} \ge I$, $k=1,2,\dots,\alpha$, and then

$$[U_{\omega_2 r_2}(k)] \le I-(I-T(\omega_2)) = T(\omega_2).$$

Similar to the above proving process, we have

$$[L_{\omega_1 r_1}(k)] \le I-(I-T(\omega_1)) = T(\omega_1),$$

where $T(\omega_1)=|1-\omega_1|I+\omega_1J(\langle A\rangle)$.

Let e denotes the vector $e=(1,1,\dots,1)^T \in V_N$ and $J_\epsilon(\langle A\rangle)=J(\langle A\rangle)+\epsilon ee^T$, since $J(\langle A\rangle)$ is nonnegative, the matrix $J_\epsilon(\langle A\rangle)$ has only positive entries and irreducible for any $\epsilon>0$. By the Perron-Frobenius theorem for any $\epsilon>0$, there is a vector $x_\epsilon>0$ such that

$$J_\epsilon(\langle A\rangle)x_\epsilon = \rho(J_\epsilon(\langle A\rangle))x_\epsilon = \rho_\epsilon x_\epsilon,$$

where $\rho_\epsilon = \rho(J_\epsilon(\langle A\rangle))$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Thus, we can get

$$|1-\omega_i| + \omega_i\rho_\epsilon < 1 \text{ for } 0 < \omega_i < \tfrac{2}{1+\rho},\ i = 1,2,$$

and then

$$\begin{array}{rcl} [H_{LBUSAOR}]x_\epsilon & \le & \sum_{k=1}^{\alpha}[E_k][U_{\omega_2 r_2}(k)][L_{\omega_1 r_1}(k)]x_\epsilon \le \sum_{k=1}^{\alpha}[E_k]T(\omega_2)T(\omega_1)x_\epsilon \\ & \le & (|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle))(|1-\omega_1|I + \omega_1 J_\epsilon(\langle A\rangle))x_\epsilon \\ & = & (|1-\omega_1| + \omega_1\rho_\epsilon)(|1-\omega_2| + \omega_2\rho_\epsilon)x_\epsilon \\ & < & x_\epsilon, \end{array}$$

then $[H_{LBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{LBUSAOR}]) < 1$. □

Theorem 4.2. *Let A be a block H-matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k = 1,2,\ldots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B], \quad k = 1,2,\ldots,\alpha, \quad 0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \le r_1 \le \omega_1, \quad 0 \le r_2 \le \omega_2,$$

then LBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$.

Proof. Since A is a block H-matrix, then, there exists a positive diagonally matrix X such that AX is a strictly block diagonally dominant matrix. Let $H_{LBUSAOR}(A)$ denote the iteration matrix of local parallel multisplitting blockwise relaxation method for blocked matrix A and $H_{LBUSAOR}(AX)$ denote the iteration matrix of local parallel multisplitting blockwise relaxation method for blocked matrix AX, respectively. By simple calculation, we have $H_{LBUSAOR}(A)$ and $H_{LBUSAOR}(AX)$ are similar. Since similar matrices have the same eigenvalues, it follows that $\rho(H_{LBUSAOR}(A)) = \rho(H_{LBUSAOR}(AX)) < 1$. □

Using GBUSAOR method, we can also get the following convergence results.

Theorem 4.3. *Let A be a strictly block diagonally dominant matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k = 1,2,\ldots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B],\ k = 1,2,\ldots,\alpha,$$

if

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \le r_1 \le \omega_1, \quad 0 \le r_2 \le \omega_2, \quad 0 < \beta < \frac{2}{1+\theta^2},$$

then GBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$ and

$$\theta = max\{|1-\omega_1| + \omega_1\rho,\ |1-\omega_2| + \omega_2\rho\}.$$

Proof. Since $\rho(H_{GBUSAOR}) \le \rho(|H_{GBUSAOR}|) \le \rho([H_{GBUSAOR}])$, the iteration (4) converges for any initial vector $x^0 \in V_N$ if and only if $\rho([H_{GBUSAOR}]) < 1$. Let

$$\theta_\epsilon = max\{|1-\omega_1| + \omega_1\rho_\epsilon,\ |1-\omega_2| + \omega_2\rho_\epsilon\},$$

similar to the proof of Theorem 4.1, we have

$$\begin{array}{rcl} [H_{GBUSAOR}]x_\epsilon & \le & \beta[\{|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle)\}\{|1-\omega_1|I + \omega_1 J_\epsilon(\langle A\rangle)\}]x_\epsilon + |1-\beta|x_\epsilon \\ & = & \beta(|1-\omega_1| + \omega_1\rho_\epsilon)\{|1-\omega_2|I + \omega_2 J_\epsilon(\langle A\rangle)\}x_\epsilon + |1-\beta|x_\epsilon \\ & = & \beta(|1-\omega_1| + \omega_1\rho_\epsilon)(|1-\omega_2| + \omega_2\rho_\epsilon)x_\epsilon + |1-\beta|x_\epsilon \\ & \le & (\beta\theta^2 + |1-\beta|)x_\epsilon \\ & = & (\beta\theta_\epsilon^2 + |1-\beta|)x_\epsilon \\ & < & x_\epsilon \end{array}$$

then $[H_{GBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{GBUSAOR}]) < 1$. □

Theorem 4.4. *Let A be a block H-matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k = 1,2,\dots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B], \quad k = 1,2,\dots,\alpha, \tag{4.1}$$

if

$$0 < \omega_1, \omega_2 < \frac{2}{1+\rho}, \quad 0 \le r_1 \le \omega_1, \quad 0 \le r_2 \le \omega_2, \quad 0 < \beta < \frac{2}{1+\theta^2},$$

then GBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$ and

$$\theta = max\{|1-\omega_1| + \omega_1\rho,\ |1-\omega_2| + \omega_2\rho\}.$$

Proof. Similar to the proof of Theorem 4.2, we can prove Theorem 4.4. □

Remark. As some special case, for local parallel multisplitting blockwise relaxation USAOR method (LBUSAOR) and global parallel multisplitting blockwise relaxation USAOR method (GBUSAOR)(see Table 1), we also have the corresponding convergence results, when $\omega_1 = \omega_2$, $r_1 = r_2$, $\omega_1 = \omega_2 = r_1 = r_2$ and so on.

Theorem 4.5. *Let A be a block H-matrix, and the collection of triples $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$ $(k = 1,2,\dots,\alpha)$ are BMM of the blocked matrix $A \in \mathbb{L}_n$. Assume that*

$$\langle A\rangle = \langle D\rangle - [L_k] - [U_k] = \langle D\rangle - [B],\ k = 1,2,\dots,\alpha,$$

if

$$0 < \omega_k, \xi_k < \frac{2}{1+\rho}, \quad 0 \le r_k \le \omega_k, \quad 0 \le \eta_k \le \xi_k, \quad 0 < \beta < \frac{2}{1+\sigma^2},$$

then GNBUSAOR method converges for any initial vector $x^0 \in V_N$, where $\rho = \rho(J(\langle A\rangle)) = \rho(\langle D\rangle^{-1}[B])$, $q(m,k) \ge 1$, $k = 1,2,\dots,\alpha$, $m = 0,1,2,\dots$ and

$$\sigma = \max_{1\le k\le\alpha}\{|1-\omega_k| + \omega_k\rho,\ |1-\xi_k| + \xi_k\rho\}.$$

Proof. We only need show $\rho([H_{GNBUSAOR}] < 1)$, when A is a strictly block diagonally dominant matrix. By the proof of Theorem 4.1, we know that

$$[(I - r_k\bar{L_k})^{-1}] \le (\langle I - r_k\bar{L_k}\rangle)^{-1} = (I - r_k[\bar{L_k}])^{-1},$$
$$[(I - \eta_k\bar{U_k})^{-1}] \le (\langle I - \eta_k\bar{U_k}\rangle)^{-1} = (I - \eta_k[\bar{U_k}])^{-1}.$$

From (6), we have

$$[P_{\omega r}] \le |1-\omega_k| I + \omega_k J_\epsilon(\langle A\rangle$$

and

$$[Q_{\xi\eta}] \le |1-\xi_k| I + \xi_k J_\epsilon(\langle A\rangle).$$

Let

$$\sigma_\epsilon = \max_{1\le k\le\alpha}\{|1-\omega_k| + \omega_k\rho_\epsilon, |1-\xi_k| + \xi_k\rho_\epsilon\}.$$

Similar to the above proving process, we get

$$\begin{array}{rcl} [H_{GNBUSAOR}]x_\epsilon & \le & \beta[\sum_{k=1}^{\alpha} E_k(P_{\omega r}Q_{\xi\eta})^{q(m,k)}]x_\epsilon + |1-\beta|x_\epsilon \\ & \le & \beta\sum_{k=1}^{\alpha}[E_k]([P_{\omega r}][Q_{\xi\eta}])^{q(m,k)}x_\epsilon + |1-\beta|x_\epsilon \\ & \le & \beta(|1-\omega_k| + \omega_k\rho_\epsilon)(|1-\xi_k| + \xi_k\rho_\epsilon)x_\epsilon + |1-\beta|x_\epsilon \\ & = & (\beta\sigma^2 + |1-\beta|)x_\epsilon \\ & < & x_\epsilon, \end{array}$$

then $[H_{GNBUSAOR}]x_\epsilon < x_\epsilon$ and $\rho([H_{GNBUSAOR}]) < 1$. □

5. Numerical example

We consider the blocked linear system [1,4]

$$Ax = b, \quad A \in \mathbb{L}_n,\ b \in V_n, \tag{5.1}$$

with $n_1 = n_2 = \ldots = n_N \equiv N$, and $n = N^2$, where

$$A = \begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B & -I \\ & & & -I & B \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}$$

and the right hand side vector b is chosen as

$$b^T = (1, \frac{1}{4}, \ldots, \frac{1}{n^2}) \in V_n.$$

Take $\alpha = 2$ and two positive integers m_1 and m_2 satisfying $1 < m_2 < m_1 < N$. Then, corresponding to the number sets $J_1 = \{1, 2, \ldots, m_1\}$ and $J_2 = \{m_2, m_2+1, \ldots, N\}$, we determine BMM $(D-L_k, U_k, E_k)$ and $(D-U_k, L_k, E_k)$, $k = 1, 2$, of the blocked matrix A in accordance with the following way:

$D = \mathrm{diag}(B, B, \ldots, B) \in \mathbb{L}_n$,

$$L_1 = (\mathscr{L}_{ij}^{(1)}) \in \mathbb{L}_n,\ \mathscr{L}_{ij}^{(1)} = \begin{cases} I & \text{for } j = i-1 \text{ and } 2 \le i \le m_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_1 = (\mathscr{U}_{ij}^{(1)}) \in \mathbb{L}_n,\ \mathscr{U}_{ij}^{(1)} = \begin{cases} I & \text{for } j = i-1 \text{ and } m_1+1 \le i \le N, \\ I & \text{for } j = i+1 \text{ and } 1 \le i \le N-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$L_2 = (\mathscr{L}_{ij}^{(2)}) \in \mathbb{L}_n,\ \mathscr{L}_{ij}^{(2)} = \begin{cases} I & \text{for } j = i-1 \text{ and } m_2 \le i \le N, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_2 = (\mathscr{U}_{ij}^{(2)}) \in \mathbb{L}_n,\ \mathscr{U}_{ij}^{(2)} = \begin{cases} I & \text{for } j = i-1 \text{ and } 2 \le i \le m_2-1, \\ I & \text{for } j = i+1 \text{ and } 1 \le i \le N-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$E_k = \mathrm{diag}(E_{11}^{(k)}, E_{22}^{(k)} \ldots, E_{NN}^{(k)}) \in \mathbb{L}_n,\ k = 1, 2,\ E_{ii}^{(1)} = \begin{cases} I & \text{for } 1 \le i \le m_2, \\ \frac{1}{2}I & \text{for } m_2 \le i \le m_1, \\ 0 & \text{for } m_1 < i \le N. \end{cases} \quad E_{ii}^{(2)} = \begin{cases} 0 & \text{for } \le i \le m_2, \\ \frac{1}{2}I & \text{for } m_2 \le i \le m_1, \\ I & \text{for } m_1 < i \le N. \end{cases}$$

In particular, we select the positive integer pair (m_1, m_2) to be

(a) $m_1 = \mathrm{Int}(\frac{2N}{3})$, $m_2 = \mathrm{Int}(\frac{N}{3})$;

(b) $m_1 = \mathrm{Int}(\frac{4N}{5})$, $m_2 = \mathrm{Int}(\frac{N}{5})$,

then we can get two kinds of concrete cases of the weighting matrices E_1 and E_2, here, $\mathrm{Int}(\cdot)$ denotes the integer part of the corresponding real number.

In our numerical experiment, the initial approximation x^0 is taken as

$$x^0 = (0.5, \ldots, 0.5)^T \in V_n.$$

Let the convergence criterion be $\|x^{k+1} - x^k\|_\infty \le 10^{-6}$. In Table 2 and Table 3, we report the number of iterations by IT.

From Table 2 and Table 3, we easily see that the multisplitting blockwise relaxation USAOR methods discussed in this paper substantially have better numerical behaviours than the multisplitting blockwise relaxation AOR methods studied in [1], which shows that our new methods are applicable and efficient.

Table 1: The iterations of LBUSAOR method for N=15

r_1	1	1.05	1.1	[0.92,1.1]	1.1	1.1	1.2
r_2	1.1	1.1	1.1	1.1	[1.0789,1.0881]	1.088	1.2
ω_1	1.406	1.406	1.6	[1.796,1.825]	1.8	1.8	1.9
ω_2	1.2	1.1	1.1	1.1	1.1	[1.096,1.1]	1.15
IT(a)	42	37	36	35	30	30	55
IT(b)	41	37	36	34	31	31	55

Table 2: The iterations of GBUSAOR method for N=15

r_1	1	1.05	1.1	[0.92,1.1]	1.1	1.1	1.2
r_2	1.1	1.1	1.1	1.1	[1.0789,1.0881]	1.088	1.2
ω_1	1.406	1.406	1.6	[1.796,1.825]	1.8	1.8	1.9
ω_2	1.2	1.1	1.1	1.1	1.1	[1.096,1.1]	1.15
β	0.8	1	1.05	1.1	1.2	[1.24,1.26]	1.3
IT(a)	51	37	35	27	25	24	90
IT(b)	52	37	35	28	26	25	64

References

[1] Z.Z. Bai, Parallel matrix multisplitting block relaxation iteration methods, *Math. Numer. Sinica* **17** (1995) 238-252.

[2] A. Berman, R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA, 1994.

[3] L. T. Zhang, T. Z. Huang, et,al, Convergence of relaxed multisplitting USAOR methods for H-matrices linear systems, *Applied Mathematics and Computation.* **202**, (2008) 121-132.

[4] Z. Z. Bai, A class of asynchronous parallel multisplitting blockwise relaxation methods *Parallel Computing* ., **25** (1999) 681-701.

[5] M. Neumann, R.J. Plemmons, Convergence of parallel multisplitting iterative methods for M-matrices, *Linear Algebra Appl.* **88** (1987) 559-573.

[6] D. P. OąŕLeary, R. E. White, Multi-splittings of matrices and parallel solution of linear systems, *SIAM Journal on Algebraic and Discrete Mathematics*, **6** (1985) 630-640.

[7] L.Elsner, Comparisons of weak regular splittings and multisplitting methods, *Numer. Math.* **56** (1989) 283-289.

[8] A. Frommer, G. Mayer, Convergence of relaxed parallel multisplitting methods, *Linear Algebra Appl.* **119** (1989) 141-152.

[9] R.E. White, Multisplitting of a symmetric positive deRnite matrix, *SIAM J. Matrix Anal. Appl.* **11** (1990) 69-82.

[10] R.E. White, Multisplitting with diĂerent weighting schemes, *SIAM J. Matrix Anal. Appl.* **10** (1989) 481-493.

[11] R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1981.

[12] D.R. Wang, On the convergence of the parallel multisplitting AOR algorithm, *Linear Algebra Appl.* **154-156** (1991) 473-486.

[13] D.G. Feingold, R.S. Varga, Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem, *Pacific J. Math.* **12** (1962) 1241-1249

[14] B.Polman, Incomplete blockwise factorizations of (block) H-matrices, *Line. Alge. Appl.* **90** (1987) 119-132

[15] R.Bru, V.MigallSn, J.Penadds, and D.B. Szyld, Parallel synchronous and asynchronous two-stage multisplitting methods, Electronic *Transactions on Numerical Analysis*, **3** (1995) 24-38.

[16] D.B. Szyld, Synchronous and asynchronous two-stage multisplitting methods, in Proceedings, *5th SIAM Conference on Applied Linear Algebra, (J. G. Lewis, Ed.)*, pp. 39-44, SIAM, Philadelphia, 1994.

[17] J.Mos, D.B. Szyed, Nonstationary Parallel Relaxed Multisplitting Methods, *Line. Alge. Appl*, **241-243** (1996) 733-747.

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The properties and iterative algorithms of circulant matrices *

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Abstract

In this paper we investigate the structure and the iterative algorithms of circulant matrices. Firstly, we show that the reduced form of a circulant matrix A is an H-matrix if the matrix A be a circulant H-matrix, moreover, we can derive that the matrix A is a circulant H-matrix if some conditions are imposed on the reduced form of the matrix A. Secondly, by using the properties of the circulant matrix A, we present two new splittings of circulant M-matrices and obtain some efficacious iterative algorithms for solving a linear system $Ax = b$.

Key words: *Circulant matrix; H-matrix; M-matrix; reduced form; splitting; iterative algorithm*

1. Introduction

Patterned matrices just like circulant matrices, symmetric matrices, Jacobi matrices, centrosymmetric matrices etc. arise in many areas of physics, electromagnetics, signal processing, statistics and applied mathematics for the investigation of problems with periodicity properties. Also, the numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve the linear systems $Ax = b$ with a patterned matrix A [1-3]. The properties of patterned matrices have been extensively investigated [4-6]. For recent years many authors have paid attention to developing iteration algorithms for solving the linear systems with patterned matrices [7-9]. Circulant matrix is a kind of very important patterned matrices.

In this paper we investigate the structure and the iterative algorithms of circulant matrices. Firstly, we discuss the properties of the circulant matrix. We prove that the reduced form of a circulant matrix A is an H-matrix if the matrix A be a circulant H-matrix. Moreover, we derive that the matrix A is a circulant H-matrix if some conditions are imposed on the reduced form of A. Secondly, by means of the properties of the circulant matrix A, we present two new splittings of circulant M-matrices and obtain some efficacious iterative algorithms for solving a linear system $Ax = b$. This paper is organized as follows. In next section, we review some basic definitions and notations. In section 3, we show some properties of circulant matrices. In section 4, a new splitting scheme is constructed, which can be deprived from a random convergent splitting of a circulant matrix A, and two new splittings of the circulant M-matrix and H-matrix are presented. The convergence of the corresponding iterative sequences is also discussed. Finally, on the basis of the opposite triangular splittings and $GMRES$ algorithm, we give three iterative algorithms to solve the linear system $Ax = b$.

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2. Preliminaries

In this section we will review some basic notations which frequently used in the following. Let $a \in R^n$ and $a = (a_0, a_1, \cdots, a_{n-1})^T$. In a circulant matrix

$$Cir(a) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

each row is a cyclic shift of the row above to the right. $Cir(a)$ is a special case of a Toeplitz matrix. It is evidently determined by its first row (or column).

Definition 2.1 A matrix $A = (a_{i,j})_{n\times n}$ is called an Z-matrix, if $a_{i,j} \leq 0 (i \neq j)$; an M-matrix if A is an Z-matrix and $A^{-1} \geq 0$. Let $\langle A\rangle = (\alpha_{i,j})_{n\times n}$, if $\alpha_{i,i} = |a_{i,i}|$, $\alpha_{i,j} = -|a_{i,j}| (i \neq j)$, then $\langle A\rangle$ is called a comparision matrix of A. A matrix A is called an H-matrix, if its comparison matrix $\langle A\rangle$ is an M-matrix.

Let $A = M - N$. Then the pairs of matrices (M, N) of A is called: a splitting of A if $\det(M) \neq 0$; a convergent splitting if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ denote the spectral radius of the matrix $M^{-1}N$; a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$; a weak regular splitting of A if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 2.2[2] Let $A = M - N$ be a weak regualr splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

Now consider the circulant matrices.

Lemma 2.3[1] (1) Let $A \in R^{n\times n}$ be a circulant matrix, then $\langle A\rangle$ is a circulant matrix. Furthermore, if A is nonsingular, then A^{-1} is a circulant matrix; (2) A^T is a circulant matrix; (3) Let $B \in R^{n\times n}$ be a circulant matrix, then $A \pm B$ and AB are also circulant matrices.

Let $G = Cir(0, 1, 0, \cdots, 0) \in R^{n\times n}$.

Lemma 2.4 $A \in R^{n\times n}$ is a circulant matrix if and only if $G^TAG = A$.

All the formulas become slightly more complicated when n is odd. For simplicity, in this paper, when $n = 2m + 1$ we restrict the circulant matrix A to be symmetric, that is

$$A = Cir((a_0, a_1, a_2, \cdots, a_m, a_m, \cdots, a_1)^T). \tag{2.1}$$

Using the partition of the matrix, the circulant matrix can be described as follows.

(i) For the case $n = 2m$, a circulant matrix can be written as the form:

$$A = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{2.2}$$

where each of the block matrices B and C is an $m \times m$ matrix.

(ii) For the case $n = 2m + 1$, a symmetric circulant matrix can be partitioned into the form:

$$A = \begin{pmatrix} B & J_m a & J_m C J_m \\ a^T J_m & \beta & a^T \\ C & a & J_m B J_m \end{pmatrix}, \tag{2.3}$$

where $J_m = (e_m, e_{m-1}, \cdots, e_1)$, e_i denotes the unit vector with ith entry 1, $B, C \in R^{m\times m}, a \in R^{m\times 1}$ and β is a scalar.

Lemma 2.5 Let A be a circulant matrix, then A is orthogonal similar to a block diagonal matrix. The block diagonal matrix can be described as follows.

(i) For $n = 2m$, let

$$P = \frac{\sqrt{2}}{2}\begin{pmatrix} I_m & I_m \\ -I_m & I_m \end{pmatrix},$$

then

$$P^T A P = \begin{pmatrix} B - C & \\ & B + C \end{pmatrix}, \tag{2.4}$$

where I_m is a mth unit matrix.

(ii) In terms of (2.1) if $n = 2m + 1$, we select the orthogonal matrix

$$P = \frac{\sqrt{2}}{2}\begin{pmatrix} I_m & & I_m \\ & \sqrt{2} & \\ -J_m & & J_m \end{pmatrix},$$

then

$$P^T A P = \begin{pmatrix} B - J_m C & & \\ & \beta & \sqrt{2}a^T J_m \\ & \sqrt{2}J_m a & B + J_m C \end{pmatrix}. \tag{2.5}$$

We call the matrix of the right side of (2.4) or (2.5) the reduced form of the circulant matrix A, corresponding to (2.2) or (2.3) respectively.

Lemma 2.6 Let A and C be M-matrices, if $A \le B \le C$, then B is also an M-matrix.

Lemma 2.7[3] Let $A^{-1} \ge 0$ and

$$A = \widetilde{M_1} - \widetilde{N}_1 = \widetilde{M_2} - \widetilde{N}_2$$

be weak regular splittings. In either of the following cases

a) $\widetilde{N}_1 \le \widetilde{N}_2$

b) $\widetilde{M_1}^{-1} \ge \widetilde{M_2}^{-1},\ N_1 \ge 0$

c) $\widetilde{M_1}^{-1} \ge \widetilde{M_2}^{-1},\ N_2 \ge 0$

the inequality

$$\rho(\widetilde{M_1}^{-1}\widetilde{N}_1) \le \rho(\widetilde{M_2}^{-1}\widetilde{N}_2)$$

holds.

Lemma 2.8[4] Let A be nonsingular and $A^{-1} \ge 0$, $A = M_l - N_l (l = 1, 2, \cdots, k)$ are k weak regualar splittings of A. Then for any qualified $E_l (l = 1, 2, \cdots, k)$, $\rho(W) < 1$, where $W = \sum_{l=1}^k E_l M_l^{-1} N_l$, $\sum_{l=1}^k E_l = I$.

Lemma 2.9[4] Let A be an H-matrix, $A = M_l - N_l (l = 1, 2, \cdots, k)$ are k splittings of A and $\langle A\rangle = \langle M_l\rangle - |N_l|$, $diag(M_l) = diag(A)$, then $\rho(W) < 1$, where $W = \sum_{l=1}^k E_l M_l^{-1} N_l$, $\sum_{l=1}^k E_l = I$.

3. Some properties of circulant matrices

In this section we will give some new properties of the reduced form of some special circulant matrices which consist in the original matrices. The following theorem is evident.

Theorem 3.1 The reduced form of a circulant matrix A is nonsingular or positive definite, respectively, if and only if A itself is nonsingular or positive definite, respectively.

Theorem 3.2 The reduced form of a circulant matrix A is an H-matrix, if the matrix A is a circulant H-matrix.

Proof Let A be a circulant H-matrix. Then, for the case of $n = 2m$ we will prove that both $B + C$ and $B - C$ are also H-matrices.

From

$$\langle B\rangle - |C| \leq \langle B - C\rangle \leq |D| \tag{3.1}$$

and

$$\langle B\rangle - |C| \leq \langle B + C\rangle \leq |D| \tag{3.2}$$

where $|D|$ is the diagonal part of the matrix $|B| + |C|$, we obtain that $\langle B\rangle - |C|$ is an M-matrix.

Since A is an H-matrix, according to the properties of the H-matrices, the comparison matrix $\langle A\rangle$ is an M-matrix, and $\langle B\rangle$ is an M-matrix too. By Lemma 2.1, $\langle A\rangle$ can be represented as

$$\langle A\rangle = \begin{pmatrix} \langle B\rangle & -|C| \\ -|C| & \langle B\rangle \end{pmatrix}. \tag{3.3}$$

Let us consider the block Gauss-Seidel splitting of the matrix $\langle A\rangle$:

$$\langle A\rangle = \begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix} - \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

In terms of

$$\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1} = \begin{pmatrix} \langle B\rangle^{-1} & 0 \\ \langle B\rangle^{-1}|C|\langle B\rangle^{-1} & 0 \end{pmatrix} \geq 0$$

and $\begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix} \geq 0$, it follows that the formula (3.4) is a regular splitting. By Lemma 2.2, the above splitting of the matrix $\langle A\rangle$ is convergent, thus

$$\begin{aligned} \rho\left(\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1} \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}\right) &= \rho\left(\begin{pmatrix} 0 & \langle B\rangle^{-1}|C| \\ 0 & (\langle B\rangle^{-1}|C|)^2 \end{pmatrix}\right) \\ &= \rho^2(\langle B\rangle^{-1}|C|) < 1. \end{aligned}$$

It is evident that $\rho(\langle B\rangle^{-1}|C|) < 1$. We obtain that

$$(\langle B\rangle - |C|)^{-1} = (I - \langle B\rangle^{-1}|C|)^{-1}\langle B\rangle^{-1} \geq 0.$$

According to the definition of M-matrix, $\langle B\rangle - |C|$ is an M-matrix. By Lemma 2.6, $\langle B - C\rangle$ and $\langle B+C\rangle$ are M-matrices, by the definition of H-matrix, both $B-C$ and $B+C$ are H-matrices.

Now we show that for the case of $n = 2m + 1$, both matrix $B - J_mC$ and

$$\begin{pmatrix} \beta & \sqrt{2}a^TJ_m \\ \sqrt{2}J_ma & B + J_mC \end{pmatrix}$$

are H-matrices.

On the base of (2.3) we get that

$$\langle A\rangle = \begin{pmatrix} \langle B\rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B\rangle J_m \end{pmatrix}. \tag{3.5}$$

As A is an H-matrix, both $\langle A\rangle$ and $\langle B\rangle$ are M-matrices. It is easy to verify that

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix} \le \begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B + J_m|C|\rangle \end{pmatrix} \le \begin{pmatrix} |\beta| & \\ & |D| \end{pmatrix}$$

where $|D|$ is the diagonal part of the matrix $|B| + J_m|C|$.

Using the equivalence conditions of M-matrices ([2]), we can easily prove that there exists a positive vector $x \in R^{2m+1}$ such that $\langle A\rangle x > 0$. Partitioning x in the form of (2.3), denoted by $(y^T, \gamma, (J_my)^T)^T$, then $\langle A\rangle x > 0$ implies that

$$\begin{pmatrix} \langle B\rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B\rangle J_m \end{pmatrix} \begin{pmatrix} y \\ \gamma \\ J_my \end{pmatrix} > 0$$

Then

$$(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)y > 0.$$

Using the equivalence of an M-matrix again, we have that $(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)$ is an M-matrix.

By

$$(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m) \le \langle B\rangle - J_m|C| \le |D|,$$

using Lemma 2.6, we find that $\langle B\rangle - J_m|C|$ is an M-matrix.

It follows from (3.5) that

$$P^T\langle A\rangle P = \begin{pmatrix} \langle B\rangle + J_m|C| & & \\ & |\beta| & -\sqrt{2}|a^T|J_m \\ & -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}.$$

As $\langle A\rangle$ is an M-matrix, all the real eigenvalues of $\langle A\rangle$ are positive, thus all the real eigenvalues of the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}$$

are positive too. According to the equivalence conditions of M-matrices, the above matrix is an M-matrix, we obtain that

$$\begin{pmatrix} \beta & \sqrt{2}a^T J_m \\ \sqrt{2}J_m a & B + J_m C \end{pmatrix}$$

is an H-matrix. We have completed the proof.

Note that converse of Theorem 3.2 does not hold in general. For example, the circulant matrix

$$A = \begin{pmatrix} 5 & 3 & 1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 5 & 3 \\ 3 & 1 & 2 & 5 \end{pmatrix}$$

is not an H-matrix, but its reduced form (2.4)

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 5 & 6 \end{pmatrix}$$

is an H-matrix.

By the above example, we find that reduced form of the circulant matrix A is an H-matrix does not imply that A is an H-matrix itself. However, if some conditions are imposed on the matrices B and C in the reduced form of A, then we can derive that the matrix A is a circulant H-matrix.

Theorem 3.3 Let A be an $n \times n$ circulant matrix.

(i) For $n = 2m$, if $\langle B \rangle - |C|$ is an M-matrix, then A is an H-matrix.

(ii) For $n = 2m+1$, if $\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B \rangle - J_m|C| \end{pmatrix}$ is an M-matrix, then A is an H-matrix.

Proof First consider the case of $n = 2m$. From the assumption that $\langle B \rangle - |C|$ is an $m \times m$ M-matrix, and

$$\langle B \rangle - |C| \leq \langle B \rangle \leq D_{\langle B \rangle},$$

where $D_{\langle B \rangle}$ denotes the diagonal part of the matrix $\langle B \rangle$, by Lemma 2.6, we find that the matrix $\langle B \rangle$ is an M-matrix.

Since $\langle B \rangle$ is an M-matrix, according to the definition of regular splitting, then $\langle B \rangle - |C|$ is a regular splitting of the M-matrix $(\langle B \rangle - |C|)$, there holds $\rho(\langle B \rangle^{-1}|C|) < 1$. By means of the proof of Theorem 3.2, the bolck Gauss-Seidel splitting of

$$\langle A \rangle = \begin{pmatrix} \langle B \rangle & 0 \\ -|C| & \langle B \rangle \end{pmatrix} - \begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}$$

is a regular splitting. Note that

$$\rho\left(\begin{pmatrix} \langle B\rangle & 0 \\ -|C| & \langle B\rangle \end{pmatrix}^{-1}\begin{pmatrix} 0 & |C| \\ 0 & 0 \end{pmatrix}\right) = \rho^2(\langle B\rangle^{-1}|C|) < 1.$$

It is well known, a Z-matrix is an M-matrix if and only if it has a convergent regular splitting, so the matrix $\langle A\rangle$ is an M-matrix, and thus A is an H-matrix.

We now turn to consider the case $n = 2m + 1$. From the hypothesis, we have that the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}$$

is an M-matrix. Therefore, the matrices $\langle B\rangle - J_m|C|, \langle B\rangle, \langle B - J_mC\rangle$ and $\langle B + J_mC\rangle$ are all M-matrices too. Moreover, the Schur complement of the matrix

$$\begin{pmatrix} |\beta| & -\sqrt{2}|a^T|J_m \\ -\sqrt{2}J_m|a| & \langle B\rangle - J_m|C| \end{pmatrix}$$

is $(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)$, which is still an M-matrix by using the property of the Schur complement of an M-matrix. Utilizing the equivalence of an M-matrix again, there exist a positive vector y such that

$$(\langle B\rangle - J_m|C| - (2/|\beta|)J_m|a||a^T|J_m)y > 0,$$

that is,

$$(\langle B\rangle - J_m|C|)y > (2/|\beta|)(|a^T|J_my)J_m|a|.$$

Select a positive scalar α such that

$$(\langle B\rangle - J_m|C|)y > \alpha J_m|a| > (2/\beta)(|a^T|J_my)J_m|a|,$$

we obtain that

$$\begin{pmatrix} \langle B\rangle & -J_m|a| & -J_m|C|J_m \\ -|a^T|J_m & |\beta| & -|a^T| \\ -|C| & -|a| & J_m\langle B\rangle J_m \end{pmatrix}\begin{pmatrix} y \\ \alpha \\ J_my \end{pmatrix} > 0,$$

which mean that the matrix $\langle A\rangle$ is an M-matrix. Thus we yield the desired results.

4. The iterative methods for circulant matrices

4.1 Construction of an arithmetic mean splitting

We will give a new splitting scheme of a circulant matrix A, which is called the arithmetic mean splitting. Let the circulant matrix $A = M - N$ be a random convergent splitting of the matrix A. An iterative sequence derived from the splitting is defined by

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b. \tag{4.1.1}$$

Since $A = G^T AG$, we can get another convergent iterative sequence:

$$x_{k+1} = G^T M^{-1} NGx_k + G^T M^{-1} Gb. \tag{4.1.2}$$

In terms of (4.1.1) and (4.1.2), we can obtain a new iterative sequence:

$$x_{k+1} = \frac{1}{2}(M^{-1}N + G^T M^{-1} NG)x_k + \frac{1}{2}(M^{-1} + G^T M^{-1} G)b. \tag{4.1.3}$$

Denote $F = \frac{1}{2}(M^{-1} + G^T M^{-1}G)$ and $H = \frac{1}{2}(M^{-1}N + G^T M^{-1}NG)$, then (4.1.3) can be written as

$$x_{k+1} = Hx_k + Fb. \tag{4.1.4}$$

If det(F)$\neq 0$, a new splitting of A can be expressed as

$$A = F^{-1} - F^{-1}H. \tag{4.1.5}$$

(4.1.5) is called the arithmetic mean splitting of the matrix A. By (4.1.3) we can derive a splitting from a random convergent splitting of the circulant matrix A.

Theorem 4.1.1 Let A be a circulant matrix and

$$A = M - N$$

be a weak regular splitting, then

$$A = G^T AG = G^T MG - G^T NG$$

is also a weak regular splitting.

Proof Since $A = M - N$ is a weak regular splitting, there holds $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. Consider that G is a permutation matrix and use the fact $G^{-1} = G^T$, then

$$(G^T MG)^{-1} = G^T M^{-1} G \geq 0,$$

and

$$(G^T MG)^{-1}(G^T NG) = G^T M^{-1} NG \geq 0,$$

Therefore $A = G^T MG - G^T NG$ is a weak regular splitting.

According to Lemma 2.8 and Theorem 4.1.1, we can get the following iterative convergent theorem.

Theorem 4.1.2 Let A be a circulant M-matrix, and $A = M - N$ be a weak regular of A, then the iterative sequence

$$x_{k+1} = \frac{1}{2}(M^{-1}N + G^T M^{-1} NG)x_k + \frac{1}{2}(M^{-1} + G^T M^{-1} G)b$$

is convergent.

Using Lemma 2.9 we can get the following result.

Theorem 4.1.3 Let A be a circulant H-matrix, $A = F - Q$ be a splitting of the matrix A and $\langle A \rangle = \langle F \rangle - |Q|$, then the iterative sequence

$$x_{k+1} = \frac{1}{2}(F^{-1}J + G^T F^{-1} JG)x_k + \frac{1}{2}(F^{-1} + G^T F^{-1} G)b$$

is convergent.

4.2 Two new splittings of circulant M-matrices

Now we will present two new splittings of the circulant matrix A and investigate their convergence.

(1) Opposite triangular splitting I:

(i) For $n = 2m$, $A = F_1 - Q_1$, where

$$F_1 = \begin{pmatrix} \hat{B}_1 & \hat{C}_1 \\ \hat{C}_1 & \hat{B}_1 \end{pmatrix}, \; Q_1 = \begin{pmatrix} B_1^* & C_1^* \\ C_1^* & B_1^* \end{pmatrix},$$

here $\hat{B}_1$ and $\hat{C}_1$ is the left lower triangular matrix of B and C respectively, and B_1^* and C_1^* is the strictly right upper triangular matrix of $-B$ and $-C$ respectively.

(ii) For $n = 2m + 1$, $A = F_2 - Q_2$, where

$$F_2 = \begin{pmatrix} \hat{B}_2 & J_m a & J_m \hat{C}_2 J_m \\ 0 & \beta & 0 \\ \hat{C}_2 & a & J_m \hat{B}_2 J_m \end{pmatrix}, \; Q_2 = \begin{pmatrix} B_2^* & 0 & J_m C_2^* J_m \\ -a^T J_m & 0 & -a^T \\ C_2^* & 0 & J_m B_2^* J_m \end{pmatrix},$$

here $\hat{B}_2$ is the left lower triangular matrix of B, $\hat{C}_2$ is the left upper triangular matrix of C, B_2^* is the strictly right upper triangular matrix of $-B$, and C_2^* is the strictly right lower triangular matrix of $-C$.

(2) Opposite triangular splitting II:

(i) For $n = 2m$, $A = R_1 - V_1$, where

$$R_1 = \begin{pmatrix} E_1 & H_1 \\ H_1 & E_1 \end{pmatrix}, \; V_1 = \begin{pmatrix} E_1^* & H_1^* \\ H_1^* & E_1^* \end{pmatrix},$$

here E_1 and H_1 is the right upper triangular matrix of B and C respectively, and E_1^* and H_1^* is the strictly left lower triangular matrix of $-B$ and $-C$ respectively.

(ii) For $n = 2m + 1$,$A = R_2 - V_2$, where

$$R_2 = \begin{pmatrix} E_2 & 0 & J_m H_2 J_m \\ a^T J_m & \beta & a^T \\ H_2 & 0 & J_m E_2 J_m \end{pmatrix}, \; V_2 = \begin{pmatrix} E_2^* & -J_m a & J_m H_2^* J_m \\ 0 & 0 & 0 \\ H_2^* & -a & J_m E_2^* J_m \end{pmatrix},$$

here E_2 is the right upper triangular matrix of B, H_2 is the right lower triangular matrix of C, E_2^* is the strictly left lower triangular matrix of $-B$, and H_2^* is the strictly left upper triangular matrix of $-C$.

In terms of the opposite triangular splitting I, we can get the following two SOR iterative sequences [5], at the same time, we can also get the similar conclusion by means of the opposite triangular splitting II.

(i) For $n = 2m$,

(a) Global SOR sequence

$$F_1 x_{k+1} = ((1 - \omega)F_1 + \omega Q_1)x_k + \omega b. \tag{4.2.1}$$

(b) Part SOR sequence

$$F_1 x_{k+1} = Q_1 x_k + b. \tag{4.2.2}$$

Thus we have

$$P^T F_1 P P^T x_{k+1} = P^T Q_1 P P^T x_k + P^T b, \tag{4.2.3}$$

$$\hat{F}_1 = P^T F_1 P = \begin{pmatrix} \hat{B}_1 - \hat{C}_1 & 0 \\ 0 & \hat{B}_1 + \hat{C}_1 \end{pmatrix} = \begin{pmatrix} \hat{T}_1 & 0 \\ 0 & \hat{T}_2 \end{pmatrix},$$

$$\hat{G}_1 = P^T Q_1 P = \begin{pmatrix} B_1^* - C_1^* & 0 \\ 0 & B_1^* + C_1^* \end{pmatrix} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix},$$

where $\hat{T}_1 = \hat{B}_1 - \hat{C}_1$ and $\hat{T}_2 = \hat{B}_1 + \hat{C}_1$ are $m \times m$ left lower triangular matrices, $\hat{H}_1 = B_1^* - C_1^*$ and $\hat{H}_2 = B_1^* + C_1^*$ are $m \times m$ strictly right upper triangular matrices.

Let $P^T x_{k+1} = y_{k+1},\ P^T x_k = y_k,\ P^T b = \hat{b}$, then (4.2.3) becomes

$$\begin{pmatrix} \hat{T}_1 & 0 \\ 0 & \hat{T}_2 \end{pmatrix} \begin{pmatrix} y_{k+1}^{(1)} \\ y_{k+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix} \begin{pmatrix} y_k^{(1)} \\ y_k^{(2)} \end{pmatrix} + \begin{pmatrix} \hat{b}^{(1)} \\ \hat{b}^{(2)} \end{pmatrix}.$$

Thus

$$\begin{cases} \hat{T}_1 y_{k+1}^{(1)} = \hat{H}_1 y_k^{(1)} + \hat{b}_1, \\ \hat{T}_2 y_{k+1}^{(2)} = \hat{H}_2 y_k^{(2)} + \hat{b}_2. \end{cases} \tag{4.2.4}$$

From (4.2.4), we can get Part SOR sequence:

$$\begin{cases} \hat{T}_1 y_{k+1}^{(1)} = ((1-\omega_1)\hat{T}_1 + \omega_1 \hat{H}_1) y_k^{(1)} + \omega_1 \hat{b}_1, \\ \hat{T}_2 y_{k+1}^{(2)} = ((1-\omega_2)\hat{T}_2 + \omega_2 \hat{H}_2) y_k^{(2)} + \omega_1 \hat{b}_2. \end{cases} \tag{4.2.5}$$

(ii) For $n = 2m+1$

(aa) Global SOR sequence

$$F_2 x_{k+1} = ((1-\omega)F_2 + \omega Q_2) x_k + \omega b. \tag{4.2.6}$$

(bb) Part SOR sequence

$$F_2 x_{k+1} = Q_2 x_k + b. \tag{4.2.7}$$

We can get the similar Part SOR iterative sequence:

$$\begin{cases} T_1^* y_{k+1}^{(1)} = ((1-\omega_1)T_1^* + \omega_1 H_1^*) y_k^{(1)} + \omega_1 b_1^*, \\ T_2^* y_{k+1}^{(2)} = ((1-\omega_2)T_2^* + \omega_2 H_2^*) y_k^{(2)} + \omega_1 b_2^*, \end{cases} \tag{4.28}$$

where $T_1^* = \hat{B}_2 - J_m\hat{C}_2$ and $T_2^* = \begin{pmatrix} \beta & 0 \\ \sqrt{2}J_m a & \hat{B}_2 + J_m\hat{C}_2 \end{pmatrix}$ are $m \times m$ left lower triangular matrices, $H_1^* = B_2^* - J_m C_2^*$ and $H_2^* = \begin{pmatrix} 0 & -\sqrt{2}a^T J_m \\ 0 & B_2^* + J_m C_2^* \end{pmatrix}$ are $m \times m$ strictly right upper triangular matrices.

Now we will discuss the convergence of the two splittings of circulant matrices and the SOR iterative sequence above.

Theorem 4.2.1 Let A be a circulant M-matrix, and $A = F - Q$ be opposite triangular splitting I or II of the matrix A, then $\rho(F^{-1}Q) < 1$.

Proof It can easily get that

$$A \leq F \leq |D|,$$

where D is the diagonal part of the matrix A. By Lemma 2.6, F is also an M-matrix, then $F^{-1} \geq 0$. On the other hand, it is evident that $Q \geq 0$. By the definition of the regular splitting, $A = F - Q$ is a regular splitting of the matrix A. Using Lemma 2.2 we have $\rho(F^{-1}Q) < 1$.

Theorem 4.2.2 Let A be a circulant M-matrix, and $A = F_1 - Q_1$ be opposite triangular splitting I of A, then

(1) if $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$; Global SOR sequence is convergent,

(2) if $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$, $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, Part SOR sequence is convergent.

Proof (1) By Theorem 4.2.1, $\rho(F_1^{-1}Q_1) < 1$. Using Lemma 6 in [6], when $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$, $\rho(H(\omega)) < 1$, where $H(\omega) = (1-\omega)I + \omega F_1^{-1}Q_1$ is the iterative matrix of global SOR sequence.

(2) Since $\rho(F_1^{-1}Q_1) < 1$, then $\rho(\hat{F}_1^{-1}\hat{Q}_1) < 1$. From (4.2.3), $\rho(\hat{T}_1^{-1}\hat{H}_1) < 1$, and $\rho(\hat{T}_2^{-1}\hat{H}_2) < 1$. By Lemma 6 of [6], when $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$ and $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, the Part SOR sequence is convergent. Similarly, (4.2.7) and (4.2.8) are convergent.

It is easy to find that the proof of the case of $n = 2m + 1$ is similar to above.

Using the same methods, we can obtain the related results of the splitting II. We will make a comparison of convergence rate of the iterative sequences. From Lemma 2.10, we can get the following two theorems.

Theorem 4.2.3 Let A be a circulant M-matrix, $A = D - (L + U)$ be Jacobi′s splitting of A and $A = F - Q$ be opposite triangular splitting I of A, then $\rho(F^{-1}Q) \leq \rho(D^{-1}(L + U))$.

Proof It can easily get that $A^{-1} \geq 0$, $A = F - Q$ and $A = D - (L + U)$ are the regular splittings of the matrix A and $0 \leq Q \leq L + U$, then by Lemma 2.7 there holds

$$\rho(F^{-1}Q) \leq \rho(D^{-1}(L + U)).$$

Example 4.2.4 Consider the circulant M-matrix

$$A = \begin{pmatrix} 2 & -1 & -0.5 & 0 \\ 0 & 2 & -1 & -0.5 \\ -0.5 & 0 & 2 & -1 \\ -1 & -0.5 & 0 & 2 \end{pmatrix}.$$

The iterative matrices of Jacobi's splitting and the opposite triangular splitting I of the matrix A can be expressed by G_J and G_I, respectively. We have $\rho(G_J) = 0.7500$ and $\rho(G_I) = 0.4444$. Thus $\rho(G_I) < \rho(G_J)$.

Theorem 4.2.5 Let A be a circulant M-matrix, $A = D-(L+U)$ and $A = R-V$ be Jacobi's splitting and opposite triangular splitting II of the matrix A respectively, then $\rho(R^{-1}V) \leq \rho(D^{-1}(L+U))$.

Proof The proof is similar to that of Theorem 4.2.3.

Example 4.2.6 Consider the circulant M-matrix

$$A = \begin{pmatrix} 8 & -1 & -2 & -4 \\ -4 & 8 & -1 & -2 \\ -2 & -4 & 8 & -1 \\ -1 & -2 & -4 & 8 \end{pmatrix}.$$

Let G_{II} be the iterative matrix of opposite triangular splitting II. We get $\rho(G_J) = 0.8750$, and $\rho(G_{II}) = 0.6944$. Thus $\rho(R^{-1}V) \leq \rho(D^{-1}(L+U))$.

Example 4.2.7 Consider the circulant M-matrix

$$A = \begin{pmatrix} 8 & -1 & -2 & -3 \\ -3 & 8 & -1 & -2 \\ -2 & -3 & 8 & -1 \\ -1 & -2 & -3 & 8 \end{pmatrix}.$$

Let the iterative matrix of Gauss-Seidel splitting be G_G. We get $\rho(G_G) = 0.5111$, and $\rho(G_I) = \rho(G_{II}) = 0.4444$. Then $\rho(F_1^{-1}J_1) = \rho(F_2^{-1}J_2) \leq \rho(D-L)^{-1}U)$, which mean that in this example, the opposite triangular splitting I and II have a better convergence rate than that of Gauss-Seidel splitting.

In fact, we can get the similar conclusion for the case of $n = 2m+1$.

4.3 Several splittings of circulant H-matrices

In this subsection, we also give two new splittings which are similar to those in Subsection 4.2. Now we only discuss their convergence, their costs of computation and store are analogous with those of Subsection 4.2.

Theorem 4.3.1 Let A be a circulant H-matrix, $A = F - Q$ be opposite triangular splitting I(II) of the matrix A, then $\rho(F^{-1}Q) < 1$.

Proof There holds $\langle A \rangle = \langle F \rangle - |Q|$, by Lemma 2.9, thus we get $\rho(F^{-1}Q) < 1$.

Theorem 4.3.2 Let A be a circulant H-matrix, $A = F - Q$ be opposite triangular splitting I(II) of A, then

(1) if $\omega \in \left(0, \frac{2}{1+\rho(F_1^{-1}Q_1)}\right)$, then Global SOR sequence is convergent;

(2) if $\omega_1 \in \left(0, \frac{2}{1+\rho(\hat{T}_1^{-1}\hat{H}_1)}\right)$, $\omega_2 \in \left(0, \frac{2}{1+\rho(\hat{T}_2^{-1}\hat{H}_2)}\right)$, then Part SOR sequence is convergent.

Proof The proof is similar to Theorem 4.2.2.

4.4 Three algorithms for the solution of $Ax = b$

Finally, we will construct three algorithms for the linear system $Ax = b$. The following algorithms 1 and 2 are based on the opposite triangular splittings in Subsections 4.2 and 4.3, and GMRES(m) algorithm is applied when the matrix A is very large and sparse.

Algorithm 1 (opposite triangular splitting I)

Step 1: Select an arbitrary starting point x_0 and a stopping criteria ε.

Step 2: Let $A = F - Q$ be the opposite triangular splitting I. Its iterative sequence is $Fx_k = Qx_{k-1} + b$, where A is a circulant M-matrix or a circulant H-matrix. By Lemma 2.5, there exists an orthogonal matrix P such that

$$P^T FPP^T x_{k+1} = P^T QPP^T x_k + P^T b.$$

It is easy to know that $\hat{F} = P^T FP$ and $\hat{Q} = P^T QP$ are a left lower triangular matrix and a right strictly upper triangular matrix, respectively. Let

$$\begin{aligned} \hat{x}_k = & \quad P^T x_k = (\hat{x}_{k,1}\hat{x}_{k,2}, \cdots, \hat{x}_{k,n})^T, \\ \hat{b} = & \quad P^T b = (\hat{b}_1\hat{b}_2, \cdots, \hat{b}_n)^T. \end{aligned}$$

Step 3: For $k = 1, 2, \cdots$, and for $j = 1$ to n, construct

$$\hat{x}_{k,i} = \frac{1}{\hat{F}_{i,i}} \left(\hat{b}_i - \sum_{s=1}^{i-1} \hat{F}_{i,s}\hat{x}_{k,s} + \sum_{s=i+1}^{n} \hat{Q}_{i,s}\hat{x}_{k-1,s} \right).$$

Step 4: If $\|\hat{x}_k - \hat{x}_{k-1}\| < \varepsilon$, then stop, let $x = P\hat{x}_k$, which is an approximate solution to the linear system $Ax = b$; Otherwise set $k = k + 1$ and return to step 3.

Algorithm 2 (opposite triangular splitting II)

Step 1: Select an arbitrary starting point x_0 and a stopping criteria ε.

Step 2: Let $A = R - V$ be the opposite triangular splitting II and its iterative sequence be $Rx_k = Vx_{k-1} + b$, where A is a circulant M-matrix or a circulant H-matrix. By Lemma 2.5, there exists an orthogonal matrix P:

$$P^T RPP^T x_{k+1} = P^T VPP^T x_k + P^T b.$$

It is easy to know that $\hat{R} = P^T RP$ and $\hat{V} = P^T VP$ are an right upper triangular matrix and a strictly left lower triangular matrix respectively. Let

$$\begin{aligned} \hat{x}_k = & \quad P^T x_k = (\hat{x}_{k,1}\hat{x}_{k,2}, \cdots, \hat{x}_{k,n})^T, \\ \hat{b} = & \quad P^T b = (\hat{b}_1\hat{b}_2, \cdots, \hat{b}_n)^T. \end{aligned}$$

Step 3: For $k = 1, 2, \cdots$, and for $j = 1$ to n, construct

$$\hat{x}_{k,i} = \frac{1}{\hat{R}_{i,i}} \left(\hat{b}_i + \sum_{s=1}^{i-1} \hat{V}_{i,s}\hat{x}_{k-1,s} - \sum_{s=i+1}^{n} \hat{R}_{i,s}\hat{x}_{k,s} \right).$$

Step 4: If $\|\hat{x}_k - \hat{x}_{k-1}\| < \varepsilon$, then stop, let $x = P\hat{x}_k$, which is an approximate solution to the linear system $Ax = b$; Otherwise set $k = k + 1$ and return to step 3.

When the circulant matrix A is very large and sparse, the GMRES(m) algorithm is very useful to solve the linear system $Ax = b$. Using the circulant property of the matrix A, we can reduce a large of the cost of computation and store by means of GMRES(m) algorithm.

Algorithm 3 (GMRES(m) algorithm)

Step 1: Reduce the linear system $Ax = b$ to

$$P^T APP^T x = P^T b.$$

Let $\tilde{A} = P^T AP$, $\tilde{x} = P^T x$, and $\tilde{b} = P^T b$.

Step 2: Choose $\tilde{x}_0 \in R^n$, calculate $r_0 = \tilde{b} - \tilde{A}\tilde{x}_0$ and $v_1 = r_0/\|r_0\|_2$.

Step 3: Choose an appropriate m, obtain $\{v_i\}_{i=1}^m$ and $\tilde{H}_m$ by the Arnoldi process.

Step 4: Calculate $y_m = min_{y \in R^k} \|\beta e_1 - \tilde{H}_m y\|_2$.

Step 5: Obtain $\tilde{x}_m = \tilde{x}_0 + V_m y_m$.

Step 6: Calculate $\|r_m\| = \|\tilde{b} - \tilde{A}\tilde{x}_m\|$. For a given $\varepsilon > 0$, if $\|r_m\| < \varepsilon$, then stop, and we can obtain the approximate solution: $x = P\tilde{x}$.

Step 7: Otherwise let $\tilde{x}_0 = \tilde{x}_m$, and $v_1 = r_m/\|r_m\|_2$, return to step 3.

References

[1] P. J. Davis, Circulant Matrices, second ed., Chelsea Publishing, New York, 1994.

[2] R. M. Gray, Toeplitz and circulant matrices: a review. Available from: <http://www-ee.stanford.edu/ gray/toeplitz.pdf>.

[3] A. Berman and R. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.

[4] J. L. Chen and X. H. Chen, Special matrices, Tsinghua University Press, Beijing, 2001.

[5] O. Rojo, H. Rojo,Some results on symmetric circulant matrices and on symmetric centrosymmetric matrices, Linear Algebra and its Applications, 392(2004):211-233.

[6] C. B. Lu and C. Q. Gu, The computation of the square roots of circulant matrices, Appl. Math. Comput. 217 (2011), 6819-6829.

[7] J. Janas and M. Moszyński, Spectral properties of Jacobi matrices by asymptotic analysis, Journal of Approximation Theory 120(2003):309-336.

[8] Y. X. Peng, X. Y. Hu and L. Zhang, An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation $AXB = C$, Applied Mathematics and Computation, 160 (2005): 763-777.

[9] L. Elsner, Comparisons of weak regular splitting and multisplitting methods, Numer. Math., 56 (1989):283-289.

[10] Y. Mei, Computing the square roots of a class of circulant matrices, Journal of Applied Mathematics, (2012), 1-15.

[11] J. W. Demmel, Applied Numerical Linear Algebra, Society for Industrial and Applied Mathematics, Philadelphia, 1997.

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Disjoint mixing weighted backward shifts on the space of all complex valued square summable sequences

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[1]

Abstract

We characterize when N-tuples $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ of powers of unilateral weighted backward shift operators defined the space of all complex valued square summable sequences exhibit the d-mixing properties, and give several equivalent conditions of d-mixing properties. At the same time, d-mixing powers of bilateral weighted backward shifts on Banach sequence spaces are also discussed in this paper.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integral numbers, $\mathbb{K}$ a real or complex scalar field, and the space of all sequences $\mathbb{K}^{\mathbb{N}} = \{(x_n)_n; x_n \in \mathbb{K}, n \in \mathbb{N}\}$.

Let $1 \leq p < \infty$. Then the space

$$l^p := \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}}; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

of p-summable sequences, endowed with the norm $\|x\| := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$, is a Banach space. In particular, l^2 is a Hilbert space with inner product defined by $\langle x, y \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$. Occasionally we let the index start with 0. The finite sequences, that is, sequences of the form $(x_1, \cdots, x_n, 0, \cdots), n \geq 1$, constitute a dense subset. Considering only the finite sequences with entries from $\mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$ we see that any l^p, $1 \leq p < \infty$, is separable. The space $l^p(\mathbb{Z})$ of p-summable sequences, indexed over $\mathbb{Z}$, is defined analogously.

The space

$$l^\infty := \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}}; \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}$$

of bounded sequences, endowed with the sup-norm $\|x\| := \sup_{n \in \mathbb{N}} |x_n|$, is a Banach space. Since it is not separable it will be of less interest to us. Instead, its closed subspace

$$c_0 := \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}}; \lim_{n \to \infty} x_n = 0 \right\}$$

of null sequences is a separable Banach space under the induced norm.

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Definition 1.1. *A Fréchet space is a vector space X, endowed with a separating increasing sequence $(p_n)_n$ (by considering $\max_{kn} p_k$, if necessary) of seminorms, which is complete in the metric given by*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x-y)\}, \quad x, y \in X.$$

Let X denote separable infinite dimensional Fréchet space, and by $L(X)$ denote the space of linear and continuous operators on a separable, infinite dimensional Fréchet space X.

Definition 1.2. *An operator $T : X \to X$ is called hypercyclic if there is some $x \in X$ whose orbit under T is dense in X. In such a case, x is called a hypercyclic vector for T. The set of hypercyclic vectors for T is denoted by $HC(T)$.*

The first examples of hypercyclic operators were on the space $X = H(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. In 1929, Birkhoff [1] showed that the operators of translation on this space are hypercyclic. In 1952, MacLane [15] showed that the operators of differentiation are also hypercyclic. In 1991, Godefroy and Shapiro [13] provided a comprehensive extension of these two results to all convolution operators (but scalar multiples of the identity). Some other related classical results have been characterized, such as in [2, 6, 7, 12, 13, 9, 10, 20] and the related references therein.

The first example of a hypercyclic operator acting on a Banach space was given by Rolewicz [16]. The example is the backward shift operator on l^p, scaled by a constant greater than 1. The backward shift operator $\mathbb{B}$ on the space l^2 of all complex valued square summable sequences is defined by

$$\mathbb{B}(a_0, a_1, a_2, \cdots) = (a_1, a_2, a_3, \cdots).$$

Since $\mathbb{B}$ is a contraction, $\mathbb{B}$ itself cannot be hypercyclic. It was shown in 1969 by Rolewicz [16] that if $\mathbb{B}$ is the backward shift, then $\lambda\mathbb{B}$ is hypercyclic if and only if $|\lambda| > 1$. It then follows easily that $\mathbb{B}$ itself is supercyclic. It was shown later in 1974 by Hilden and Wallen [14] that any (unilateral) backward weighted shift is supercyclic. In [18] and [19], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. And, in [8], bilateral weighted backward shifts on l^2 spaces are also discussed and hypercyclic and supercyclic properties are characterized, respectively.

Recently, there have been an increasing interest in studying the disjoint hypercyclicity acting on different spaces of holomorphic functions. In [3] and [4], disjoint hypercyclic unilateral weighted backward shifts on $c_0(\mathbb{N})$ or $l^p(\mathbb{N})$ are characterized and disjoint hypercyclic properties of unilateral weighted backward shifts are also discussed. In 2012, in [5], the authors have showed that every separable infinite-dimensional Fréchet space supports an arbitrarily large finite and commuting disjoint mixing collection of operators. When this space is a Banach space, it supports an arbitrarily large finite disjoint mixing collection of C_0-semigroups.

In this paper, we will discuss disjoint mixing powers of unilateral weighted backward shifts on $l^2(\mathbb{N})$ and characterize the equivalent conditions of disjoint mixing powers of bilateral weighted backward shifts on $l^2(\mathbb{Z})$.

2 Preliminary definitions

Definition 2.1. *We say that $N \geq 2$ sequences of operators $(T_{1,j})_{j=1}^{\infty}, \cdots, (T_{N,j})_{j=1}^{\infty}$ on separable infinite dimensional Fréchet space X, are d-topologically transitive (respectively, d-mixing), provided for every non-empty open subsets $V_0, V_1, \cdots, V_N$ of X, there exists $m \in \mathbb{N}$,*

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so that

$$\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap \cdots \cap T_{N,m}^{-1}(V_N)$$

(respectively, so that $\emptyset \neq V_0 \cap T_{1,j}^{-1}(V_1) \cap \cdots \cap T_{N,j}^{-1}(V_N)$*, for* $\forall j \geq m$*); Also, we say that* $T_1, \cdots, T_N$ *are d-topologically transitive (respectively d-mixing), provided* $(T_1^j)_{j=1}^{\infty}, \cdots,$ $(T_N^j)_{j=1}^{\infty}$ *are d-topologically transitive sequences (respectively, d-mixing sequences).*

Definition 2.2. *We say that* $N \geq 2$ *hypercyclic operators* $T_1, T_2, \cdots, T_N$ *acting on separable infinite dimensional Fréchet space* X *are disjoint , or diagonally hypercyclic (in short, d-hypercyclic), provided there is some vector* $(z, z, \cdots, z)$ *in the diagonal of* $X^N = X \times X \times \cdots \times X$ *such that*

$$\{(z, z, \cdots, z), (T_1 z, T_2 z, \cdots, T_N z), (T_1^2 z, T_2^2 z, \cdots, T_N^2 z), \cdots\} \quad (z \in X)$$

is dense in X^N*. We call the vector* $z \in X$ *a d-hypercyclic vector associated to the operators* $T_1, T_2, \cdots, T_N$*.*

Definition 2.3. *We say that* $N \geq 2$ *sequences of operators* $(T_{1,j})_{j=1}^{\infty}, \cdots, (T_{N,j})_{j=1}^{\infty}$ *on separable infinite dimensional Fréchet space* X *are d-universal (respectively densely d-universal) if*

$$\{(T_{1,j} z, \cdots, T_{N,j} z) : j \in \mathbb{N}\}$$

is dense in X^N *for some vector* $z \in X$ *(respectively for each vector* z *in a given dense subset of* X *). We call such vector* z *a d-universal vector for* $(T_{1,j})_{j=1}^{\infty}, \cdots, (T_{N,j})_{j=1}^{\infty}$*. Also, we say that* $(T_{1,j})_{j=1}^{\infty}, \cdots, (T_{N,j})_{j=1}^{\infty}$ *are hereditarily universal (respectively hereditarily densely d-universal) provided for each increasing sequence of positive integers* (n_k) *the sequences* $(T_{1,n_k})_{k=1}^{\infty}, \cdots, (T_{N,n_k})_{k=1}^{\infty}$ *are d-universal (respectively densely d-universal).*

3 Disjoint mixing unilateral weighted backward shifts on $l^2(\mathbb{N})$

In this section, $X = l^2(\mathbb{N})$ over the real or complex scalar field $\mathbb{K}$ and, given a bounded sequence $a = (a_k)_k$ of non-zero weights, let $B_a : X \to X$ be the unilateral weighted shift:

$$x = (x_0, x_1, \cdots) \overset{B_a}{\to} (a_1 x_1, a_2 x_2, \cdots)$$

Theorem 3.1. *Let* $X = l^2(\mathbb{N})$*, and let integers* $1 \leq r_1 < r_2 < \cdots < r_N$ *be given. For each* $1 \leq l \leq \mathbb{N}$*, let* $a_l = (a_{l.k})_{k=1}^{\infty}$ *be a weight sequence and* $B_{a_l} : X \to X$ *be the corresponding unilateral backward shift*

$$x = (x_0, x_1, \cdots) \overset{B_{a_l}}{\to} (a_{l,1} x_1, a_{l,2} x_2, \cdots)$$

And, let $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *be mixing on* $X = l^2(\mathbb{N})$*. Then the following are equivalent:*

(a) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *are d-mixing.*

(b) The following conditions hold:

$$\lim_{n \to +\infty} |a_{l,1} \cdots a_{l,r_l n}| = +\infty, \quad (1 \leq l \leq N)$$

$$\lim_{n \to +\infty} \frac{|a_{l,1} \cdots a_{l,r_l n}|}{\left|a_{s,1+(r_l - r_s)n} \cdots a_{s,r_l n}\right|} = +\infty, \quad (1 \leq s < l \leq N).$$

(c) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *satisfy the d-Hypercyclicity Criterion with respect to some syndetic sequence.*

(d) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ *are hereditarily densely d-hypercyclic with respect to the sequence* (n)*.*

To prove the theorem, let us state d-Hypercyclicity Criterion and a couple of results, which are used in the proofs of the main theorems.

Definition 3.2. *Let (n_k) be a strictly increasing sequences of positive integers. We say that $T_1, T_2, \cdots ., T_N \in L(X)$ satisfy the d-Hypercyclicity Criterion with respect to (n_k) provided there exist dense subsets $X_0, X_1, \cdots, X_N$ of X and mappings $S_{l,k} : X_l \to X (1 \le l \le N, k \in \mathbb{N})$ satisfying*

$$(i)\ \ T_l^{n_k} \underset{k\to\infty}{\to} 0 \ \ \textit{pointwise on} \ \ X_0, \tag{3.1}$$

$$(ii)\ \ S_{l,k} \underset{k\to\infty}{\to} 0 \ \ \textit{pointwise on} \ \ X_l, \ \textit{and} \tag{3.2}$$

$$(iii)\ \ (T_l^{n_k} S_{i,k} - \delta_{i,l} Id_{X_l}) \underset{k\to\infty}{\to} 0 \ \ \textit{pointwise on} \ \ X_l (1 \le i \le N). \tag{3.3}$$

In general, we say that $T_1, T_2, \cdots ., T_N \in L(X)$ satisfy the d-Hypercyclicity Criterion if there exists some sequence (n_k) for which the above is satisfied.

Lemma 3.3. *[4, Proposition 7] Let $T_1, T_2, \cdots ., T_N$ satisfy the d-Hypercyclicity Criterion with respect to a sequence $\{n_k\}$. Then the sequences $\{T_1^{n_k}\}_{k=1}^{\infty}, \cdots, \{T_N^{n_k}\}_{k=1}^{\infty}$ are d-mixing. In particular, $T_1, T_2, \cdots ., T_N$ are d-hypercyclic.*

An increasing sequence of positive integers $\{n_k\}$ is syndetic if

$$\sup_k \{n_{k+1} - n_k\} < \infty.$$

We say that T satisfies the Hypercyclicity Criterion for a syndetic sequence if the sequence $\{n_k\}$ is syndetic in the above criterion . Notice that a large class of hypercyclic operators satisfies the Hypercyclicity Criterion for a syndetic sequence, for instance: λB where $|\lambda| > 1$ and B is the backward shift on $\ell^2 = \ell^2(\mathbb{N})$ (the Hilbert space of square summable sequences).

Theorem 3.4. *Let $T_1, T_2, \cdots, T_N$ be operators acting on separable infinite dimensional Fréchet space X. Assume that $T_1, T_2, \cdots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to some syndetic sequence $\{n_k\}$. Then $T_1, T_2, \cdots, T_N$ are d-mixing.*

Proof. Let $V_0, V_1, \cdots, V_N$ be open and non-empty subsets of X. Since the sequence $\{n_k\}$ in the d-Hypercyclicity Criterion is syndetic, there is some positive integer m such that

$$n_{k+1} - n_k \le m, \ \ \forall k \ge 0.$$

For $i = 0, 1, \cdots, m$ and $l = 1, \cdots, N$, consider open sets $V_{l,i}$ such that $T^i(V_{l,i}) = V_l$. Pick $y_0 \in V_0 \cap X_0$ and take $\varepsilon > 0$ such that the ball $B(y_0, (N+1)\varepsilon) \subset V_0$. Also, for each $i = 0, 1, \cdots, m$ and $l = 1, \cdots, N$, take $y_{l,i} \in V_{l,i} \cap X_l$ and we may assume that ε is small enough such that the ball $B(y_{l,i}, (N+1)\varepsilon) \subset V_{l,i}$. In what follows we write where $\|x\| = d(x, 0)$ is the complete invariant metric of the Fréchet space. By (3.1), (3.2) and (3.3), there exists $k_0 \in \mathbb{N}$ so that $T_l^{n_k} y_0, \ \ S_{l,k} y_{l,i}$ and $(T_l^{n_k} S_{j,k} y_{j,i} - \delta_{j,l} y_{j,i})$ belong to $B(0, \varepsilon)$ for $\forall k \ge k_0, \ \ 1 \le j \le N$, and $0 \le i \le m$.

Set $N_1 = n_{k_0}$, and let $n \ge N_1$. It is followed that there is some n_k with $k \ge k_0$ and $0 \le r \le m$ such that

$$n = n_k + r.$$

Then $y_n := y_0 + \sum_{j=1}^{N} S_{j,n_k} y_{j,r} \in V_0$ and

$$T_l^n y_n = T_l^n y_0 + \sum\nolimits_{j=1}^{N} T_l^n S_{j,n_k} y_{j,r} \subset T_l^r (B(y_{l,r}, (N+1)\varepsilon)) \subset V_l \ \ (1 \le l \le N).$$

That is,

$$V_0 \cap T_1^{-n}(V_1) \cap ... \cap T_N^{-n}(V_N) \ne \emptyset \ \ \text{for each} \ \ n \ge N_1.$$

So $T_1, T_2, \cdots, T_N$ are d-mixing. □

Now, we can prove Theorem 3.1.

Proof. of Theorem 3.1.

$(a) \Rightarrow (b)$. Let $\varepsilon > 0$ and pick $0 < \delta < 1$ with $\frac{\delta}{1-\delta} < \varepsilon$. Since $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are d-mixing, there exists $m \in \mathbb{N}$, for all $j \geq m$, all $1 \leq l \leq N$ so that

$$V_0 \cap \bigcap_{l=1}^{N} B_{a_l}^{-\mathrm{jr_l}} (V_l) \neq \phi.$$

In particular, let $V_0 = V_1 = \cdots = V_N = \{x \in X; \|x - e_0\| < \delta\}$. Then there exists a vector $x = (x_0, x_1, \cdots)$ and $m \in \mathbb{N}$ $(m > 1)$, for all $n \geq m > 1$, for $k \geq r_1 n$, we have

$$|x_k| < \delta$$

$$\left\|B_{a_l}^{r_l n} x - e_0\right\| < \delta \, (1 \leq l \leq N) \tag{3.4}$$

Then for $l = 1, \cdots, N$, we get

$$1 - \delta < |a_{l,1} \cdots a_{l,r_l n} x_{r_l n}| < 1 + \delta,$$

$$|a_{l,i+1} \cdots a_{l,i+r_l n} x_{i+r_l n}| < \delta, \quad i > 0. \tag{3.5}$$

Now, let $1 \leq l \leq N$ be fixed. Combining (3.4) with (3.5), we get

$$|a_{l,1}, \cdots a_{l,r_l n}| > \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}.$$

And, for $1 \leq s < l \leq N$, it follows from (3.4) and (3.5) that

$$\begin{aligned} \frac{|a_{l,1} \cdots a_{l,r_l n}|}{\left|a_{s,(r_l - r_s)n+1} \cdots a_{s,r_l n}\right|} &= \frac{|a_{l,1} \cdots a_{l,r_l n} x_{r_l n}|}{\left|a_{s,(r_l - r_s)n+1} \cdots a_{s,r_l n} x_{(r_l - r_s)n + r_s n}\right|} \\ &> \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}. \end{aligned}$$

$(b) \Rightarrow (c)$. Now, let $X_0 = span\{e_0, e_1 \cdots\}$. Notice that X_0 is dense in X, and that $B_{a_l}^{r_l n} \underset{n\to\infty}{\to} 0$ pointwise on X_0 $(1 \leq l \leq N)$, consider the mappings $S_{l,n} : X_0 \to X$ given by

$$S_{l,n}(x_0, x_1 \cdots) = \left(\overbrace{0, \cdots, 0}^{r_l n}, \frac{x_0}{a_{l,1} a_{l,2} \cdots a_{l,r_l n}}, \cdots, \frac{x_j}{a_{l,1+j} a_{l,2+j} \cdots a_{l,r_l n + j}}, \cdots \right)$$

Therefore, $B_{a_l}^{r_l n} S_{l,n} = Id_{X_0}$ and since (b) holds, $S_{l,n} \underset{n\to\infty}{\to} 0$ pointwise on X_0 $(1 \leq l \leq N)$. Now, for $1 \leq s < l \leq N$ and since $r_s < r_l$, we get $B_{a_l}^{r_l n} S_{s,n} \underset{n\to\infty}{\to} 0$. And $B_{a_l}^{r_s n} S_{l,n} \underset{n\to\infty}{\to} 0$ on X_0. Then, $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Hypercyclicity Criterion with respect to the sequence (n).

$(c) \Rightarrow (a)$. By Theorem 3.4, it is obvious.

$(d) \Leftrightarrow (a)$. An application of the Baire theorem shows that if X is Baire and second countable, then a sequence $\{(T_{1,n}, T_{2,n}, \cdots, T_{k,n})\}_{n\in\mathbb{Z}_+}$ is d-transitive if and only it is densely d-universal. Clearly, a sequence $\{(T_{1,n}, T_{2,n}, \cdots, T_{k,n})\}_{n\in\mathbb{Z}_+}$ is d-mixing if and only if its every subsequence is d-transitive. Again, we know that $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are d-mixing if and only if $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are hereditarily densely d-hypercyclic with respect to the full sequence (n). □

4 Disjoint mixing bilateral weighted backward shifts on $l^2(\mathbb{Z})$

In this section, we characterize the equivalent conditions of disjoint mixing bilateral weighted backward shifts on $l^2(\mathbb{Z})$.

Theorem 4.1. *Let $X = l^2(\mathbb{Z})$. For each $1 \le l \le \mathbb{N}$, let $a_l = (a_{l,j})_{j\in\mathbb{Z}}$ be a bounded bilateral sequence of non-zero scalars, and $B_{a_l} : X \to X$ be the corresponding backward shift given by*

$$B_{a_l} e_k = a_{l,k} e_{k-1}\,(k \in \mathbb{Z})$$

For any integers $1 \le r_1 < r_2 < \cdots < r_N$, let $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ be mixing on $X = l^2(\mathbb{Z})$. Then, the following are equivalent:

(a) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are d-mixing.

(b) The following conditions hold:

$$\left\{\begin{array}{l} \lim\limits_{p\to+\infty} \left|\prod_{i=1}^{r_l p} a_{l,i}\right| = +\infty \\ \lim\limits_{p\to+\infty} \left|\prod_{i=1-r_l p}^{0} a_{l,i}\right| = 0 \end{array}\right., 1 \le l \le N \tag{4.1}$$

$$\left\{\begin{array}{l} \lim\limits_{p\to+\infty} \frac{\left|\prod_{i=1}^{r_l p} a_{l,i}\right|}{\left|\prod_{i=(r_l-r_s)p+1}^{r_l p} a_{s,i}\right|} = \infty \\ \lim\limits_{p\to+\infty} \frac{\left|\prod_{i=1-(r_l-r_s)p}^{r_s p} a_{l,i}\right|}{\left|\prod_{i=1}^{r_s p} a_{s,i}\right|} = 0 \end{array}\right., 1 \le s < l \le N \tag{4.2}$$

(c) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Hypercyclicity Criterion with respect to some syndetic sequence.

(d) $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are hereditarily densely d-hypercyclic with respect to the sequence (n).

Proof. $(a) \Rightarrow (b)$. Let $\varepsilon > 0$ and pick $0 < \delta < 1$ with $\frac{\delta}{1-\delta} < \varepsilon$. Since $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are d-mixing, there exists a vector $x = (x_0, x_1 \cdots)$ so that

$$|x - e_0| < \delta. \tag{4.3}$$

and $m \in \mathbb{N}$ $(m > 0)$, for all $p \ge m > 0$, for $k \ge r_1 p$, we have

$$|x_k| < \delta,$$

$$\left\|B_{a_l}^{r_l p} x - e_0\right\| < \delta \quad (1 \le l \le N)\,. \tag{4.4}$$

It follows from (4.3) that

$$\left\{\begin{array}{c} |x_0 - 1| < \delta, \\ |x_i| < \delta, \quad \text{if}\,|i| > 0. \end{array}\right., \tag{4.5}$$

Now, by the definition of bilateral shifts, (4.4) and (4.5), we get

$$\left|\prod_{i=1}^{r_l p} a_{l,i}\right| > \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}$$

Similarly, since $|pr_l| > 0$, we have

$$\left|\prod_{i=1-r_l p}^{0} a_{l,i}\right| < \frac{\delta}{1-\delta} < \varepsilon$$

Besides, if $1 \leq s < l \leq N$, by he definition of bilateral shifts, (4.4) and (4.5), we easily get

$$\frac{|\prod_{i=1}^{r_l p} a_{l,i}|}{\left|\prod_{i=(r_l-r_s)p+1}^{r_l p} a_{s,i}\right|} > \frac{|(\prod_{i=1}^{r_l p} a_{l,i})\, x_{pr_l}|}{\delta} > \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}$$

Similarly, if $1 \leq s < l \leq N$, we get

$$\left|\prod_{i=1-(r_l-r_s)p}^{r_s p} a_{l,i}\right| < \varepsilon \left|\prod_{i=1}^{r_s p} a_{s,i}\right|$$

$(b) \Rightarrow (c)$. By assumption, (4.1) and (4.2) hold. Now, let $X_0 := span\{e_k : k \in \mathbb{Z}\}$. By the definition of bilateral shifts, (4.1) and (4.2), we get $\left(B_{a_l}^{r_l}\right)^n \underset{n\to\infty}{\to} 0$ pointwise on X_0. And for each $1 \leq l \leq N$, let $S_{l,n} : X_0 \to X$ be the linear map given by

$$S_{l,n} e_k = \frac{1}{a_{l,k+1} a_{l,k+2} \cdots a_{l,k+r_l n}} e_{k+r_l n} \,(k \in \mathbb{Z})\,.$$

By (4.1), $S_{l,n} \underset{n\to\infty}{\to} 0$ pointwise on X_0. And $\left(B_{a_l}^{r_l}\right)^n S_{l,n} = Id_{X_0}$. At the same time, for $1 \leq s < l \leq N$, we know

$$\left(B_{a_l}^{r_l}\right)^n S_{s,n} e_k = \frac{\prod_{i=k-(r_l-r_s)n+1}^{k+r_s n} a_{l,i}}{\prod_{i=k+1}^{k+r_s n} a_{s,i}} e_{k+r_s n - r_l n}, (k \in \mathbb{Z})\,.$$

Therefore, By (4.2), $\left(B_{a_l}^{r_l}\right)^n S_{s,n} \underset{n\to\infty}{\to} 0$. Similarly, for $1 \leq l < s \leq N$, we have

$$\left(B_{a_s}^{r_s}\right)^n S_{l,n} e_k = \frac{\prod_{i=k+(r_l-r_s)n+1}^{k+r_l n} a_{s,i}}{\prod_{i=k+1}^{k+r_l n} a_{l,i}} e_{k+r_l n - r_s n}, (k \in \mathbb{Z})\,.$$

By (4.2), $\left(B_{a_s}^{r_s}\right)^n S_{l,n} \underset{n\to\infty}{\to} 0$ pointwise on X_0. So $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ satisfy the d-Hypercyclicity Criterion with respect to the sequence (n).

$(c) \Rightarrow (a)$. By Theorem 3.4, $B_{a_1}^{r_1}, \cdots, B_{a_N}^{r_N}$ are d-mixing.

$(d) \Leftrightarrow (a)$. It is obvious. □

References

[1] G.D. Birkhoff, Démonstration dun théorème élémentaire sur les fonctions entières, C.R. Acad. Sci. Paris 189(1929) 473-475.

[2] P. S. Bourdon, J. H. Shapiro, Cyclic phenomena for composition operators, Mem. Amer. Math. Soc. 596 (1997).

[3] J. Bès, A. Peris, Disjointness in hypercyclicity, J. Math. Anal. Appl. 336 (2007), 297-315.

[4] J. Bès, Ö. Maritin, A. Peris, Disjoint hypercyclic linear fractional composition operators, J. Math. Anal. Appl. 381 (2011), 843-856.

[5] J.Bès, Ö. Maritin, A. Peris c, S. Shkarind, Disjoint mixing operators, J. Funct. Anal. 263 (2012), 1283-1322.

[6] K. C. Chan, J. H. Shapiro, The cyclic behavior of translation operators on Hilbert spaces of entire functions, Indiana Univ. Math. J. 40 (1991), 1421-1449.

[7] R. Y. Chen and Z.H.Zhou, Hypercyclicity of weighted composition operators on the unit ball of $\mathbb{C}^N$, J. Korean Math. Soc. 48(5) (2011), 969-984.

[8] N. Feldman, Hypercyclicity and Supercyclicity for invertible bilateral weighted shifts, Proc. Amer. Math. Soc. 131(2)(2003), 479-485.

[9] K. G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36(3) (1999), 345-381.

[10] K. G. Grosse-Erdmann, Recent developments in hypercyclicity, Rev. R. Acad. Cien. Serie A. Mat. 97 (2) (2003), 273-286.

[11] K. G. Grosse-Erdmann, A. P. Manguillot, Linear Chaos, Springer, New York, 2011.

[12] R. M. Gethner, J. H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. 100 (1987), 281-288.

[13] G. Godefroy, J. H. Shapiro, Operators with dense invariant cyclic vector manifolds, J. Funt. Anal. 98 (1991), 229-269.

[14] H. M. Hilden, L. J. Wallen, Some cyclic and non-cyclic vectors of certain operators, Ind. Univ. Math. J. 23 (1974), 557-565.

[15] G. R. MacLane, Sequences of derivatives and normal families, J. Anal. Math. 2 (1952), 72-87.

[16] S. Rolewicz, On orbits of elements, Studia Math. 32 (1969), 17-22.

[17] H. Rezaei, Notes on dynamics of the adjoint of a weighted composition operator, Taiwanese J. Math. 14(4) (2010), 1377–1384.

[18] H. N. Salas, Hypercyclic weighted shifts, Trans Amer. Math. Soc. 347 (1995), 993-1004.

[19] H. N. Salas, Supercyclicity and weighted shifts, Studia Math. 135(1) (1999), 55-74.

[20] B. Yousefi, H. Rezaei, Some necessary and sufficient conditions for Hypercyclicity Criterion, Proc. India Acad. Sci. (Math. Sci.) 115(2) (2005), 209-216.

SOME IDENTITIES INVOLVING ASSOCIATED SEQUENCES OF SPECIAL POLYNOMIALS

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Abstract. In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of polynomials, we derive some interesting identities of special polynomials.

1. Introduction

For $r \in \mathbb{R}$, the *Bernoulli polynomials* of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \ (r \in \mathbb{R}), \ (\text{see [12,13,14,18,21]}). \tag{1.1}$$

In the special case, $x=0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the n-th Bernoulli numbers of order r. It is also well known that the *Euler polynomials* of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \ (r \in \mathbb{R}), \ (\text{see [9,10,11,19,20]}). \tag{1.2}$$

Let $x=0$. Then $E_n^{(r)}(0) = E_n^{(r)}$ are called the *n-th Euler numbers* of order r.

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \ \middle| \ a_k \in \mathbb{C} \right\}. \tag{1.3}$$

Let $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional L on a polynomial $p(x)$ is defined by $\langle L \mid p(x)\rangle$ and the vector space structure on $\mathbb{P}^*$ is derived by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\,\langle L|p(x)\rangle$, where c is a complex constant.

For $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \ (n \geq 0), \ (\text{see [3,8,17]}). \tag{1.4}$$

By (1.3) and (1.4), we get

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k}, \ (n,k \geq 0), \ (\text{see [4,5,7,10,17,18]}), \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}t^k$. Then, by (1.5), we get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, we see that $f_L(xt) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$

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onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional (see [4, 6, 8, 17, 18]). We call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [4, 10, 17]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t))=1$, then $f(t)$ is called a *delta series* and if $o(f(t))=0$, then $f(t)$ is called an *invertible series*. Let $o(f(t))=1$ and $o(g(t))=0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle = n!\delta_{n,k}$ where $n,k\geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t),f(t))$, which is denoted by $S_n(x)\sim(g(t),f(t))$. If $S_n(x)\sim(1,f(t))$, then $S_n(x)$ is called the *associated sequence* for $f(t)$. By (1.5), we see that $\langle e^{yt}|\, p(x)\rangle = p(y)$.

Let $f(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$. Then we have

$$f(t)=\sum_{k=0}^{\infty}\frac{\left\langle f(t)|x^k\right\rangle}{k!}t^k,\ p(x)=\sum_{k=0}^{\infty}\frac{\left\langle t^k|p(x)\right\rangle}{k!}x^k,\ \text{(see [10,17])}, \tag{1.6}$$

and

$$\langle f_1(t)f_2(t)\cdots f_m(t)|\, x^n\rangle = \sum_{i_1+\cdots+i_m=n}\binom{n}{i_1,\ldots,i_m}\left\langle f_1(t)|\, x^{i_1}\right\rangle\cdots\left\langle f_m(t)|\, x^{i_1}\right\rangle, \tag{1.7}$$

where $f_1(t),f_2(t),\cdots,f_m(t)\in\mathcal{F}$, (see [4, 10, 17]). From (1.6), we have

$$p^{(k)}(0)=\left\langle t^k|p(x)\right\rangle,\ \left\langle 1\left|p^{(k)}(x)\right.\right\rangle = p^{(k)}(0). \tag{1.8}$$

Thus, by (1.8), we get

$$t^kp(x)=p^{(k)}(x)=\frac{d^kp(x)}{dx^k},\ (k\geq 0),\ \text{(see [17])}. \tag{1.9}$$

For $S_n(x)\sim(g(t),f(t))$, we have the following equations:

$$S_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(y)S_{n-k}(x),\ \text{where } p_k(y)=g(t)S_k(y), \tag{1.10}$$

and

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)}=\sum_{k=0}^{\infty}\frac{S_k(y)}{k!}t^k,\ \text{for all } y\in\mathbb{C}\ \text{(see [10,15,16,17,18])}, \tag{1.11}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

Let $p_n(x)\sim(1,f(t))$ and $q_n(x)\sim(1,g(t))$. Then the transfer formula for associated sequence implies that, for $n\in\mathbb{N}$,

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x),\ \text{(see [11,17,22])}. \tag{1.12}$$

Now we introduce several important sequences which are used to derive our results in this paper (see [10, 11, 17]):

(The Poisson-Charlier sequences)

$$C_n(x;a)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a^{-k}(x)_k\sim\left(e^{a(e^t-1)},a(e^t-1)\right),$$

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where $a \neq 0$, $(x)_n = x(x-1)\cdots(x-n+1)$,

$$\sum_{k=0}^{n} C_n(k;a)\frac{t^k}{k!}e^{-t} = \left(\frac{t-a}{a}\right)^n, \ (a \neq 0),\ n \in \mathbb{N} \cup \{0\}, \tag{1.13}$$

(The Abel sequences)

$$A_n(x;b) = x(x-bn)^{n-1} \sim \left(1, te^{bt}\right), \ (b \neq 0), \tag{1.14}$$

(The Mittag-Leffler sequences)

$$M_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n-1)_{n-k}2^k(x)_k \sim \left(1, \frac{e^t-1}{e^t+1}\right), \tag{1.15}$$

(The exponential sequences)

$$\phi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \sim (1, \log(1+t)), \tag{1.16}$$

and

(The Laguerre sequences)

$$L_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1}\frac{n!}{k!}(-x)^k \sim \left(1, \frac{t}{t-1}\right). \tag{1.17}$$

In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of specials polynomials, we derive some interesting identities involving associated sequences of special polynomials.

2. Associated sequences of special polynomials.

As is well known, the Bessel differential equation is given by

$$x^2y^{''} + 2(x+1)y^{'} + n(n+1)y = 0, \ (\text{see [1,2]}). \tag{2.1}$$

From (2.1), we have the solution of (2.1) as follows:

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k, \ (\text{see [1,2]}). \tag{2.2}$$

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1, t - \frac{t^2}{2}\right), \ x^n \sim (1,t), \ (\text{see [1,2,10,17]}). \tag{2.3}$$

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From (1.12) and (2.3), for $n \in \mathbb{N}$, we have

$$\begin{aligned} p_n(x) &= x\left(\frac{t}{t-\frac{t^2}{2}}\right)^n x^{-1}x^n = x\left(1-\frac{t}{2}\right)^{-n} x^{n-1} \\ &= x\sum_{k=0}^{\infty}\binom{-n}{k}(-1)^k\left(\frac{t}{2}\right)^k x^{n-1} \\ &= x\sum_{k=0}^{n-1}\binom{n+k-1}{k}\left(\frac{1}{2}\right)^k (n-1)_k x^{n-1-k} \\ &= \sum_{k=0}^{n-1}\frac{(n+k-1)!}{k!(n-1-k)!}\left(\frac{1}{2}\right)^k x^{n-k} \\ &= \sum_{k=1}^{n}\frac{(2n-k-1)!}{(n-k)!(k-1)!}\left(\frac{1}{2}\right)^{n-k} x^k. \end{aligned} \tag{2.4}$$

By (2.2) and (2.4), we get

$$p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right) \sim \left(1, t-\frac{t^2}{2}\right). \tag{2.5}$$

From (1.11) and (2.5), we can derive the following generating function of $p_n(x)$:

$$\sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!} = e^{x\left(1-(1-2t)^{\frac{1}{2}}\right)}, \tag{2.6}$$

and, by (1.10), we get

$$(x+y)^n y_{n-1}\left(\frac{1}{x+y}\right) = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} y_{k-1}\left(\frac{1}{x}\right) y_{n-k-1}\left(\frac{1}{y}\right). \tag{2.7}$$

By (1.12) and (2.3), we get

$$x^n = x\left(\frac{t-\frac{t^2}{2}}{t}\right)^n x^{-1}p_n(x) = x\left(\frac{t-2}{-2}\right)^n x^{-1}p_n(x). \tag{2.8}$$

Thus, by (1.13), (2.4) and (2.8), we get

$$\begin{aligned} (-1)^n x^{n-1} &= \left(\frac{t-2}{2}\right)^2 x^{-1}p_n(x) = \sum_{k=0}^{\infty} C_n(k;2)\frac{t^k}{k!}e^{-t}\left(x^{-1}p_n(x)\right) \\ &= \sum_{k=0}^{n-1} C_n(k;2)\frac{t^k}{k!}(x-1)^{-1}p_n(x-1) \\ &= \sum_{k=0}^{n-1} C_n(k;2)\frac{t^k}{k!}\sum_{l=1}^{n}\frac{(2n-l-1)!}{(l-1)!(n-l)!}\left(\frac{1}{2}\right)^{n-l}(x-1)^{l-1} \\ &= \sum_{k=0}^{n-1} C_n(k;2)\sum_{l=k+1}^{n}\frac{(2n-l-1)!}{(l-1)!(n-l)!}\binom{l-1}{k}\left(\frac{1}{2}\right)^{n-l}(x-1)^{l-1-k} \\ &= \sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1} C_n(k;2)\binom{m+k}{k}\frac{(2n-m-k-2)!}{(m+k)!(n-m-k-1)!}\left(\frac{1}{2}\right)^{n-m-k-1}(x-1)^m. \end{aligned} \tag{2.9}$$

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From (2.8), we have

$$\begin{aligned} x^{n-1} &= \left(1-\frac{t}{2}\right)^n x^{-1}p_n(x) = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{t}{2}\right)^k x^{-1}p_n(x) \\ &= \sum_{k=0}^{n-1}\sum_{l=k+1}^{n}\binom{n}{k}(l-1)_k\frac{(2n-l-1)!}{(l-1)!(n-l)!}(-1)^k\left(\frac{1}{2}\right)^{n+k-l}x^{l-1-k} \\ &= \sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1}\binom{n}{k}(m+k)_k\frac{(2n-m-k-2)!(-1)^k}{(m+k)!(n-m-1-k)!}\left(\frac{1}{2}\right)^{n-m-1}x^m \\ &= \sum_{m=0}^{n-1}\left\{\sum_{k=0}^{n-m-1}(-1)^k\left(\frac{1}{2}\right)^{n-m-1}\binom{n}{k}\frac{(2n-m-k-2)!}{m!(n-m-k-1)!}\right\}x^m. \end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.1. *For $n\in\mathbb{N}$, we have*

$$(-1)^n x^{n-1} = \sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1}C_n(k;2)\binom{m+k}{k}\frac{(2n-m-k-2)!}{(m+k)!(n-m-k-1)!}\left(\frac{1}{2}\right)^{n-m-k-1}(x-1)^m.$$

Moreover,

$$\sum_{k=0}^{n-m-1}(-1)^k\left(\frac{1}{2}\right)^{n-m-1}\binom{n}{k}\frac{(2n-m-k-2)!}{m!(n-m-k-1)!}=0,$$

where $0\le m\le n-2$.

Let us consider the following associated sequences:

$$p_n(x)\sim\left(1,te^{c(e^t-1)}\right),\ c\neq 0,\ A_n(x;b)=x(x-bn)^{n-1}\sim\left(1,te^{bt}\right),\ b\neq 0. \tag{2.11}$$

By (1.12) and (2.11), we get

$$p_n(x)=x\left(\frac{t}{te^{c(e^t-1)}}\right)^n x^{-1}x^n = x\sum_{k=0}^{\infty}\frac{(-nc)^k}{k!}(e^t-1)^k x^{n-1}. \tag{2.12}$$

We recall that Newton's difference operator Δ is defined by $\Delta f(x)=f(x+1)-f(x)$. For $n\in\mathbb{N}$, we easily see that

$$\Delta^n p(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}p(x+k). \tag{2.13}$$

By (2.13), we get

$$(e^t-1)^k p(x)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}p(x)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{l-k}p(x+l)=\Delta^k p(x). \tag{2.14}$$

In particular, if we take $p(x)=x^{n-1}$, then we have

$$(e^t-1)^k x^{n-1}=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(x+j)^{n-1}. \tag{2.15}$$

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From (2.12) and (2.15), we have

$$p_n(x) = x\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{(-1)^j(nc)^k}{k!}\binom{k}{j}(x+j)^{n-1} \sim \left(1, te^{c(e^t-1)}\right),\ c \neq 0. \tag{2.16}$$

Therefore, by (2.16), we obtain the following lemma.

Lemma 2.2. *For $c \neq 0$ and $n \in \mathbb{N}$, let $p_n(x) \sim \left(1, te^{c(e^t-1)}\right)$. Then we have*

$$p_n(x) = x\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{(-1)^j(nc)^k}{k!}\binom{k}{j}(x+j)^{n-1}.$$

From the definition of Abel sequences and (2.11), we note that

$$\begin{aligned} A_n(x;b) = x(x-bn)^{n-1} &= x\left(\frac{te^{c(e^t-1)}}{te^{bt}}\right)^n x^{-1}p_n(x) \\ &= x\left(e^{-bt-c(1-e^t)}\right)^n x^{-1}p_n(x) = x\left(\sum_{l=0}^{\infty}\frac{a_l^{(-b)}(-c)}{l!}t^l\right)^n x^{-1}p_n(x), \end{aligned} \tag{2.17}$$

where $a_n^{(\beta)}(x) \sim \left((1-t)^{-\beta}, \log(1-t)\right)$ is the *actuarial polynomial* with the generating function given by

$$\sum_{l=0}^{\infty}\frac{a_l^{(\beta)}(x)}{l!}t^l = e^{\beta t + x(1-e^t)}.$$

By Lemma 2.2 and (2.17), we get

$$\begin{aligned} &A_n(x;b) \\ =&x\left\{\sum_{m=0}^{\infty}\sum_{l_1+\cdots+l_n=m}\binom{m}{l_1,\ldots,l_n}a_{l_1}^{(-b)}(-c)\cdots a_{l_n}^{(-b)}(-c)\frac{t^m}{m!}\right\} \\ &\times\left\{\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{(-1)^j(nc)^k}{k!}\binom{k}{j}(x+j)^{n-1}\right\} \\ =&x\sum_{m=0}^{n-1}\sum_{l_1+\cdots+l_n=m}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1}{m}\binom{m}{l_1,\ldots,l_n}\binom{k}{j}\left(\prod_{j=1}^{n}a_{l_j}^{(-b)}(-c)\right)\left(\frac{(-1)^j(nc)^k}{k!}\right)(x+j)^{n-1-m}. \end{aligned} \tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, $b \neq 0$, $c \neq 0$, we have*

$$\begin{aligned} &A_n(x;b) \\ =&x\sum_{m=0}^{n-1}\sum_{l_1+\cdots+l_n=m}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1}{m}\binom{m}{l_1,\ldots,l_n}\binom{k}{j}\left(\prod_{j=1}^{n}a_{l_j}^{(-b)}(-c)\right)\left(\frac{(-1)^j(nc)^k}{k!}\right)(x+j)^{n-1-m}. \end{aligned}$$

For (2.17), we note that

$$\begin{aligned} A_n(x;b) &= x\left(e^{c(e^t-1)}\right)^n e^{-nbt}x^{-1}p_n(x) \\ &= x\left(e^{c(e^t-1)}\right)^n (x-nb)^{-1}p_n(x-nb). \end{aligned} \tag{2.19}$$

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By (1.16) and Lemma 2.2, we easily see that the generating function of exponential sequences is given by

$$\sum_{k=0}^{\infty}\phi_k(x)\frac{t^k}{k!}=e^{x(e^t-1)}. \tag{2.20}$$

From (2.19) and (2.20), we have

$$\begin{aligned}
&A_n(x;b)\\
&=x\left\{\sum_{m=0}^{\infty}\sum_{l_1+\cdots+l_n=m}\binom{m}{l_1,\ldots,l_n}\left(\prod_{j=1}^{n}\phi_{l_j}(c)\right)\frac{t^m}{m!}\right\}(x-nb)^{-1}p_n(x-nb)\\
&=x\sum_{m=0}^{\infty}\sum_{l_1+\cdots+l_n=m}\binom{m}{l_1,\ldots,l_n}\left(\prod_{j=1}^{n}\phi_{l_j}(c)\right)\frac{t^m}{m!}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{(-1)^j(nc)^k}{k!}\binom{k}{j}(x-nb+j)^{n-1}\\
&=x\sum_{m=0}^{n-1}\sum_{l_1+\cdots+l_n=m}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1}{m}\binom{m}{l_1,\ldots,l_n}\binom{k}{j}\left(\prod_{j=1}^{n}\phi_{l_j}(c)\right)\frac{(-1)^j(nc)^k}{k!}(x-nb+j)^{n-m-1}.
\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following corollary.

Corollary 2.4. *For $n\geq 1$, $b\neq 0$, $c\neq 0$, we have*

$$\begin{aligned}
&A_n(x;b)\\
&=x\sum_{m=0}^{n-1}\sum_{l_1+\cdots+l_n=m}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{n-1}{m}\binom{m}{l_1,\ldots,l_n}\binom{k}{j}\left(\prod_{j=1}^{n}\phi_{l_j}(c)\right)\frac{(-1)^j(nc)^k}{k!}(x-nb+j)^{n-m-1}.
\end{aligned}$$

Note that $x^n\sim(1,t)$. By (1.12), (1.13) and (1.17), we get

$$\begin{aligned}
L_n(x)&=x\left(\frac{t}{\frac{t}{t-1}}\right)^n x^{-1}x^n=x(t-1)^n x^{n-1}\\
&=x\left(\sum_{k=0}^{n-1}C_n(k;1)\frac{t^k}{k!}e^{-t}\right)x^{n-1}=x\sum_{k=0}^{n-1}C_n(k;1)\frac{t^k}{k!}(x-1)^{n-1}\\
&=x\sum_{k=0}^{n-1}C_n(k;1)\binom{n-1}{k}(x-1)^{n-1-k}=x\sum_{k=0}^{n-1}\binom{n-1}{k}C_n(n-1-k;1)(x-1)^k.
\end{aligned} \tag{2.22}$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 2.5. *For $n\geq 1$, we have*

$$L_n(x)=x\sum_{k=0}^{n-1}\binom{n-1}{k}C_n(n-1-k;1)(x-1)^k.$$

Mott considered the associated sequences for $f(t)=\frac{-2t}{1-t^2}$. That is, the Mott sequence is given by

$$S_n(x)\sim\left(1,\frac{-2t}{1-t^2}\right). \tag{2.23}$$

From (2.23), we note that the generating function of Mott sequences is given by

$$\sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!} = \exp\left(x\left(\frac{1-\sqrt{1+t^2}}{t}\right)\right).$$

By (1.12), (1.17) and (2.23), we get

$$\begin{aligned}
S_n(x) &= x\left(\frac{\frac{t}{t+1}}{\frac{-2t}{1-t^2}}\right)^n x^{-1}L_n(-x) = x\left(\frac{t-1}{2}\right)^n x^{-1}L_n(-x)\\
&= 2^{-n}x(t-1)^n x^{-1}L_n(-x) = 2^{-n}x\left(\sum_{k=0}^{n-1} C_n(k;1)\frac{t^k}{k!}e^{-t}\right)x^{-1}L_n(-x)\\
&= 2^{-n}x\sum_{k=0}^{n-1} C_n(k;1)\frac{t^k}{k!}(x-1)^{-1}L_n(1-x)\\
&= 2^{-n}x\sum_{k=0}^{n-1} C_n(k;1)\frac{1}{k!}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}t^k(x-1)^{l-1}\\
&= \frac{n!}{2^n}\sum_{k=0}^{n-1}\sum_{l=1}^{n}\binom{n-1}{l-1}\binom{l-1}{k}\frac{C_n(k;1)}{l!}x(x-1)^{l-1-k}.
\end{aligned} \tag{2.24}$$

Thus, by (2.24), we obtain the following lemma.

Lemma 2.6. *For $n \in \mathbb{N}$, let $S_n(x) \sim \left(1, \frac{-2t}{1-t^2}\right)$. Then we have*

$$S_n(x) = \frac{n!}{2^n}\sum_{k=0}^{n-1}\sum_{l=1}^{n}\binom{n-1}{l-1}\binom{l-1}{k}\frac{C_n(k;1)}{l!}x(x-1)^{l-1-k}.$$

As is known, we have

$$xB_{n-1}^{(an)}(x) \sim \left(1, t\left(\frac{e^t-1}{t}\right)^a\right),\ xE_{n-1}^{(bn)}(x) \sim \left(1, t\left(\frac{e^t+1}{2}\right)^b\right), \tag{2.25}$$

where a, b are positive integers (see [10, 11, 17]). For $n \geq 1$, by (1.12) and (2.25), we get

$$\begin{aligned}
xE_{n-1}^{(bn)}(x) &= x\left(\frac{t\left(\frac{e^t-1}{t}\right)^a}{t\left(\frac{e^t+1}{2}\right)^b}\right)^n x^{-1}B_{n-1}^{(an)}(x)\\
&= x\left(\frac{e^t+1}{2}\right)^{-bn}\left(\frac{e^t-1}{t}\right)^{an}B_{n-1}^{(an)}(x).
\end{aligned} \tag{2.26}$$

Thus, by (2.26), we get

$$\left(\frac{e^t+1}{2}\right)^{bn}E_{n-1}^{(bn)}(x) = \left(\frac{e^t-1}{t}\right)^{an}B_{n-1}^{(an)}(x). \tag{2.27}$$

$$\begin{aligned}\text{LHS of (2.27)} &= 2^{-bn}\left(e^t+1\right)^{bn} E_{n-1}^{(bn)}(x) = 2^{-bn}\sum_{k=0}^{bn}\binom{bn}{k}e^{kt}E_{n-1}^{(bn)}(x)\\ &= 2^{-bn}\sum_{k=0}^{bn}\binom{bn}{k}E_{n-1}^{(bn)}(x+k).\end{aligned}\tag{2.28}$$

$$\begin{aligned}\text{RHS of (2.27)} &= \left(\frac{1}{t}\right)^{an}(an)!\sum_{l=an}^{\infty}S_2(l,an)\frac{t^l}{l!}B_{n-1}^{(an)}(x)\\ &= \sum_{l=0}^{n-1}\frac{(an)!}{(l+an)!}S_2(l+an,an)(n-1)_l B_{n-1-l}^{(an)}(x)\\ &= (n-1)!\sum_{l=0}^{n-1}\frac{(an)!}{(l+an)!(n-1-l)!}S_2(l+an,an)B_{n-1-l}^{(an)}(x),\end{aligned}\tag{2.29}$$

where $S_2(n,k)$ is the Stirling number of the second kind. Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.7. *For $n \geq 1$, $a, b \in \mathbb{N}\cup\{0\}$, we have*

$$\sum_{k=0}^{bn}\binom{bn}{k}E_{n-1}^{(bn)}(x+k) = 2^{bn}(n-1)!\sum_{l=0}^{n-1}\frac{(an)!}{(l+an)!(n-1-l)!}S_2(l+an,an)B_{n-1-l}^{(an)}(x).$$

The Pidduck sequences is given by

$$P_n(x) \sim \left(\frac{2}{e^t+1}, \frac{e^t-1}{e^t+1}\right).\tag{2.30}$$

From (2.30), we can derive the generating function of the Pidduck sequences as follows:

$$\sum_{k=0}^{\infty}P_k(x)\frac{t^k}{k!} = (1-t)^{-1}\left(\frac{1+t}{1-t}\right)^x.\tag{2.31}$$

Let $S_n(x) \sim \left(1, \frac{2t}{e^t+1}\right)$. Then, from (1.12), (1.15) and (2.30), we have

$$\begin{aligned}M_n(x) = \frac{2}{e^t+1}P_n(x) &= x\left(\frac{\frac{2t}{e^t+1}}{\frac{e^t-1}{e^t+1}}\right)^n x^{-1}S_n(x)\\ &= 2^n x\left(\frac{t}{e^t-1}\right)^n x^{-1}S_n(x).\end{aligned}\tag{2.32}$$

By (1.12), we easily get

$$\begin{aligned}S_n(x) &= x\left(\frac{\frac{2t}{e^t+1}}{t}\right)^{-n}x^{-1}x^n = 2^{-n}x(e^t+1)^n x^{n-1}\\ &= 2^{-n}x\sum_{j=0}^{n}\binom{n}{j}e^{jt}x^{n-1} = 2^{-n}x\sum_{j=0}^{n}\binom{n}{j}(x+j)^{n-1}.\end{aligned}\tag{2.33}$$

From (2.32) and (2.33), we have

$$\begin{aligned} M_n(x) &= 2^n x \left(\frac{t}{e^t-1}\right)^n \left(2^{-n}\sum_{j=0}^{n}\binom{n}{j}(x+j)^{n-1}\right) \\ &= \sum_{j=0}^{n}\binom{n}{j}x\left(\frac{t}{e^t-1}\right)^n (x+j)^{n-1} = \sum_{j=0}^{n}\binom{n}{j}xB_{n-1}^{(n)}(x+j). \end{aligned} \tag{2.34}$$

By (2.32) and (2.34), we get

$$\begin{aligned} P_n(x) &= \frac{1}{2}(e^t+1)M_n(x) = \frac{1}{2}(e^t+1)\sum_{j=0}^{n}\binom{n}{j}xB_{n-1}^{(n)}(x+j) \\ &= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left\{(x+1)B_{n-1}^{(n)}(x+1+j)+xB_{n-1}^{(n)}(x+j)\right\} \\ &= \frac{1}{2}\left\{\sum_{j=1}^{n}\left(\binom{n}{j-1}(x+1)+\binom{n}{j}x\right)B_{n-1}^{(n)}(x+j)+(x+1)B_{n-1}^{(n)}(x+n+1)+xB_{n-1}^{(n)}(x)\right\} \\ &= \frac{1}{2}\sum_{j=0}^{n+1}\left\{\binom{n+1}{j}x+\binom{n}{j-1}\right\}B_{n-1}^{(n)}(x+j). \end{aligned} \tag{2.35}$$

Therefore, by (2.35), we obtain the following theorem.

Theorem 2.8. *For $n \geq 1$, we have*

$$P_n(x) = \frac{1}{2}\sum_{j=0}^{n+1}\left\{\binom{n+1}{j}x+\binom{n}{j-1}\right\}B_{n-1}^{(n)}(x+j).$$

Let us consider the following two associated sequences:

$$S_n(x) \sim \left(1, \frac{2t}{e^t+1}\right),\ M_n(x) \sim \left(1, \frac{e^t-1}{e^t+1}\right). \tag{2.36}$$

For $n \geq 1$, by (1.12), we get

$$\begin{aligned} S_n(x) &= x\left(\frac{\frac{e^t-1}{e^t+1}}{\frac{2t}{e^t+1}}\right)^n x^{-1}M_n(x) \\ &= 2^{-n}x\left(\frac{e^t-1}{t}\right)^n x^{-1}M_n(x). \end{aligned} \tag{2.37}$$

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By (2.34) and (2.37), we get

$$
\begin{aligned}
S_n(x) &= 2^{-n}x\frac{1}{t^n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}x^{-1}M_n(x)\\
&= 2^{-n}x\sum_{l=0}^{n-1}\frac{n!S_2(l+n,n)}{(l+n)!}t^l(x^{-1}M_n(x))\\
&= 2^{-n}x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)\sum_{j=0}^{n}\binom{n}{j}(n-1)_l B_{n-1-l}^{(n)}(x+j)\\
&= 2^{-n}xn!(n-1)!\sum_{l=0}^{n-1}\sum_{j=0}^{n}\frac{\binom{n}{j}S_2(l+n,n)}{(l+n)!(n-l-1)!}B_{n-1-l}^{(n)}(x+j).
\end{aligned}
\tag{2.38}
$$

Therefore, by (2.33) and (2.36), we obtain the following theorem.

Theorem 2.9. *For $n \geq 1$, we have*

$$\sum_{j=0}^{n}\binom{n}{j}(x+j)^{n-1} = n!(n-1)!\sum_{l=0}^{n-1}\sum_{j=0}^{n}\frac{\binom{n}{j}S_2(l+n,n)}{(l+n)!(n-l-1)!}B_{n-1-l}^{(n)}(x+j).$$

Moreover,

$$\sum_{j=0}^{n}\binom{n}{j}j^{n-1} = n!(n-1)!\sum_{l=0}^{n-1}\sum_{j=0}^{n}\frac{\binom{n}{j}S_2(l+n,n)}{(l+n)!(n-l-1)!}B_{n-1-l}^{(n)}(j).$$

By (1.15), we get

$$x^{-1}M_n(x) = \sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k\sum_{j=0}^{k-1}S_1(k-1,j)(x-1)^j, \tag{2.39}$$

where $S_1(k,j)$ is the Stirling number of the first kind. From (2.38) and (2.39), we can derive

$$
\begin{aligned}
S_n(x) &= 2^{-n}x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)t^l(x^{-1}M_n(x))\\
&= 2^{-n}x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)t^l\sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k\sum_{j=0}^{k-1}S_1(k-1,j)(x-1)^j\\
&= 2^{-n}xn!(n-1)!\sum_{j=0}^{n-1}\sum_{l=0}^{n-1}\sum_{k=j+1}^{n}\frac{\binom{n}{k}2^kS_2(l+n,n)S_1(k-1,j)}{(l+n)!(k-1)!}t^l(x-1)^j\\
&= 2^{-n}xn!(n-1)!\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{k=j+1}^{n}\frac{\binom{n}{k}2^kS_2(l+n,n)S_1(k-1,j)}{(l+n)!(k-1)!}(j)_l(x-1)^{j-l}.
\end{aligned}
\tag{2.40}
$$

Therefore, by (2.33) and (2.40), we obtain the following theorem.

Theorem 2.10. *For $n \geq 1$, we have*

$$\sum_{j=0}^{n}\binom{n}{j}(x+j)^{n-1} = n!(n-1)!\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{k=j+1}^{n}\frac{\binom{n}{k}2^kS_2(l+n,n)S_1(k-1,j)j!}{(l+n)!(k-1)!(j-l)!}(x-1)^{j-l}.$$

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Remark. From (2.34), we note that

$$x^{-1}M_n(x)=\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}. \tag{2.41}$$

By (2.38) and (2.41), we get

$$\begin{aligned}
S_n(x)&=2^{-n}x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)t^l(x^{-1}M_n(x))\\
&=2^{-n}x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)t^l\left(\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}\right)\\
&=2^{-n}xn!\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{k=0}^{n}\frac{\binom{n}{l}S_2(l+n,n)S_1(n-1,j)}{(l+n)!}(j)_l(x+k-1)^{j-l}\\
&=2^{-n}xn!\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{k=0}^{n}\frac{\binom{n}{k}S_2(l+n,n)S_1(n-1,j)j!}{(l+n)!(j-l)!}(x+k-1)^{j-l}.
\end{aligned} \tag{2.42}$$

So, by (2.33) and (2.42), we get

$$\sum_{j=0}^{n}\binom{n}{j}(x+j)^{n-1}=n!\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{k=0}^{n}\frac{\binom{n}{k}S_2(l+n,n)S_1(n-1,j)j!}{(l+n)!(j-l)!}(x+k-1)^{j-l}. \tag{2.43}$$

The Narumi polynomials $N_n^{(a)}(x)$ of order a is defined by the generating function to be

$$\sum_{k=0}^{\infty}\frac{N_k^{(a)}(x)}{k!}t^k=\left(\frac{\log(1+t)}{t}\right)^a(1+t)^x. \tag{2.44}$$

Thus, from (2.44), we see that

$$N_n^{(a)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^a,e^t-1\right),\ \text{(see [17,18])}. \tag{2.45}$$

In the special case, $x=0$, $N_k^{(a)}(0)=N_k^{(a)}$ are called the *k-th Narumi numbers* of order a. If $a=1$ in (2.45), then we will write $N_n(x)$ and N_n for $N_n^{(1)}(x)$ and $N_n^{(1)}$.

By (1.12) and (1.16), we get

$$\begin{aligned}
\phi_n(x)&=x\left(\frac{t}{\log(1+t)}\right)^n x^{-1}x^n=x\left(\frac{t}{\log(1+t)}\right)^n x^{n-1}\\
&=x\left(\sum_{k=0}^{\infty}\frac{N_k^{(-n)}(0)}{k!}t^k\right)x^{n-1}=x\sum_{k=0}^{n-1}\frac{N_k^{(-n)}}{k!}(n-1)_kx^{n-1-k}\\
&=\sum_{k=0}^{n-1}\binom{n-1}{k}N_k^{(-n)}x^{n-k}=\sum_{k=1}^{n}\binom{n-1}{k-1}N_{n-k}^{(-n)}x^k.
\end{aligned} \tag{2.46}$$

Therefore, by (1.16) and (2.46), we obtain the following lemma.

Lemma 2.11. *For $n,k\in\mathbb{N}$ with $k\le n$, we have*

$$S_2(n,k)=\binom{n-1}{k-1}N_{n-k}^{(-n)}.$$

By (1.12), (1.16) and (1.17), we get

$$\begin{aligned}\phi_n(x) &= x\left(\frac{t}{(1+t)\log(1+t)}\right)^n x^{-1}L_n(-x)\\ &= x\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{-n}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}x^{l-1}\\ &= x\left(\sum_{k=0}^{\infty}\frac{N_k^{(-n)}(-n)}{k!}t^k\right)\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}x^{l-1}\\ &= x\sum_{k=0}^{n-1}\frac{N_k^{(-n)}(-n)}{k!}\sum_{l=k+1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}(l-1)_k x^{l-1-k}\\ &= n!\sum_{k=0}^{n-1}\sum_{l=k+1}^{n}\frac{\binom{n-1}{l-1}\binom{l-1}{k}}{l!}N_k^{(-n)}(-n)x^{l-k}\\ &= n!\sum_{k=0}^{n-1}\sum_{m=1}^{n-k}\frac{\binom{n-1}{k+m-1}\binom{k+m-1}{k}}{(k+m)!}N_k^{(-n)}(-n)x^{m}\\ &= n!\sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\frac{\binom{n-1}{k+m-1}\binom{k+m-1}{k}}{(k+m)!}N_k^{(-n)}(-n)\right\}x^{m}.\end{aligned} \tag{2.47}$$

From (1.16) and (2.47), we have

$$S_2(n,m) = n!\sum_{k=0}^{n-m}\frac{\binom{n-1}{k+m-1}\binom{k+m-1}{k}}{(k+m)!}N_k^{(-n)}(-n), \tag{2.48}$$

where $1 \le m \le n$.

Therefore, by Lemma 2.11 and (2.48), we obtain the following theorem.

Theorem 2.12. *For $m,n \in \mathbb{N}$ with $m \le n$, we have*

$$\binom{n-1}{m-1}N_{n-m}^{(-n)} = n!\sum_{k=0}^{n-m}\frac{\binom{n-1}{k+m-1}\binom{k+m-1}{k}}{(k+m)!}N_k^{(-n)}(-n).$$

It is well known that

$$\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty}B_k^{(k-n+1)}(x)\frac{t^k}{k!},\ \text{(see [17])}. \tag{2.49}$$

Thus, by (2.44) and (2.49), we get

$$\sum_{k=0}^{\infty}B_k^{(k-n+1)}(x)\frac{t^k}{k!} = \left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty}N_k^{(-n)}(x-1)\frac{t^k}{k!}. \tag{2.50}$$

By comparing the coefficients on the both sides of (2.50), we get

$$B_k^{(k-n+1)}(x) = N_k^{(-n)}(x-1). \tag{2.51}$$

Therefore, by (2.51), we obtain the following corollary.

Corollary 2.13. *For $m,n \in \mathbb{N}$ with $m \le n$, we have*

$$\binom{n-1}{m-1}B_{n-m}^{(-m+1)}(1) = n!\sum_{k=0}^{n-m}(\frac{\binom{n-1}{k+m-1}\binom{k+m-1}{k}}{(k+m)!}B_k^{(k-n+1)}(-n+1).$$

Let us consider the following associated sequence:

$$S_n(x) \sim (1, t(1+t)^a), \ a \neq 0. \tag{2.52}$$

Then, by (1.12) and (2.52), we get

$$\begin{aligned} S_n(x) &= x\left(\frac{t}{t(1+t)^a}\right)^n x^{-1}x^n = x(1+t)^{-an}x^{n-1} \\ &= x\sum_{l=0}^{n-1}\binom{-an}{l}t^l x^{n-1} = \sum_{l=0}^{n-1}\binom{-an}{l}(n-1)_l x^{n-l} \\ &= \sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}x^l. \end{aligned} \tag{2.53}$$

For $n \geq 1$, from (1.16) and (2.52), we have

$$\begin{aligned} \phi_n(x) &= x\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n x^{-1}S_n(x) \\ &= x\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{an}x^{-1}S_n(x) \\ &= x\left(\sum_{k=0}^{n-1}\frac{N_k^{(-n)}(an)}{k!}t^k\right)x^{-1}S_n(x). \end{aligned} \tag{2.54}$$

By (2.53) and (2.54), we get

$$\begin{aligned} \phi_n(x) &= x\sum_{k=0}^{n-1}\frac{N_k^{(-n)}(an)}{k!}\sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}t^k x^{l-1} \\ &= x\sum_{k=0}^{n-1}\frac{N_k^{(-n)}(an)}{k!}\sum_{l=k+1}^{n}\binom{-an}{n-l}(n-1)_{n-l}(l-1)_k x^{l-1-k} \\ &= (n-1)!\sum_{k=0}^{n-1}\sum_{l=k+1}^{n}\frac{\binom{-an}{n-l}\binom{l-1}{k}}{(l-1)!}N_k^{(-n)}(an)x^{l-k} \\ &= (n-1)!\sum_{k=0}^{n-1}\sum_{m=1}^{n-k}\frac{\binom{-an}{n-k-m}\binom{k+m-1}{k}}{(k+m-1)!}N_k^{(-n)}(an)x^m \\ &= (n-1)!\sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\frac{\binom{-an}{n-k-m}\binom{k+m-1}{k}}{(k+m-1)!}N_k^{(-n)}(an)\right\}x^m \\ &= (n-1)!\sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\frac{\binom{-an}{n-k-m}\binom{k+m-1}{k}}{(k+m-1)!}B_k^{(k-n+1)}(an+1)\right\}x^m. \end{aligned} \tag{2.55}$$

Therefore, by (1.16) and (2.55), we obtain the following theorem.

Theorem 2.14. *For $m, n \in \mathbb{N}$ with $m \leq n$, we have*

$$\binom{n-1}{m-1}B_{n-m}^{(-m+1)}(1) = (n-1)!\sum_{k=0}^{n-m}\frac{\binom{-an}{n-k-m}\binom{k+m-1}{k}}{(k+m-1)!}B_k^{(k-n+1)}(an+1).$$

REMARKS (I). For $n \geq 1$, we have

$$\begin{aligned} x^n &= x\left(\frac{\log(1+t)}{t}\right)^n x^{-1}\phi_n(x) \\ &= \sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}\right\}x^m. \end{aligned} \tag{2.56}$$

By comparing the coefficients on the both sides of (2.56), we get

$$\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)} = \delta_{m,n}, \tag{2.57}$$

where $1 \leq m \leq n$.

(II). For $n \geq 1$, we have

$$\begin{aligned} L_n(-x) &= x\left(\frac{\log(1+t)}{\frac{t}{1+t}}\right)^n x^{-1}\phi_n(x) \\ &= \sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(n)\right\}x^m. \end{aligned} \tag{2.58}$$

By (1.17) and (2.58), we get

$$\binom{n-1}{m-1}\frac{n!}{m!} = \sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(n), \tag{2.59}$$

where $1 \leq m \leq n$.

As is known, the Laguerre polynomials of order α are given by Sheffer sequences to be

$$L_n^{(\alpha)}(x) \sim \left((1-t)^{-\alpha-1}, \frac{t}{t-1}\right). \tag{2.60}$$

Thus, by the definition of Sheffer sequence, we get

$$\left\langle (1+t)^{-\alpha-1}\left(\frac{t}{t+1}\right)^k \middle| L_n^{(\alpha)}(-x)\right\rangle = n!\delta_{n,k} \ (n,k \geq 0). \tag{2.61}$$

By (2.61), we easily see that

$$L_n(-x) = (1+t)^{-\alpha-1}L_n^{(\alpha)}(-x) \sim \left(1, \frac{t}{1+t}\right). \tag{2.62}$$

From (2.58) and (2.62), we have

$$\begin{aligned} &(1+t)^{-\alpha-1}L_n^{(\alpha)}(-x) = L_n(-x) \\ &= \sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(n)\right\}x^m. \end{aligned} \tag{2.63}$$

Thus, by (2.63), we get

$$\begin{aligned} L_n^{(\alpha)}(-x) &= (1+t)^{\alpha+1}\sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(n)\right\}x^m \\ &= \sum_{l=0}^{n}\left\{\sum_{m=l}^{n}\sum_{k=0}^{n-m}\binom{k+m-1}{k}\binom{\alpha+1}{m-l}(m)_{m-l}S_2(n,k+m)N_k^{(n)}(n)\right\}x^l. \end{aligned} \tag{2.64}$$

It is known that

$$L_n^{(\alpha)}(-x) = \sum_{l=0}^{n}\binom{n+\alpha}{n-l}\frac{n!}{l!}x^l. \tag{2.65}$$

By (2.64) and (2.65), we get

$$\binom{n+\alpha}{n-l}\frac{n!}{l!} = \sum_{m=l}^{n}\sum_{k=0}^{n-m}\binom{k+m-1}{k}\binom{\alpha+1}{m-l}(m)_{m-l}S_2(n,k+m)N_k^{(n)}(n),$$

where $0 \le l \le n$.

Finally, we consider the following associated sequences:

$$S_n(x) = \sum_{k=1}^{n}\binom{-an}{n-k}(n-1)_{n-k}x^k \sim \left(1, t(1+t)^a\right),\ a \neq 0. \tag{2.66}$$

Thus, by (1.12) and (2.66), we get

$$\begin{aligned} S_n(x) &= x\left(\frac{\log(1+t)}{t(1+t)^a}\right)^n x^{-1}\phi_n(x) \\ &= x\left(\frac{\log(1+t)}{t}\right)^n (1+t)^{-an}x^{-1}\phi_n(x) \\ &= x\sum_{k=0}^{n-1}\frac{N_k^{(n)}(-an)}{k!}t^k\sum_{l=0}^{n}S_2(n,l)x^{l-1} \\ &= \sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(-an)\right\}x^m. \end{aligned} \tag{2.67}$$

From (2.66) and (2.67), we have

$$\binom{-an}{n-m}(n-1)_{n-m} = \sum_{k=0}^{n-m}\binom{k+m-1}{k}S_2(n,k+m)N_k^{(n)}(-an),$$

where $m, n \in \mathbb{N}$ with $m \le n$ and $a \neq 0$.

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References

[1] S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. 22 (2012), no. 3, 399-406.

[2] S. Araci, M. Acikgoz, *Extended q-Dedekind-type Daehee-Changhee sums associated with Extended q-Euler polynomials*, arXiv:1211.1233

[3] L. Carlitz, *Some generating functions for Laguerre polynomials*, Duke Math. J., **35** (1968) 825-827.

[4] L. Carlitz, *A note on the Bessel polynomials*, Duke Math. J., **24** (1957) 151-162.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 16, NO.4, 2014, COPYRIGHT 2014 EUDOXUS PRESS, LLC

[5] B. Diarra, *Ultrametric umbral calculus in characteristic p*, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007) 845-869.

[6] T. Ernst, *Examples of a q-umbral calculus*, Adv. Stud. Contemp. Math. **16** no. 1, (2008) 1-22.

[7] Q. Fang and T. Wang, *Umbral calculus and invariant sequences*, Ars Combin., **101** (2011), 257-264.

[8] D. S. Kim, T.Kim, S.-H. Lee and S.-H. Rim, *Frobenius-Euler polynomials and umbral calculus in the p-adic case*, Adv. Difference Equ. 2012, **2012**:222.

[9] D. S. Kim and T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, J. of Ineq. and Appl., 2012, **2012**:307.

[10] D. S. Kim and T. Kim, *Applications of Umbral Calculus Associated with p-Adic Invariant Integrals on* $\mathbb{Z}_p$, Abstract and Applied Analysis 2012 (2012), Article ID 865721, 12 pages.

[11] D. S. Kim, T.Kim, S.-H. Lee and S.-H. Rim, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Difference Equ. 2012, **2012**:196.

[12] D. S. Kim, T. Kim, S.-H. Lee and Y.-H. Kim, *Some identities for the product of two Bernoulli and Euler polynomials*, Adv. Difference Equ. 2012, **2012**:95.

[13] T. Kim, S.-H. Rim, D. V. Dolgy and S.-H. Lee, *Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials*, Adv. Difference Equ. 2012, **2012**:201.

[14] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009) no. 4, 484-491.

[15] T. Kim, *On the weighted q-Bernoulli numbers and polynomials*, Adv. Stud. Contemp. Math. **21** (2011), no. 2, 207-215.

[16] A. K. Kwaśniewski, *More on the Bernoulli-Taylor formula for extended umbral calculus*, Adv. Appl. Clifford Algebr., **16** (2006), no. 1, 29-39.

[17] T. J. Robinson, *Formal calculus and umbral calculus*, Electron. J. Combin., **17** (2010), no. 1, Research Paper 95, 31 pp.

[18] S. Roman, *More on the umbral calculus, with emphasis on the q-umbral calculus*, J. Math. Anal. Appl., **107** (1985), 222-254.

[19] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[20] C. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math., **21** (2011), no. 2, 271-223.

[21] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math., **16** (2008), no. 2, 251-278 .

[22] X.-H. Sun, *On umbral calculus*, I, J. Math. Anal. Appl., **224** (2000), no. 2, 279-290.

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SOME NEW INTEGRAL INEQUALITIES OF THE TYPE OF HERMITE-HADAMARD'S FOR THE MAPPINGS WHOSE ABSOLUTE VALUES OF THEIR DERIVATIVE ARE CONVEX

MUHAMMAD IQBAL, MUHAMMAD IQBAL BHATTI, AND MUHAMMAD MUDDASSAR

Abstract. In this paper, We establish various inequalities for some differentiable mappings that are linked with the illustrious Hermite-Hadamard integral inequality for mappings whose absolute values of derivatives are convex. The new integral inequalities are then applied to some special means and numerical integration to get some better estimate than some already presented.

1. Introduction

The role of mathematical inequalities within the mathematical branches as well as in its enormous applications should not be underestimated. The appearance of the new mathematical inequality often puts on firm foundation for the heuristic algorithms and procedures used in applied sciences. Among others one of the main inequality, which gives us an explicit error bounds in the trapezoidal and midpoint rules of a smooth function, called Hermit-Hadamard's inequality defined as [9, p. 53]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : [a, b] \to \mathbf{R}$ is a convex function. Both inequalities hold in the reversed direction for f to be concave. We note that Hermit-Hadamard's inequality (1) may be regarded as a refinement of the concept of convexity and it follows easily from Jensens inequality. Inequality (1) has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found [1, 2, 3, 4, 8] and the references cited therein.

One of the refinements of the celebrated Hermit-Hadamard's inequality (1) is given in [9, Page 55] as follows:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} &\leq \frac{1}{b-a}\int_a^b f(u)du \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{2}$$

where $f : [a, b] \to \mathbf{R}$ is convex function.

M. A. Latif et al. [7] discussed some new estimations regarding 2^{nd} term and 3^{rd} term in (2) for differentiable functions whose absolute values are convex.

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In this paper, some new Hermite-Hadamard type inequalities involving differentiable functions whose absolute values are convex and concave. Our established results provides estimates of left Hermite-Hadamard inequality (1) as obtained in [3, 5, 6] and regarding 2^{nd} and 3^{rd} term of inequality (2) as obtained in [7] at end-points and mid-point of interval [a, b], respectively.

This work is organized in the following way. After this Introduction, in Section 2 main results are given. In Section 3 some applications for some special means are provided. In the last Section 4, error is estimated for the generalized quadrature formula.

2. Main Results

In order to prove our main theorems, we first prove the following lemma:

Lemma 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on I°,the interior of I where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then*

$$\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du =$$
$$\frac{(x-a)^2}{4(b-a)}\left\{\int_0^1 (1-t)f'\left(ta+(1-t)\frac{a+x}{2}\right)dt + \int_0^1 (t-1)f'\left(tx+(1-t)\frac{a+x}{2}\right)dt\right\}+$$
$$\frac{(b-x)^2}{4(b-a)}\left\{\int_0^1 (1-t)f'\left(tx+(1-t)\frac{b+x}{2}\right)dt + \int_0^1 (t-1)f'\left(tb+(1-t)\frac{b+x}{2}\right)dt\right\},$$

for all $x \in [a, b]$.

Proof. Integrating by parts and making use of the substitution $u = ta + (1-t)\frac{a+x}{2}$, we have

$$\frac{(x-a)^2}{4(b-a)}\int_0^1 (1-t)f'\left(ta+(1-t)\frac{a+x}{2}\right)dt$$
$$= \frac{(x-a)^2}{4(b-a)}\left\{\frac{2(1-t)f\left(ta+(1-t)\frac{a+x}{2}\right)}{a-x}\Bigg|_0^1 - \frac{2}{a-x}\int_0^1 (-1)f\left(ta+(1-t)\frac{a+x}{2}\right)dt\right\}$$
$$= \frac{x-a}{2(b-a)} f\left(\frac{a+x}{2}\right) - \frac{1}{b-a}\int_a^{\frac{a+x}{2}} f(u)du.$$

Analogously:

$$\frac{(x-a)^2}{4(b-a)}\int_0^1 (t-1)f'\left(tx+(1-t)\frac{a+x}{2}\right)dt = \frac{x-a}{2(b-a)} f\left(\frac{a+x}{2}\right) - \frac{1}{b-a}\int_{\frac{a+x}{2}}^{x} f(u)du,$$
$$\frac{(b-x)^2}{4(b-a)}\int_0^1 (1-t)f'\left(tx+(1-t)\frac{b+x}{2}\right)dt = \frac{b-x}{2(b-a)} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_x^{\frac{b+x}{2}} f(u)du,$$

and

$$\frac{(b-x)^2}{4(b-a)}\int_0^1 (t-1)f'\left(tb+(1-t)\frac{b+x}{2}\right)dt = \frac{b-x}{2(b-a)} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_{\frac{b+x}{2}}^{b} f(u)du.$$

Adding above equalities, we get the desired equality. This completes the proof of the lemma. □

Theorem 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\frac{(x-a)^2}{b-a}\left[\frac{|f'(a)|+4|f'\left(\frac{a+x}{2}\right)|+|f'(x)|}{24}\right]+\frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+4|f'\left(\frac{b+x}{2}\right)|+|f'(b)|}{24}\right],$$

for each $x \in [a,b]$.

Proof. By using the convexity of $|f'|$, the properties of modulus on lemma 1, we have

$$\left|\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{4(b-a)}\int_0^1 (1-t)\left\{\left|f'\left(ta+(1-t)\frac{a+x}{2}\right)\right| + \left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|\right\} dt$$
$$+ \frac{(b-x)^2}{4(b-a)}\int_0^1 (1-t)\left\{\left|f'\left(tx+(1-t)\frac{b+x}{2}\right)\right| + \left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|\right\} dt \tag{3}$$
$$\leq \frac{(x-a)^2}{4(b-a)}\int_0^1 (1-t)\left\{t|f'(a)| + 2(1-t)\left|f'\left(\frac{a+x}{2}\right)\right| + t|f'(x)|\right\} dt$$
$$+ \frac{(b-x)^2}{4(b-a)}\int_0^1 (1-t)\left\{t|f'(x)| + 2(1-t)\left|f'\left(\frac{b+x}{2}\right)\right| + t|f'(b)|\right\} dt$$
$$= \frac{(x-a)^2}{b-a}\left[\frac{|f'(a)|+4\left|f'\left(\frac{a+x}{2}\right)\right|+|f'(x)|}{24}\right]+\frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+4|f'\left(\frac{b+x}{2}\right)|+|f'(b)|}{24}\right],$$

which completes the proof. ∎

Corollary 1. *Under the conditions of theorem 1, the followings hold:*

$$\left|\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2\left[|f'(a)| + |f'(x)|\right] + (b-x)^2\left[|f'(x)| + |f'(b)|\right]}{8(b-a)}, \tag{4}$$

for each $x \in [a,b]$.

Remark 1. *By setting $x = a$ (or $x = b$), inequality* (4) *reduces to* [5, Theorem 2.2].

Remark 2. *By setting $x = (a+b)/2$, theorem 1 reduces to* [7, Theorem 1]. *Moreover inequality* (4) *reduces to:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{32}\left[|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right], \tag{5}$$

which gives sharp bound than as in [7, Corollary 1].

Theorem 2. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then*

$$\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{2}{p+1}\right)^{1/p} \times$$
$$\left[\frac{(x-a)^2}{8(b-a)}\left\{\left(|f'(a)|^q+\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(x)|^q+\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}\right\}+\right.$$
$$\left.\frac{(b-x)^2}{8(b-a)}\left\{\left(|f'(x)|^q+\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(b)|^q+\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}\right\}\right],$$

for each $x \in [a,b]$.

Proof. Using the well-known Hölder integral inequality in (3), we have

$$\begin{aligned}
&\left|\frac{x-a}{b-a}\, f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(ta+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tx+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q}.
\end{aligned}$$

By convexity of $|f'|^{\frac{p}{p-1}}$ and the Hermite-Hadamard's inequality, we have

$$\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{2}{p+1}\right)^{1/p} \times$$
$$\left[\frac{(x-a)^2}{8(b-a)}\left\{\left(|f'(a)|^q+\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(x)|^q+\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}\right\}+\right.$$
$$\left.\frac{(b-x)^2}{8(b-a)}\left\{\left(|f'(x)|^q+\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(b)|^q+\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}\right\}\right],$$

which completes the proof. □

Corollary 2. *Under the conditions of theorem 2, the followings hold:*

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{\left[(x-a)^2(|f'(a)|+|f'(x)|)+(b-x)^2(|f'(x)|+|f'(b)|)\right]}{2^{2(p-1)/p}(b-a)(p+1)^{1/p}},
\end{aligned} \tag{6}$$

for each $x \in [a,b]$.

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Proof. The proof follows from theorem 2, simply applying convexity on factors $\left|f'\left(\frac{a+x}{2}\right)\right|^q$ and $\left|f'\left(\frac{b+x}{2}\right)\right|^q$ and the fact

$$\sum_{k=1}^{n}(u_k+v_k)^s \le \sum_{k=1}^{n}(u_k)^s + \sum_{k=1}^{n}(v_k)^s, \quad u_k, v_k \ge 0; 1 \le k \le n; 0 \le s < 1.$$

□

Remark 3. *By setting $x = a$ (or $x = b$), inequality* (6) *reduces to* [5, Theorem 2.4].

Remark 4. *By setting $x = (a+b)/2$, theorem 2 reduces to* [7, Theorem 2].

Theorem 3. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then*

$$\left|\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{1}{3}\right)^{1/q} \times$$
$$\left[\frac{(x-a)^2}{8(b-a)}\left\{\left(|f'(a)|^q + 2\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q} + \left(|f'(x)|^q + 2\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}\right\} + \right.$$
$$\left.\frac{(b-x)^2}{8(b-a)}\left\{\left(|f'(x)|^q + 2\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q} + \left(|f'(b)|^q + 2\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}\right\}\right],$$

for each $x \in [a,b]$.

Proof. Using the well-known Power-mean integral inequality for $q \ge 1$ in (3), we have

$$\begin{aligned}
&\left|\frac{x-a}{b-a}\, f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(ta + (1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+ \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(tx + (1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+ \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(tx + (1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+ \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(tb + (1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q}.
\end{aligned}$$

By convexity of $|f'|^q$

$$\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+x}{2}\right)\right|^q dt \le \frac{|f'(a)|^q + 2\left|f'\left(\frac{a+x}{2}\right)\right|^q}{6},\\
&\int_0^1 (1-t)\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt \le \frac{|f'(x)|^q + 2\left|f'\left(\frac{a+x}{2}\right)\right|^q}{6},\\
&\int_0^1 (1-t)\left|f'\left(tx+(1-t)\frac{b+x}{2}\right)\right|^q dt \le \frac{|f'(x)|^q + 2\left|f'\left(\frac{b+x}{2}\right)\right|^q}{6},\\
&\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt \le \frac{|f'(b)|^q + 2\left|f'\left(\frac{b+x}{2}\right)\right|^q}{6}.
\end{aligned}$$

Combining all the obtained inequalities, we get

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{1}{3}\right)^{1/q}\times\\
&\left[\frac{(x-a)^2}{8(b-a)}\left\{\left(|f'(a)|^q+2\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(x)|^q+2\left|f'\left(\frac{a+x}{2}\right)\right|^q\right)^{1/q}\right\}+\right.\\
&\left.\frac{(b-x)^2}{8(b-a)}\left\{\left(|f'(x)|^q+2\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}+\left(|f'(b)|^q+2\left|f'\left(\frac{b+x}{2}\right)\right|^q\right)^{1/q}\right\}\right],
\end{aligned}$$

which completes the proof. □

Corollary 3. *Under the conditions of theorem 3, the followings hold:*

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\le \frac{3^{(q-1)/q}[(x-a)^2(|f'(a)|+|f'(x)|)+(b-x)^2(|f'(b)|+|f'(x)|)]}{8(b-a)}, \qquad (7)
\end{aligned}$$

for each $x \in [a,b]$.

Remark 5. *By setting* $x = a$ (*or* $x = b$), *inequality* (7) *reduces to* [6, Theorem 2.1].

Remark 6. *By setting* $x = \frac{a+b}{2}$, *theorem 3 reduces to* [7, Theorem 3].

Theorem 4. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is concave on* $[a,b]$ *for some fixed* $q > 1$, *then*

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{1}{p+1}\right)^{1/p}\left[\frac{(x-a)^2}{4(b-a)}\right.\\
&\left.\left\{\left|f'\left(\frac{3a+x}{4}\right)\right|+\left|f'\left(\frac{a+3x}{4}\right)\right|\right\}+\frac{(b-x)^2}{4(b-a)}\left\{\left|f'\left(\frac{b+3x}{4}\right)\right|+\left|f'\left(\frac{3b+x}{4}\right)\right|\right\}\right],
\end{aligned}$$

for each $x \in [a,b]$.

Proof. Using the well-known Hölder integral inequality for $q > 1$ in inequality (3), we have

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq \frac{(x-a)^2}{4(b-a)}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(ta+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(x-a)^2}{4(b-a)}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tx+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q} \\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1(1-t)^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{1/q}.
\end{aligned}$$

By concavity of $|f'|^q$ and using the Hermite-Hadamard inequality (1), we have

$$\begin{aligned}
\int_0^1\left|f'\left(ta+(1-t)\frac{a+x}{2}\right)\right|^q dt &\leq \left|f'\left(\frac{3a+x}{4}\right)\right|^q, \\
\int_0^1\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt &\leq \left|f'\left(\frac{a+3x}{4}\right)\right|^q, \\
\int_0^1\left|f'\left(tx+(1-t)\frac{b+x}{2}\right)\right|^q dt &\leq \left|f'\left(\frac{b+3x}{4}\right)\right|^q, \\
\int_0^1\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt &\leq \left|f'\left(\frac{3b+x}{4}\right)\right|^q.
\end{aligned}$$

Combining all the above inequalities gives the desired result. □

Remark 7. *By setting $x = a$ (or $x = b$), theorem 4 reduces to* [3, Theorem 5].

Remark 8. *By setting $x = (a+b)/2$ theorem 4 reduces to* [7, Theorem 4].

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|$ is linear map, then*

$$\begin{aligned}
&\left|\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}\ f\left(\frac{b+x}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\qquad\qquad \leq \frac{(x-a)^2\left|f'\left(a+x\right)\right|+(b-x)^2\left|f'\left(b+x\right)\right|}{8(b-a)},
\end{aligned}$$

for each $x \in [a,b]$.

Proof. First, we note that by the concavity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we note that

$$\begin{aligned}
|f'(tx+(1-t)y)|^q &\geq t|f'(x)|^q+(1-t)|f'(y)|^q \\
&\geq (t|f'(x)|+(1-t)|f'(y)|)^q
\end{aligned}$$

and hence

$$|f'(tx+(1-t)y)| \geq t|f'(x)|+(1-t)|f'(y)|$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. This shows that $|f'|$ is also concave on $[a,b]$. Now, using the Jensen's integral inequality in inequality (3), we have

$$\left|\frac{x-a}{b-a} f\left(\frac{a+x}{2}\right) + \frac{b-x}{b-a} f\left(\frac{b+x}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \left(\int_0^1 (1-t)dt\right)\left[\frac{(x-a)^2}{4(b-a)}\left\{\left|f'\left(\frac{(1-t)\left(ta+(1-t)\frac{a+x}{2}\right)}{\int_0^1 (1-t)dt}\right)\right| + \right.\right.$$
$$\left.\left|f'\left(\frac{(1-t)\left(tx+(1-t)\frac{a+x}{2}\right)}{\int_0^1 (1-t)dt}\right)\right|\right\} + \frac{(b-x)^2}{4(b-a)}\left\{\left|f'\left(\frac{(1-t)\left(tx+(1-t)\frac{b+x}{2}\right)}{\int_0^1 (1-t)dt}\right)\right|\right.$$
$$\left.\left. + \left|f'\left(\frac{(1-t)\left(tb+(1-t)\frac{b+x}{2}\right)}{\int_0^1 (1-t)dt}\right)\right|\right\}\right]$$

$$\leq \frac{(x-a)^2}{8(b-a)}\left\{\left|f'\left(\frac{2a+x}{3}\right)\right| + \left|f'\left(\frac{a+2x}{3}\right)\right|\right\} +$$
$$\frac{(b-x)^2}{8(b-a)}\left\{\left|f'\left(\frac{2x+b}{3}\right)\right| + \left|f'\left(\frac{x+2b}{3}\right)\right|\right\},$$

which completes the proof. □

Remark 9. *By setting $x = a$ (or $x = b$), theorem 5 reduces to* [6, Theorem 2.2].

Remark 10. *By setting $x = (a+b)/2$ in theorem 5, the followings hold*

$$\left|\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{16}|f'(a+b)|, \tag{8}$$

which gives sharp bound than as was obtained in [7, Corollary 5].

3. Applications to Some Special Means

We now consider the applications to the following special means.
The arithmetic mean

$$A(a,b) = \frac{a+b}{2}, \qquad a, b \in \mathbf{R}$$

The harmonic mean

$$H(a,b) = \frac{2ab}{a+b}, \qquad a, b \in \mathbf{R}\backslash\{0\}$$

The logarithmic mean

$$L(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{b-a}{lnb - lna} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

Generalized logarithmic mean

$$L_n(a,b) = \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & \text{if } a \neq b \end{cases}, n \in \mathbf{Z}\backslash\{-1,0\}; \quad a, b > 0$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 1. *Let $a, b \in \mathbf{R}, a < b, 0 \notin [a, b]$ and $n \in \mathbf{Z}, |n| \geq 2$, then*

$$\left| A\left(\left(\frac{3a+b}{4}\right)^n, \left(\frac{a+3b}{4}\right)^n\right) - L_n^n(a,b)\right| \leq |n| \left(\frac{b-a}{16}\right) \left[\; A\left(|a|^{n-1}, |b|^{n-1}\right) + A^{n-1}(a,b)\right]$$

Proof. Follows by inequality (5), setting $f(x) = x^n, x \in \mathbf{R}, n \in \mathbf{Z}$. ∎

Proposition 2. *Let $a, b \in \mathbf{R}$, $a < b$, $0 \notin [a, b]$. Then for all $q \geq 1$*

$$\left| H^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) - L(a,b)\right| \leq \left(\frac{3^{1-(1/q)}}{8}\right)(b-a) A\left(|a|^{-2}, |b|^{-2}\right).$$

Proof. Follows by corollary 3 with $x = (a+b)/2$, setting $f(x) = \frac{1}{x}$. ∎

4. The quadrature formula

Let $d : a = x_0 < x_1 < x_2 < ... < x_n = b$ be a division of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = Q(f,d) + E(f,d) \tag{9}$$

where

$$Q(f,d) = \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{3x_i + x_{i+1}}{4}\right)\right](x_{i+1} - x_i)$$

and $E(f, d)$ denotes the approximation error. Here, we derive some error estimates for quadrature formula (9).

Proposition 3. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $|f'|$ is convex on $[a, b]$, then in (9), for every division d of $[a, b]$, we have*

$$|E(f,d)| \leq \frac{1}{32}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^2 \left[\; |f'(x_i)| + 2\left|f'\left(\frac{x_i + x_{i+1}}{2}\right) + |f'(x_{i+1})|\right| \;\right]$$

Proof. On applying inequality (5) on the subinterval $[x_i, x_{i+1}](i = 0, 1, 2, ..., n-1)$ of the division d, we have

$$\left|\frac{f'\left(\frac{3x_i+x_{i+1}}{4}\right) + f'\left(\frac{x_i+3x_{i+1}}{4}\right)}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{(x_{i+1}-x_i)}{32}\left[|f'(x_i)| + 2\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right| + |f'(x_{i+1}|\right].$$

Now

$$|E(f,d)| = \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f'\left(\frac{3x_i+x_{i+1}}{4}\right) + f'\left(\frac{x_i+3x_{i+1}}{4}\right)}{2}(x_{i+1} - x_i)\right\}\right|$$

$$\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f'\left(\frac{3x_i+x_{i+1}}{4}\right)+f'\left(\frac{x_i+3x_{i+1}}{4}\right)}{2}\right|$$

$$\leq \frac{1}{32}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\ |f'(x_i)|+2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+|f'(x_{i+1})|\ \right]$$

which completes the proof of the proposition. □

Proposition 4. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable function on I^o such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$ and $|f'|^{p/p-1}$ is convex on $[a,b]$, where $p > 1$. Then in* (9)*, for every division d of $[a,b]$, we have*

$$|E(f,d)| \leq \left(\frac{4}{p+1}\right)^{1/p}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{16}\ [\ |f'(x_i)|+ \\ 2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+|f'(x_{i+1})|\ ]$$

Proof. The proof is similar to that of proposition 3, applying the subinterval $[x_i, x_{i+1}](i = 0,1,2,...,n-1)$ of the division d, with $x = (x_i + x_{i+1})/2$ on corollary 2. □

Proposition 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q \geq 1$ and $|f'|$ is linear map, then for every division d of $[a,b]$, then the following inequality holds:*

$$|E(f,d)| \leq \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{16}\,|f'(x_i+x_{i+1})|$$

Proof. The proof is similar to that of proposition 3 and using inequality (8). □

References

[1] M. Avci, H. Kavurmaci, and M. E. Özdemir, New inequalities Of Hermite-Hadamard type via $s-$convex functions in the second sense, Appl. Math. Comput., 217(2011) 5171−5176.

[2] S.S. Dragomir and C.E.M. Pearce, Selected topics on Hermite−Hadamard inequalities and applications, RGMIA Monographs, Victoria University, 2000.
Online: http://www.sta.vu.edu.au/RGMIA/monographs/hermite_hadamard.html.

[3] S. Hussain, M.I. Bhatti and M. Iqbal, Hadamard-type inequalities for s-convex functions I, Punjab Univ. Jour. Math., 41 (2009) 51-60.

[4] H. Kavurmaci, M. Avci, and M. E. Özdemir, New inequalities Of Hermite-Hadamard type for convex functions with applications, Jour. Ineq. Appl.,(2011).

[5] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Applied Mathematics and Computation 147 (2004) 137 − 146.

[6] U. S. Kirmaci and M.E. Özdemir, IOn some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Applied Mathematics and Computation 153 (2004) 361 − 368.

[7] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications to special means and to general quadrature formula, RGMIA Research Report Collection, 14(2011) Preprint. Available Online: http://ajmaa.org/RGMIA/papers/v14/v14a103.pdf.

[8] C.E.M. Pearce and J.E. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett., 13 (2000) 51 − 55.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 16, NO.4, 2014, COPYRIGHT 2014 EUDOXUS PRESS, LLC

[9] C.Niculescu and L. E. Persson, Convex functions and their applications, Springer, Berlin Heidelberg NewYork, (2004).
On-line: http://web.cs.dal.ca/ jborwein/Preprints/Books/CUP/CUPold/np-convex.pdf.

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A GENERALIZED ADDITIVE FUNCTIONAL INEQUALITY IN BANACH SPACES

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the following function inequality

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| \quad (1 < |a + b + c|)$$

in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \le \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Th.M. Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

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for all $x \in E$. *If* $p < 0$ *then inequality* (1.1) *holds for all* $x, y \neq 0$, *and* (1.2) *for* $x \neq 0$. *Also, if the function* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *into* E' *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E$, *then* T *is* $\mathbb{R}$-*linear.*

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. On the other hand, J.M. Rassias [5] generalized the Hyers-Ulam stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.2. ([6, 7]) *If it is assumed that there exist constants* $\Theta \geq 0$ *and* $p_1, p_2 \in \mathbb{R}$ *such that* $p = p_1 + p_2 \neq 1$, *and* $f : E \to E'$ *is a mapping from a norm space* E *into a Banach space* E' *such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \Theta \|x\|^{p_1} \|y\|^{p_2}$$

for all $x, y \in E$, *then there exists a unique additive mapping* $T : E \to E'$ *such that*

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p} \|x\|^p,$$

for all $x \in E$. *If, in addition,* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E$, *then* T *is* $\mathbb{R}$-*linear*

More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [8]–[22].

In [23], Park et al. investigated the following inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\| 2f\left(\frac{x+y+z}{2}\right) \right\|,$$

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|,$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\|$$

in Banach spaces. Recently, Cho et al. [24] investigated the following functional inequality

$$\|f(x) + f(y) + f(z) \leq \left\| Kf\left(\frac{x+y+z}{K}\right) \right\| \qquad (0 < |K| < |3|)$$

in non-Archimedean Banach spaces. Lu and Park [25] investigated the following functional inequality

$$\left\| \sum_{i=1}^{N} f(x_i) \right\| \leq \left\| Kf\left(\frac{\sum_{i=1}^{N}(x_i)}{K}\right) \right\| \qquad (0 < |K| \leq N)$$

in Fréchet spaces.

In [26], Lu and Park investigated the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\| Kf\left(\frac{x+y+z}{K}\right) \right\| \qquad (0 < |K| < 3), \tag{1.3}$$

$$\|f(x) + f(y) + Kf(z)\| \leq \left\| Kf\left(\frac{x+y}{K} + z\right) \right\| \qquad (0 < K \neq 2) \tag{1.4}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces.

Li et al. [27] considered the following functional inequalities

$$\|af(x)+bf(y)+cf(z)\| \leq \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \quad (0<|K|<|a+b+c|), \tag{1.5}$$

$$\|af(x)+bf(y)+Kf(z)\| \leq \left\|Kf\left(\frac{ax+by}{K}+z\right)\right\| \quad (0<K<|a+b+K|), \tag{1.6}$$

where a,b,c are nonzero real numbers, in quasi-Banach spaces.

In this paper, we consider the following functional inequality

$$\|af(x)+bf(y)+cf(z)\| \leq \|f(\alpha x+\beta y+\gamma z)\|, \quad (1<|a+b+c|), \tag{1.7}$$

where a,b,c and α,β,γ are nonzero real number, and prove the Hyers-Ulam stability of the functional inequality (1.7) in Banach spaces.

2. Hyers-Ulam stability of the functional inequality (1.7)

Throughout this section, assume that X is a normed space and that Y is a Banach space.

Proposition 2.1. *Let $f: X \to Y$ be a mapping such that*

$$\|af(x)+bf(y)+cf(z)\| \leq \|f(\alpha x+\beta y+\gamma z)\| \tag{2.1}$$

for all $x,y,z \in X$. Then the mapping $f: X \to Y$ is additive.

Proof. Letting $x=y=z=0$ in (2.1), we get

$$\|(a+b+c)f(0)\| \leq \|f(0)\|.$$

So $f(0)=0$.

Letting $z=0$ and $y=-\frac{\alpha}{\beta}x$ in (2.1), we get

$$\left\|af(x)+bf\left(-\frac{\alpha}{\beta}x\right)\right\| \leq \|f(0)\|=0$$

for all $x \in X$. So $f(x)=-\frac{b}{a}f(-\frac{\alpha}{\beta}x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y=0$ and $z=\frac{\alpha}{\gamma}x$ in (2.1), we get

$$\left\|af(-x)+cf\left(\frac{\alpha}{\gamma}x\right)\right\| \leq \|f(0)\|=0$$

for all $x \in X$. So $f(-x)=-\frac{c}{a}f(\frac{\alpha}{\gamma}x)$ for all $x \in X$. Then we get

$$\begin{aligned}\|f(x)+f(-x)\| &= \left\|-\frac{b}{a}f\left(-\frac{\alpha}{\beta}x\right)-\frac{c}{a}f\left(\frac{\alpha}{\gamma}x\right)\right\| \\ &= \frac{1}{|a|}\left\|af(0)+bf\left(-\frac{\alpha}{\beta}x\right)+cf\left(\frac{\alpha}{\gamma}x\right)\right\| \\ &\leq \frac{1}{|a|}\left\|f\left(\alpha\cdot 0-\beta\frac{\alpha}{\beta}x+\gamma\frac{\alpha}{\gamma}x\right)\right\|=0\end{aligned}$$

and so $f(-x) = -f(x)$ for all $x \in X$.

$$\begin{aligned}\|f(x)+f(y)-f(x+y)\| &= \|f(x)+f(y)+f(-x-y)\| \\ &= \left\|-\frac{a}{a}f(-\frac{\alpha}{\alpha}x) - \frac{b}{a}f(-\frac{\alpha}{\beta}y) - \frac{c}{a}f(\frac{\alpha x+\alpha y}{\gamma})\right\| \\ &= \frac{1}{|a|}\left\|af(-\frac{\alpha}{\alpha}x) + bf(-\frac{\alpha}{\beta}y) + cf(\frac{\alpha x+\alpha y}{\gamma})\right\| \\ &= \frac{1}{|a|}\left\|f\left(\alpha\cdot(-\frac{\alpha}{\alpha}x) + \beta\cdot(-\frac{\alpha}{\beta}x) + \gamma\cdot\frac{\alpha(x+y)}{\gamma}\right)\right\| = 0\end{aligned}$$

for all $x, y \in X$. Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

Theorem 2.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| + \phi(x,y,z), \tag{2.2}$$

where $\phi : X^3 \to [0,\infty)$ satisfies $\phi(0,0,0) = 0$ and

$$\widetilde{\phi}(x,y,z) := \sum_{j=0}^{\infty}\left(\frac{c}{a}\right)^j \phi\left(\left(\frac{\alpha}{\gamma}\right)^j x, \left(\frac{\alpha}{\gamma}\right)^j y, \left(\frac{\alpha}{\gamma}\right)^j z\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|a|}\left[\widetilde{\phi}\left(x, -\frac{\alpha}{\beta}x, 0\right) + \widetilde{\phi}\left(0, -\frac{\alpha}{\beta}x, \frac{\alpha}{\gamma}x\right)\right] \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.2), we get $\|(a+b+c)f(0)\| \le \|f(0)\| + \phi(0,0,0) = \|f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ and $z = -\frac{\alpha}{\gamma}x$ in (2.2), we get

$$\left\|af(x) + cf\left(-\frac{\alpha}{\gamma}x\right)\right\| \le \phi\left(x, 0, -\frac{\alpha}{\gamma}x\right)$$

for all $x \in X$. So $\left\|f(x) + \frac{c}{a}f(-\frac{\alpha}{\gamma}x)\right\| \le \frac{1}{|a|}\phi\left(x, 0, -\frac{\alpha}{\gamma}x\right)$ for all $x \in X$.

Letting $y = -\frac{\alpha}{\beta}x$ and $z = 0$ in (2.2), we obtain

$$\left\|f(x) + \frac{b}{a}f\left(-\frac{\alpha}{\beta}x\right)\right\| \le \frac{1}{|a|}\phi\left(x, -\frac{\alpha}{\beta}x, 0\right)$$

for all $x \in X$. So

$$\begin{aligned}\left\|f(x) - \frac{c}{a}f\left(\frac{\alpha}{\gamma}x\right)\right\| &= \left\|f(x) + \frac{b}{a}f\left(-\frac{\alpha x}{\beta}\right) - \frac{b}{a}f\left(-\frac{\alpha x}{\beta}\right) - \frac{c}{a}f\left(\frac{\alpha}{\gamma}x\right)\right\| \\ &\le \left(\left\|f(x) + \frac{b}{a}f\left(-\frac{\alpha x}{\beta}\right)\right\| + \left\|\frac{b}{a}f\left(-\frac{\alpha x}{\beta}\right) + \frac{c}{a}f\left(\frac{\alpha}{\gamma}x\right)\right\|\right) \qquad (2.4) \\ &\le \frac{1}{|a|}\left[\phi\left(x, -\frac{\alpha x}{\beta}, 0\right) + \phi\left(0, -\frac{\alpha x}{\beta}, \frac{\alpha x}{\gamma}\right)\right]\end{aligned}$$

for all $x \in X$.

It follows from (2.4) that

$$\begin{aligned}
&\left\|(\frac{c}{a})^l f\left((\frac{\alpha}{\gamma})^l x\right) - (\frac{c}{a})^m f\left((\frac{\alpha}{\gamma})^m x\right)\right\| \\
&\leq \sum_{j=l}^{m-1} \left\|(\frac{c}{a})^j f\left((\frac{\alpha}{\gamma})^j x\right) - (\frac{c}{a})^{j+1} f\left((\frac{\alpha}{\gamma})^{j+1} x\right)\right\| \\
&\leq \frac{1}{|a|}\sum_{j=l}^{m-1}\left(\frac{c}{a}\right)^j \left[\phi\left(\left(\frac{\alpha}{\gamma}\right)^j x, -\frac{\alpha}{\beta}\left(\frac{\alpha}{\gamma}\right)^j x, 0\right) + \phi\left(0, -\frac{\alpha}{\beta}\left(\frac{\alpha}{\gamma}\right)^j x, \left(\frac{\alpha}{\gamma}\right)^{j+1} x\right)\right]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{c}{a})^n f((\frac{\alpha}{\gamma})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{c}{a})^n f((\frac{\alpha}{\gamma})^n x)\}$ converges. We define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}\{(\frac{c}{a})^n f((\frac{\alpha}{\gamma})^n x)\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.3).

Next, we show that $A : X \to Y$ is an additive mapping.

$$\begin{aligned}
\|A(x) + A(-x)\| &= \lim_{n\to\infty}(\frac{c}{a})^n \left\|f\left(\frac{\alpha^n x}{\gamma^n}\right) + f\left(\frac{-\alpha^n x}{\gamma^n}\right)\right\| \\
&\leq \lim_{n\to\infty}(\frac{c}{a})^n \left[\left\|f\left(\frac{\alpha^n x}{\gamma^n}\right) + \frac{b}{a} f\left(-\frac{\alpha}{\beta}\cdot\frac{\alpha^n x}{\gamma^n}\right)\right\|\right. \\
&+ \left\|f\left(-\frac{\alpha^n x}{\gamma^n}\right) + \frac{c}{a} f\left(\frac{\alpha}{\gamma}\cdot\frac{\alpha^n x}{\gamma^n}\right)\right\| \\
&+ \left.\left\|\frac{b}{a} f\left(-\frac{\alpha}{\beta}\cdot\frac{\alpha^n x}{\gamma^n}\right) + \frac{c}{a} f\left(\frac{\alpha}{\gamma}\cdot\frac{\alpha^n x}{\gamma^n}\right)\right\|\right] \\
&\leq \frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n \left[\phi\left(\frac{\alpha^n x}{\gamma^n}, -\frac{\alpha}{\beta}\frac{\alpha^n x}{\gamma^n}, 0\right) + \phi\left(-\frac{\alpha^n x}{\gamma^n}, 0, \frac{\alpha^{n+1} x}{\gamma^{n+1}}\right)\right. \\
&+ \left.\phi\left(0, -\frac{\alpha}{\beta}\frac{\alpha^n x}{\gamma^n}, \frac{\alpha^{n+1} x}{\gamma^{n+1}}\right)\right] = 0
\end{aligned}$$

and so $A(-x) = -A(x)$ for all $x \in X$.

$$\begin{aligned}
&\|A(x) + A(y) - A(x+y)\|| = \lim_{n\to\infty}(\frac{c}{a})^n \left\|f\left(\frac{\alpha^n x}{\gamma^n}\right) + f\left(\frac{\alpha^n y}{\gamma^n}\right) - f\left(\frac{\alpha^n (x+y)}{\gamma^n}\right)\right\| \\
&= \lim_{n\to\infty}(\frac{c}{a})^n \left[\left\|f\left(\frac{\alpha^n x}{\gamma^n}\right) + \frac{b}{a} f\left(-\frac{\alpha}{\beta}\frac{\alpha^n x}{\gamma^n}\right)\right\|\right. \\
&+ \left\|f\left(\frac{\alpha^n y}{\gamma^n}\right) + \frac{c}{a} f\left(-\frac{\alpha^{n+1} y}{\gamma^{n+1}}\right)\right\| \\
&+ \left.\left\|f\left(\frac{\alpha^n (x+y)}{\gamma^n}\right) + \frac{b}{a} f\left(-\frac{\alpha}{\beta}\frac{\alpha^n x}{\gamma^n}\right) + \frac{c}{a} f\left(-\frac{\alpha^{n+1} y}{\gamma^{n+1}}\right)\right\|\right] \\
&\leq \frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n \left[\phi\left(\frac{\alpha^n x}{\gamma^n}, -\frac{\alpha}{\beta}(\frac{\alpha^n x}{\gamma^n}), 0\right) + \phi\left(\frac{\alpha^n y}{\gamma^n}, 0, -\frac{\alpha}{\gamma}(\frac{\alpha^n x}{\gamma^n})\right)\right. \\
&+\left.\phi\left(\frac{\alpha^n (x+y)}{\gamma^n}, -\frac{\alpha}{\beta}(\frac{\alpha^n x}{\gamma^n}), -\frac{\alpha}{\gamma}(\frac{\alpha^n x}{\gamma^n})\right)\right] = 0
\end{aligned}$$

for all $x, y \in X$. Thus the mapping $A : X \to Y$ is additive.

Now, we prove the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (2.3). Then we obtain

$$\begin{aligned}
\|A(x) - T(x)\| &= (\frac{c}{a})^n \left\| A\left((\frac{\alpha}{\gamma})^n x\right) - T\left((\frac{\alpha}{\gamma})^n x\right)\right\| \\
&\le (\frac{c}{a})^n \left[\left\| A\left((\frac{\alpha}{\gamma})^n x\right) - f\left((\frac{\alpha}{\gamma})^n x\right)\right\|\right. \\
&+ \left.\left\| T\left((\frac{\alpha}{\gamma})^n x\right) - f\left((\frac{\alpha}{\gamma})^n x\right)\right\|\right] \\
&\le \frac{2}{|a|}\left[\widetilde{\phi}\left(x, -\frac{\alpha}{\beta}x, 0\right) + \widetilde{\phi}(0, -\frac{\alpha}{\beta}x, \frac{\alpha}{\gamma}x)\right]
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Then we can conclude that $A(x) = T(x)$ for all $x \in X$. This complete the proof. $\square$

Corollary 2.3. *Assume that $1 \le \frac{c}{a} < \frac{\alpha}{\gamma}$ or $-1 \ge \frac{c}{a} > \frac{\alpha}{\gamma}$. Let p and θ be positive real numbers with $p > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(\alpha x + \beta y + \gamma z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|a|}\left(1 + \left|\frac{\alpha}{\gamma}\right|^p + 2\left|\frac{\alpha}{\beta}\right|^p\right) \frac{\left|\frac{\alpha}{\gamma}\right|^p}{\left|\frac{\alpha}{\gamma}\right|^p - \frac{c}{a}} \theta\|x\|^p$$

for all $x \in X$.

Proof. Defining $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ in Theorem 2.2, we get the desired result. $\square$

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References

[1] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.
[2] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
[3] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[4] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.
[5] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math. **108** (1984), 445–446.
[6] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.

[7] J.M. Rassias, *On a new approximation of approximately linear mappings by linear mappings*, Discuss. Math. **7** (1985), 193–196.

[8] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.

[9] G. Lu, C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.

[10] I. Chang, *Stability of higher ring derivations in fuzzy Banach algebras,* J. Computat. Anal. Appl. **14** (2012), 1059–1066.

[11] I. Cho, D. Kang, H. Koh, *Stability problems of cubic mappings with the fixed point alternative,* J. Computat. Anal. Appl. **14** (2012), 132–142.

[12] M. Eshaghi Gordji, M. Bavand Savadkouhi, M. Bidkham, *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **12** (2010), 454–462.

[13] M. Eshaghi Gordji, A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 724–729.

[14] M. Eshaghi Gordji, R. Farokhzad Rostami, S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras,* J. Computat. Anal. Appl. **13** (2011), 734–742.

[15] M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi, M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras,* J. Computat. Anal. Appl. **12** (2010), 463–470.

[16] H.A. Kenary, J. Lee, C. Park, *Non-Archimedean stability of an AQQ-functional equation,* J. Computat. Anal. Appl. **14** (2012), 211–227.

[17] C. Park, Y. Cho, H.A. Kenary, *Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **14** (2012), 526–535.

[18] C. Park, S. Jang, R. Saadati, *Fuzzy approximate of homomorphisms,* J. Computat. Anal. Appl. **14** (2012), 833–841.

[19] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 1097–1105.

[20] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Computat. Anal. Appl. **13** (2011), 1106–1114.

[21] C. Park, *Homomorphisms between Poisson JC^*-algebra*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[22] C. Park, *Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras*, Bull. Sci. Math. **132** (2008), 87–96.

[23] C. Park, Y. Cho, M. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).

[24] Y. Cho, C. Park, R. Saadati, *Functional inequalities in non-Archimedean Banach spaces*, Appl. Math. Lett. **23** (2010), 1238–1242.

[25] G. Lu, Y. Jiang, C. Park, *Functional inequality in Fréchet spaces*, J. Computat. Anal. Appl. **15** (2013), 369–373.

[26] G. Lu, C. Park, *Additive functional inequalities in Banach spaces*, J. Inequal. Appl. **2012**, Art. No. 294 (2012).

[27] L. Li, G. Lu, C. Park, D. Shin, *Additive functional inequalities in generalized quasi-Banach spaces*, J. Computat. Anal. Appl. **15** (2013), 1165–1175.

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An Efficient Spectral Collocation Algorithm for Solving Neutral Functional-Differential Equations

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Abstract. In this article, a spectral collocation method based on the Chebyshev polynomials is investigated for the approximate solution of a class of neutral functional-differential equations with variable coefficients, which have many applications in mathematical physics. A Chebyshev collocation method based on Chebyshev Gauss-Lobatto quadrature points is utilized to reduce the solution of such problem to a system of algebraic equations. In addition, accurate approximation is obtained by selecting few Chebyshev Gauss-Lobatto collocation points. Comparing the numerical results with those of known techniques shows that the present method is better in terms of accuracy over the other methods mentioned in this paper.

keyword: Neutral functional-differential equations; Proportional delays; Collocation method; Shifted Chebyshev-Gauss-Lobatto quadrature.

1 Introduction

In this paper, we discuss the numerical solution of the neutral functional-differential equations (NFDEs) with proportional delays

$$(u(x)+a(x)u(\gamma_m x))^{(m)} = \beta u(x) + \sum_{n=0}^{m-1} b_n(x)u^{(n)}(\gamma_n x) + f(x), x \geq 0, \tag{1.1}$$

with the initial conditions

$$\sum_{n=0}^{m-1} \eta_{in} u^{(n)}(0) = \lambda_i, \quad i = 0, 1, \cdots, m-1, \tag{1.2}$$

where a and $b_n(n = 0, 1, \cdots, m-1)$ are analytical functions, and $\beta,\ p_n,\ \eta_{in},\ \lambda_i$ are constants with $0 < p_n < 1(n = 0, 1, \cdots, m)$. In fact, NFDEs play an important role in the mathematical modeling of real-world phenomena (see, [12, 19]).

Over the years, it was found that most of delay differential equations cannot be solved exactly. Therefore, within the past few years, several fast and accurate numerical methods have been proposed for implementing approximations of such equations (see, for instance, [7, 15, 22]). In [13] and [14], the authors proposed the rational approximation and the spectral collocation approach to obtain numerical solutions of delay differential equations, respectively. In [26], Yalcinbas et al. developed the Hermite collocation approximation for tackling a class of delay differential equation with variable coefficients. Two efficient algorithms for solving pantograph equations are given in [27, 29], meanwhile, Yuzbasi et al. developed the Bessel collocation method for solving such equations in [28]. Recently, Chen and Wang [6] investigated the variational iteration method for the solution of NFDEs with proportional delays. Hu et al. [20] introduced linear multi-step scheme to present numerical solutions for NFDEs. The reproducing kernel Hilbert space method has been applied in [17] for introducing a numerical solution of NFDEs (1.1)-(1.2). Wang and his collaborators obtained numerical algorithms for NFDEs by using continuous Runge-Kutta methods [23], and one-leg θ-method [24, 25].

Our main aim of this paper is to propose an orthogonal collocation approach for the numerical solution of the neutral functional-differential equations with variable coefficients on the interval $[0, L]$. This approach is based on expanding the approximate solution as the members of a complete set of Chebyshev polynomials, and then the $(N-m+1)$ nodes of the shifted Chebyshev-Gauss-Lobatto quadrature are satisfied Eq. (1.1) to produce $(N-m+1)$ algebraic equations. These equations together with m additional algebraic equations from Eq. (1.1), constitute $(N+1)$ linear algebraic system of equations. The structure of the resulted matrix system is discussed. The main attractive property of the applying the Chebyshev collocation method is that the Gauss type quadrature nodes and weights of Chebyshev polynomials are explicitly and exactly known. This supplies a very compelling motivation for the use of Chebyshev polynomials. Finally, we implemented three numerical examples to demonstrate that the Chebyshev-Gauss-Lobatto collocation method is better in terms of accuracy over the other methods mentioned in this paper [1, 6, 24, 25, 17, 21].

The paper is organized as follows. Section 2 is for preliminary needed hereafter, In Section 3, we design the shifted Chebyshev Gauss-Lobatto collocation technique for NFDEs with proportional delays. In Section 4, we present some numerical results demonstrating the efficiency of suggested numerical algorithm. Concluding remarks are given in Section 5.

2 Preliminaries

This section is devoted to the study of the properties of Chebyshev orthogonal polynomials [16, 3, 9, 10]. The well-known Chebyshev polynomials are orthogonal with respect to the weight function $\omega(t) = (1-t^2)^{\frac{-1}{2}}$ in the interval $(-1, 1)$ and can be determined with the aid of the three-term recurrence relation reads:

$$T_{i+1}(t) = 2tT_i(t) - T_{i-1}(t), \quad i = 1, 2, \cdots,$$

where $T_0(t) = 1$ and $T_1 = t$.

The Chebyshev polynomials are eigenfunctions of the Sturm-Liouville problem:

$$(1-t^2)^{\frac{1}{2}} \left((1-t^2)^{\frac{1}{2}} T_i(t)\right)' + i^2 T_i(t) = 0, \quad t \in [-1, 1].$$

Now, we present the so-called shifted Chebyshev polynomials defined on the interval $(0, L)$, by introducing the change of variable $t = \frac{2x}{L} - 1$, denoting by $T_{L,i}(x)$ the shifted Chebyshev polynomials which can be evaluated from the recurrence formula:

$$T_{L,i+1}(x) = 2(\frac{2x}{L} - 1)T_{L,i}(x) - T_{L,i-1}(x), \quad i = 1, 2, \cdots,$$

According to the properties of the standard Chebyshev polynomials, we deduce that

$$\begin{aligned} &T_{L,i}(0) = (-1)^i, \quad T_{L,i}(L) = 1, \\ &D^q T_{L,i}(0) = \frac{(-1)^{i-q} i(i+q-1)!}{\Gamma(q+\frac{1}{2})(i-q)! L^q} \sqrt{\pi}, q \leq i. \end{aligned} \tag{2.1}$$

Next, let $\omega_L(x) = \dfrac{1}{\sqrt{Lx - x^2}}$, then we define the weighted space $L^2_{\omega_L}[0, L]$. The shifted Chebyshev polynomials form a complete $L^2_{\omega_L}[0, L]$-orthogonal system, i.e.,

$$\int_0^L T_{L,k}(x) T_{L,j}(x) \omega_L(x) dx = h_k \delta_{k,j},$$

where

$$h_k = \begin{cases} \frac{C_k}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases} \quad C_0 = 2, \quad C_k = 1, \quad k \geq 1. \tag{2.2}$$

Lemma 2.1. *The high-order derivatives of shifted Chebyshev polynomial can be expressed in terms of the shifted Chebyshev polynomials themselves as*

$$D^q T_{L,k}(x) = \sum_{\substack{i=0 \\ (k+i-q) \text{ even}}}^{k-q} C_q(k, i) T_{L,i}(x), \quad k \geq q, \tag{2.3}$$

where

$$C_q(k, i) = \frac{2^{2q}\, k(p-i+q-1)!\,(p+q-1)!}{L^q\, c_i\,(q-1)!\,(p-i)!\,p!}, \tag{2.4}$$

and $2p = k + i - q$, $c_0 = 2, c_i = 1;\ i \geq 1$.

For the proof of the previous relation see, [8].□

3 Shifted Chebyshev-Gauss collocation method

In the collocation methods [2, 5, 11, 18], one needs to exactly satisfy the differential equation at specified collocation points in the domain of solution. Generally, the distribution of the collocation nodes can be freely chosen, but an accurate approximations are obtained by selecting the collocation nodes as the zeros of the orthogonal polynomials. For shifted Chebyshev polynomials, two commonly used quadrature and collocation nodes, namely: (i) shifted Chebyshev-Gauss nodes (in the interior of the domain), and (ii) shifted Chebyshev-Gauss-Lobatto nodes (in the interior and at the two endpoints of the domain).

Now, we will present the shifted Chebyshev-Gauss-Lobatto type quadratures. Let $x_{N,j},\ 0 \leqslant j \leqslant N$, be the nodes of the standard Chebyshev-Gauss-Lobatto interpolation on $(-1,1)$ and $\varpi_{N,j},\ 0 \leqslant j \leqslant N$, be the corresponding weights. Throughout this paper, we assume that $x_{L,N,j},\ 0 \leqslant j \leqslant N$ stands for the nodes of the shifted Chebyshev-Gauss-Lobatto interpolation on the interval $(0,L)$. Thus $x_{L,N,j} = \frac{L}{2}(x_{N,j}+1)$, and their corresponding wights are $\varpi_{L,N,j} = \varpi_{N,j},\ 0 \leqslant j \leqslant N$. Let $S_N(0,L)$ be the set of all polynomials of degree less than or equal to N. In virtue of the property of the standard Chebyshev-Gauss-Lobatto quadrature, one gets for any $\phi \in S_{2N-1}(0,L)$,

$$\begin{aligned}\int_0^L \omega_L(x)\phi(x)dx &= \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\phi\left(\frac{L}{2}(x+1)\right)dx \\ &= \sum_{j=0}^N \varpi_{N,j}\phi\left(\frac{L}{2}(x_{N,j}+1)\right) = \sum_{j=0}^N \varpi_{L,N,j}\phi(x_{L,N,j}).\end{aligned} \tag{3.1}$$

Associating with this quadrature rule, we denote by $I_N^{T_L}$ the shifted Chebyshev-Gauss-Lobatto interpolation,

$$I_N^{T_L}\, u(x_{L,N,j}) = u(x_{L,N,j}), \quad 0 \le k \le N.$$

In this section, we use the shifted Chebyshev-Gauss collocation method to solve numerically the model problem of (1.1), (1.2). We set

$$S_N(0,L) = \operatorname{span}\{T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,N}(x)\}. \tag{3.2}$$

The shifted Chebyshev-Gauss collocation method for solving (1.1) and (1.2) is to seek $u_N(x) \in S_N(0,L)$, such that

$$\begin{aligned}(u(x_{L,N-m,k}) + a(x_{L,N-m,k})u(\gamma_m x_{L,N-m,k}))^{(m)} &= \beta u(x_{L,N-m,k}) \\ &+ \sum_{n=0}^{m-1} b_n(x_{L,N-m,k})u^{(n)}(\gamma_n x_{L,N-m,k}) \\ &+ f(x_{L,N-m,k}), \quad k = 0,1,\cdots,N-m, \\ &\sum_{n=0}^{m-1} \eta_{in}u^{(n)}(0) = \lambda_i, \quad i = 0,1,\cdots,m-1,\end{aligned} \tag{3.3}$$

where the $x_{L,N-m,k};\ k = 1,2,\ldots,N-m-1$ are distinct and lie between 0 and L, $x_{L,N-m,0} = 0$ and $x_{L,N-m,N-m} = L$. For simplicity in presentation and without loss of generality, assume $a(x) \equiv 1$. We now derive the collocation algorithm for solving (1.1) and (1.2). To do this, consider the solution is approximated by a truncated Chebyshev expansion

$$u_N(x) = \sum_{j=0}^N a_j T_{L,j}(x), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^T. \tag{3.4}$$

Let us firstly introduce the following corollary which will be of fundamental importance in what follows

Corollary 3.1. *The q-th order derivative of shifted Chebyshev polynomials with proportional delay can be written as*

$$D^q T_{L,k}(\gamma_r x) = \sum_{\substack{i=0 \\ (k+i-q)\ even}}^{k-q} \gamma_r^q C_q(k,i) T_{L,i}(\gamma_r x), \quad k \ge q, \tag{3.5}$$

where $C_q(k,i)$ *is defined in* (2.4). □

Now, we approximate $u(x)$ and $u^q(x),\ q=1,2,\dots,m$, as (3.4) and (2.3), and in virtue of Corollary 3.1 for obtaining $u^q(\gamma_r x),\ q=1,2,\dots,m$, then Eq. (1.1) can be written as

$$\sum_{j=0}^{N} a_j D^{(m)} T_{L,j}(x) + \sum_{j=0}^{N} a_j D^{(m)} T_{L,j}(\gamma_m x) = \beta \sum_{j=0}^{N} a_j T_{L,j}(x) + \sum_{n=0}^{m-1}\sum_{j=0}^{N} b_n(x) a_j D^{(n)} T_{L,j}(\gamma_n x) + f(x). \tag{3.6}$$

According to (2.3), we deduce that

$$\sum_{j=0}^{N}\left(\sum_{\substack{\rho=0 \\ (j+\rho-m)\ \text{even}}}^{j-m} \left(C_m(j,\rho) T_{L,\rho}(x) + (\gamma_m)^m C_m(j,\rho) T_{L,\rho}(\gamma_m x)\right) a_j \right) = \beta \sum_{j=0}^{N} a_j T_{L,j}(x) + \sum_{n=0}^{m-1}\sum_{j=0}^{N} \sum_{\substack{\rho=0 \\ (j+\rho-n)\ \text{even}}}^{j-n} a_j b_n(x) (\gamma_n)^n C_n(j,\rho) T_{L,\rho}(\gamma_n x) + f(x). \tag{3.7}$$

Also, by substituting Eq. (3.4) in Eq. (1.2) we obtain

$$\sum_{n=0}^{m-1}\sum_{j=0}^{N} \eta_{in} a_j D^{(n)} T_{L,j}(0) = \lambda_i. \tag{3.8}$$

To find the solution $u_N(x)$, we first collocate Eq. (3.7) at the $(N-m+1)$ shifted Chebyshev roots, yields

$$\begin{aligned}\sum_{j=0}^{N}&\left(\sum_{\substack{\rho=0 \\ (j+\rho-m)\ \text{even}}}^{j-m} \left(C_m(j,\rho) T_{L,\rho}(x_{L,N-m,k}) + (\gamma_m)^m C_m(j,\rho) T_{L,\rho}(\gamma_m x_{L,N-m,k})\right) \right) a_j \\ &= \beta \sum_{j=0}^{N} a_j T_{L,j}(x_{L,N-m,k}) \\ &+ \sum_{n=0}^{m-1}\sum_{j=0}^{N} \sum_{\substack{\rho=0 \\ (j+\rho-n)\ \text{even}}}^{j-n} a_j b_n(x_{L,N-m,k}) (\gamma_n)^n C_n(j,\rho) T_{L,\rho}(\gamma_n x_{L,N-m,k}) + f(x_{L,N-m,k}), \\ &\qquad k=0,1,\dots,N-m.\end{aligned} \tag{3.9}$$

Next, Eq. (3.8), after using (2.1), can be written as

$$\sum_{n=0}^{m-1}\sum_{j=0}^{N} (-1)^{j-n} \frac{j(j+n-1)!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})(j-n)!L^n} \eta_{in} a_j = \lambda_i, \qquad i=0,1,\dots,m-1. \tag{3.10}$$

Let us denote

$$\begin{aligned} \mathbf{a} &= (a_0, a_1, \cdots, a_N)^T, \\ f_k &= f(x_{L,N-m,k}), \quad k=0,1,\cdots,N-m, \\ \mathbf{f} &= (f_0, f_1, \cdots, f_{N-m}, \lambda_0, \cdots, \lambda_{m-1})^T. \end{aligned}$$

The matrix system associated with (3.9) and (3.10) becomes

$$\left(A + \gamma_m^m\, B + \beta C + \sum_{n=0}^{m-1} \gamma_n^n\, D_n + E\right) \mathbf{a} = \mathbf{f}, \tag{3.11}$$

where the matrices $A,\ B,\ C,\ D_i,\ i=1,2,\dots,m-1$ and E are given explicitly in the following theorem.

Theorem 3.2. *If we denote $A=(a_{kj})_{0<k,j<N},\ B=(b_{kj})_{0<k,j<N},\ C=(c_{kj})_{0<k,j<N},\ D_n=(d^n_{kj})_{0<k,j<N};$ $n=1,2,\cdots,m-1,$ and $E=(e_{kj})_{0<k,j<N}$, then the elements $a_{kj},\ b_{kj},\ c_{kj}$, and d^n_{kj} are given by*

$$a_{kj} = \begin{cases} \displaystyle\sum_{\substack{\rho=0 \\ (j+\rho-m)\ even}}^{j-m} C_m(j,\rho) T_{L,\rho}(x_{L,N-m,k}), & k=0,1,\cdots,N-m,\ j=0,1,\cdots,N, \\ 0, & k=N-m+1,\dots,N,\ j=0,1,\dots,N, \end{cases}$$

$$b_{kj}=\begin{cases}\sum\limits_{\substack{\rho=0\\(j+\rho-m)\ even}}^{j-m} C_m(j,\rho)T_{L,\rho}(\gamma_m x_{L,N-m,k}), & k=0,1,\cdots,N-m,\ j=0,1,\cdots,N,\\ 0, & k=N-m+1,\dots,N,\ j=0,1,\dots,N,\end{cases}$$

$$c_{kj}=\begin{cases}-T_{L,j}(x_{L,N-m,k}), & k=0,1,\cdots,N-m,\ j=0,1,\cdots,N,\\ 0, & k=N-m+1,\dots,N,\ j=0,1,\dots,N,\end{cases}$$

$$d^n_{kj}=\begin{cases}-\sum\limits_{\substack{\rho=0\\(j+\rho-n)\ even}}^{j-n} b_n(x_{L,N-m,k})C_n(j,\rho)T_{L,\rho}(\gamma_n x_{L,N-m,k}), & k=0,1,\cdots,N-m,\ j=0,1,\cdots,N,\\ 0, & k=N-m+1,\dots,N,\ j=0,1,\dots,N.\end{cases}$$

Moreover, the elements of the matrix corresponding to the mixed initial conditions are given by

$$e_{kj}=\begin{cases}0, & k=0,1,\cdots,N-m,\ j=0,1,\cdots,N,\\ \sum\limits_{n=0}^{m-1}(-1)^{j-n}\frac{j(j+n-1)!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})(j-n)!L^n}\eta_{k-N-m+1,n}, & k=N-m+1,\dots,N,\ j=0,1,\dots,N.\end{cases}$$

Proof. The proof of this theorem is not difficult, and it can be accomplished by using Lemma 2.1 and Corollary 3.1.□

Remark 3.3. *It is worth mention here that the previous procedure can be implemented to NFFE subject to boundary conditions.*

Remark 3.4. *In the case of $a(x)\neq 0,\ b_n(x)\neq 0, n=0,1,\cdots,m-1,$ and $\beta\neq 0$, the linear system* (3.11), *can be solved by forming explicitly the LU factorization; i.e, $A+\gamma_m^m B+\beta C+\sum\limits_{n=0}^{m-1}\gamma_n^n D_n+E=LU$. The expense of calculating LU factorization is $O(N^3)$ operations and the expense of solving the linear system* (3.11)*, provided that the factorization is known, is $O(N^2)$.*

4 Numerical results

In this section, we will carry out three test examples to study the validity and effectiveness of the proposed method and also, shown that high accurate solutions are achieved using a few number of the Chebyshev Gauss-Lobatto points. Moreover, comparisons with other methods reveal that the present method is accurate and convenient. All the numerical computations have been performed by the symbolic computation software Mathematica 8.0.

Example 1. *Let us first consider first-order NFFE [1, 6, 24, 25, 21]*

$$\begin{cases}u'(x)=-u(x)+\frac{1}{2}u(\frac{x}{2})+\frac{1}{2}u'(\frac{x}{2}), & x\in[0,1],\\ u(0)=1.\end{cases}\tag{4.1}$$

The analytic solution of the aforementioned problem is $u(x)=e^{-x}$. In Table 4.1, we compare the errors of the present method with two-stage order-one Runge-Kutta (RK) method [1], the one-legθ method [24, 25] with $\theta=0.8$, variational iteration (VI) method [6] and shifted Chebyshev operational matrix (SCOM) [21]. The Graph of analytical solution and approximate solution at $x=5$ for $N=18$ is displayed in Fig. 1 to make it easer to compare with analytical solution.

Example 2. *Consider the second-order NFFE with proportional delays [1, 6, 24, 25, 21]*

$$\begin{cases}u''(x)=\frac{3}{4}u(x)+u(\frac{x}{2})+u'(\frac{x}{2})+\frac{1}{2}u''(\frac{x}{2})-x^2-x+1, & x\in[0,1],\\ u(0)=u'(0)=0,\end{cases}\tag{4.2}$$

The exact solution is is $u(x)=x^2$.

Table 4.1: Comparison of the absolute errors for Example 1.

x	RK method	θ-method with $\theta = 0.8$	VI method $m = 8$	SCOM method	SLC method $N = 14$
0.2	$8.24.10^{-4}$	$8.86.10^{-3}$	$7.08.10^{-4}$	$4.83.10^{-11}$	$1.48.10^{-16}$
0.4	$1.35.10^{-3}$	$2.66.10^{-2}$	$1.29.10^{-3}$	$3.36.10^{-11}$	$2.11.10^{-17}$
0.6	$1.66.10^{-3}$	$4.58.10^{-2}$	$1.76.10^{-3}$	$1.18.10^{-11}$	$7.57.10^{-17}$
0.8	$1.81.10^{-3}$	$6.29.10^{-2}$	$2.15.10^{-3}$	$5.25.10^{-11}$	$1.10.10^{-17}$
1.0	$1.85.10^{-3}$	$7.66.10^{-2}$	$2.47.10^{-3}$	$2.40.10^{-12}$	$2.13.10^{-15}$

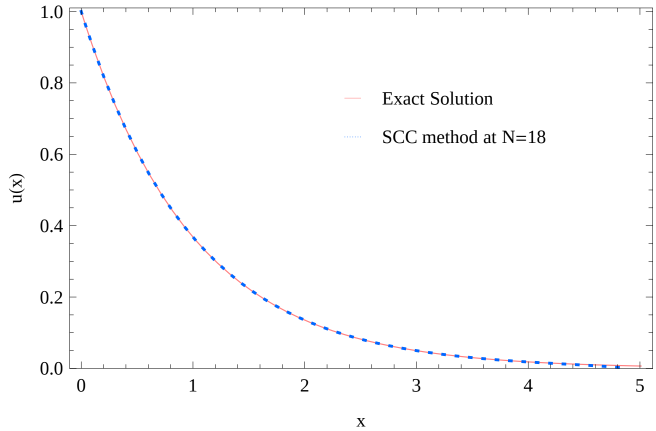

Figure 1: Graph of exact solution and approximate solution at $x = 5$ and $N = 18$ for Example 1.

In Table 4.2, we compare the errors of the present method with two-stage order-one Runge-Kutta (RK) method of [1], the one-legθ-method of [24, 25] with $\theta = 0.8$, variational iteration (VI) method [6], the reproducing kernel Hilbert space method (RKHSM) [17] and shifted Chebyshev operational matrix (SCOM) [21]. The Graph of analytical solution and approximate solution at $t = 100$ for $N = 4$ is displayed in Fig. 2 to make it easer to compare with analytical solution. Moreover, absolute errors obtained by the SCC method, with $N = 5$, are plotted in Fig. 3.

Example 3. *Consider the following third-order NFFE with proportional delays*

$$\begin{cases} u^{'''}(x) = u(x) + u^{'}(\frac{x}{2}) + u^{''}(\frac{x}{3}) + \frac{1}{2}u^{'''}(\frac{x}{4}) - x^4 - \frac{x^3}{2} - \frac{4}{3}x^2 + 21x, & x \in [0,1], \\ u(0) = u^{'}(0) = u^{''}(0) = 0, \end{cases} \tag{4.3}$$

whose exact solution is $u(x) = x^4$.

The graph of the approximate solution and the exact solution are displayed in Fig. 4. In Table 4.3, we compare the errors of the present method with two-stage order-one Runge-Kutta method of [1] and the

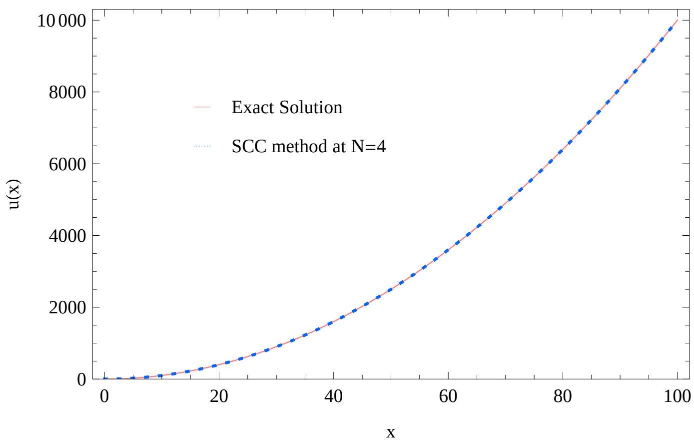

Figure 2: Graph of exact solution and approximate solution at $t = 100$ and $N = 4$ for Example 2.

Table 4.2: Comparison of the absolute errors for Example 2.

x	RK method	θ-method with $\theta = 0.8$	VI method $m = 6$	RKHSM $n = 100$	SCOM method	SCC method $N = 5$
0.1	$1.00.10^{-3}$	$6.10.10^{-3}$	$1.67.10^{-4}$	$9.57.10^{-6}$	$2.08.10^{-17}$	$5.50.10^{-18}$
0.2	$2.02.10^{-3}$	$2.58.10^{-2}$	$7.15.10^{-4}$	$1.95.10^{-4}$	$4.16.10^{-17}$	$3.61.10^{-17}$
0.3	$3.07.10^{-3}$	$6.47.10^{-2}$	$1.73.10^{-3}$	$2.94.10^{-4}$	$4.16.10^{-17}$	$2.21.10^{-17}$
0.4	$4.17.10^{-3}$	$1.37.10^{-1}$	$3.30.10^{-3}$	$3.93.10^{-4}$	$5.55.10^{-17}$	$4.37.10^{-17}$
0.5	$5.34.10^{-3}$	$2.81.10^{-1}$	$5.55.10^{-3}$	$4.92.10^{-4}$	$1.11.10^{-16}$	$5.17.10^{-17}$

Table 4.3: Comparison of the absolute errors for Example 3.

x	Two-stage order-one Runge-Kutta method	Variational iteration method			SCC method $N = 5$
		$n = 4$	$n = 5$	n=6	
0.1	$4.79.10^{-5}$	$2.46.10^{-8}$	$3.07.10^{-9}$	$9.09.10^{-12}$	$3.41.10^{-16}$
0.2	$4.43.10^{-4}$	$4.03.10^{-7}$	$5.04.10^{-8}$	$2.98.10^{-11}$	$4.16.10^{-17}$
0.3	$1.57.10^{-3}$	$2.09.10^{-6}$	$2.62.10^{-7}$	$2.33.10^{-9}$	$4.16.10^{-17}$
0.4	$3.85.10^{-3}$	$6.80.10^{-6}$	$8.49.10^{-7}$	$1.01.10^{-8}$	$1.38.10^{-17}$
0.5	$7.78.10^{-3}$	$1.71.10^{-5}$	$2.13.10^{-6}$	$3.20.10^{-8}$	$1.66.10^{-16}$
0.6	$1.39.10^{-2}$	$3.64.10^{-5}$	$4.55.10^{-6}$	$8.24.10^{-8}$	$1.90.10^{-16}$
0.7	$2.28.10^{-2}$	$6.96.10^{-5}$	$8.69.10^{-6}$	$1.85.10^{-7}$	$2.42.10^{-16}$
0.8	$3.53.10^{-2}$	$1.23.10^{-4}$	$1.53.10^{-5}$	$3.76.10^{-7}$	$1.38.10^{-16}$
0.9	$5.19.10^{-2}$	$2.03.10^{-4}$	$2.54.10^{-5}$	$7.09.10^{-7}$	$6.59.10^{-17}$
1.0	$7.34.10^{-2}$	$3.21.10^{-4}$	$4.01.10^{-5}$	$1.26.10^{-6}$	$2.77.10^{-17}$

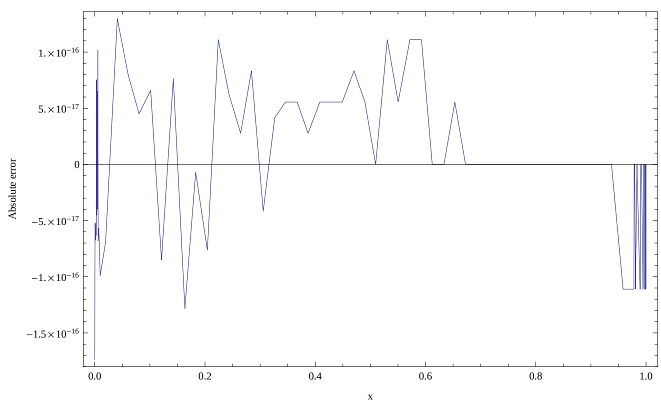

Figure 3: Graph of absolute errors at $N = 5$ for Example 2.

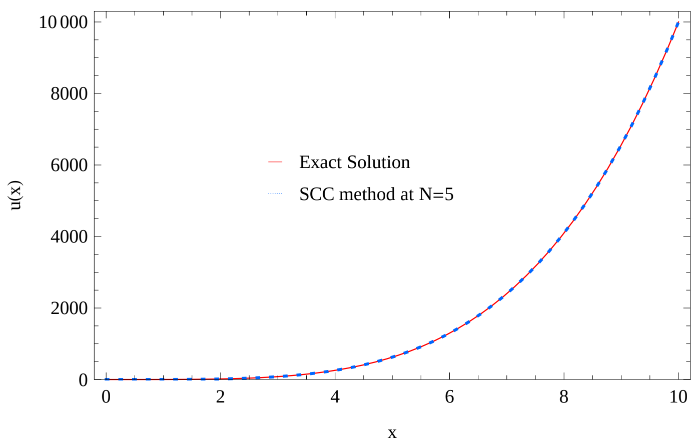

Figure 4: Graph of exact solution and approximate solution at $t = 100$ and $N = 5$ for Example 3.

variational iteration method ($n = 4, 5, 6$) [6]. From this table, it shown that the present method is better in terms of accuracy over the other methods mentioned in this example.

5 Conclusions

We proposed a high accuracy numerical algorithm for the solution of neutral functional-differential equation. In this algorithm we implemented a Chebyshev collocation method based on Chebyshev Gauss-Lobatto points. The main advantage of the proposed method is that, high accurate solutions are achieved using a few number of the Chebyshev Gauss-Lobatto points. Through the comparisons among the exact solutions and the approximate solutions of two-stage order-one Runge-Kutta method [1], variational iteration method [6], one-leg θ-method [24, 25], the reproducing kernel Hilbert space method (RKHSM) [17] and shifted Chebyshev operational matrix method [21] and the current method, it has been shown that the presented method has provided the most accurate solutions for the neutral functional-differential equation with proportional delays to date.

References

[1] A. Bellen and M. Zennaro, Numerical methods for delay differential equations, Numerical mathematics and scientific computation, New York, The Clarendon Press Oxford University Press, 2003.

[2] A.H. Bhrawy and A.S. Alofi, A Jacobi-Gauss Collocation Method for Solving Nonlinear Lane-Emden type Equations, Commun. Nonlinear. Sci. Numer. Simulat. **17** (2012) 62-70

[3] A.H. Bhrawy and A.S. Alofi, The operational matrix of fractional integration for shifted Chebyshev polynomials, Applied Mathematics Letters, **26** (2013) 25-31

[4] A. H. Bhrawy, M.M. Tharwat and Yildirim, A new formula for fractional integrals of Chebyshev polynomials: Application for solving multi-term fractional differential equations, Appl. Math. Model. **37** (2013) 4245-4252

[5] A.H. Bhrawy and M.A. Alghamdi, A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals, Boundary Value Problems **2012** (2012) 62.

[6] X. Chen and L. Wang, The variational iteration method for solving a neutral functional-differential equation with proportional delays, Comput. Math. Appl. **59** (2010) 2696-2702.

[7] M. Dehghan, F. Shakeri, The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics, Phys. Scr. **78** (2008) 11

[8] E.H. Doha, On the construction of recurrence relations for the expansion and connection coeffi-cients in series of Jacobi polynomials, J. Phys. A: Math. Gen. **37** (2004) 657-675.

[9] E.H. Doha, A.H Bhrawy, Efficient spectral-Galerkin algorithms for direct solution of the integrated forms of second-order equations using ultraspherical polynomials ANZIAM J. 48 (2007) 361-386

[10] E.H. Doha, A.H. Bhrawy and S.S. Ezzeldeen, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, Comput. Math. Appl. **62** (2011) 2364-2373

[11] E.H. Doha, A.H. Bhrawy and S.S. Ezzeldeen, A New Jacobi Operational Matrix: An Application For Solving Fractional Differential Equation, Appl. Math. Modell. **36** (2012) 4931-4943

[12] L. Fox, D.F. Mayers, J.A. Ockendon, A.B. Tayler, On a functional differential equation, J. Inst. Math. Appl. **8** (1971) 271-307.

[13] E. Ishiwata and Y. Muroya, Rational approxination method for delay differential equations with proportional delay, Appl. Math. Comput. **187** (2007) 741-747.

[14] E. Ishiwata, Y. Muroya and H. Brunner, A super-attainable order in collocation methods for differential equations with proportional delay, Appl. Math. Comput. **198** (2008) 227-236.

[15] M.Z. Liu, D.S. Li, Properties of analytic solution and numerical solution of multi-pantograph equation, Appl. Math. Comput. **155** (2004) 853-71.

[16] Y. Luke, The Special Functions and Their Approximations, vol. 2, Academic Press, New York, 1969.

[17] X. Lv and Y. Gao, The RKHSM for solving neutral functional-differential equations with proportional delays, Math. Meth. Appl. Sci., (2012) DOI: 10.1002/mma.2612.

[18] M. Maleki, I. Hashim, M.T. Kajani, S. Abbasbandy, An adaptive pseudospectral method for fractional order boundary value problems, Abstract and Applied Analysis 2012 (2012) 381708

[19] J.R. Ockendon, A.B. Tayler, The dynamics of a current collection system for an electric locomotive Proc. Roy. Soc. London A **322** (1971) 447-468.

[20] H. Peng, C. Huang and S. Wu, Asymptotic stability of linear multistep methods for nonlinear delay differential equations, Appl. Math. Comput. **211** (2009) 95-101.

[21] S. Sedaghat, Y. Ordokhani and M. Dehghan, Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials, Commun. Nonlinear Sci. Numer. Simulat. **17** (2012) 4815-4830.

[22] E. Tohidi, A. H. Bhrawy, Kh. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, Appl. Math. Model. 37 (2013) 4283-4294

[23] W. Wang, Y. Zhang and S. Li, Stability of continuous Runge-Kutta-type methods for nonlinear neutral delay-differential equations, Appl. Math. Model., **33** (2009) 3319-3329.

[24] W. Wang and S. Li, On the one-leg θ-methods for solving nonlinear neutral function differential equations, Appl. Math. Comput., **193** (2007) 285-301.

[25] W. Wang, T. Qin and S. Li, Stability of one-leg θ-methods for nonlinear neutral differential equations with proportional delay, Appl. Math. Comput. **213** (2009) 177-183.

[26] S. Yalcinbas, M. Aynigul and M. Sezer, A collocation method using Hermite polynomials for approximate solution of pantograph equations, J. Frank. Inst. **348** (2011) 1128-1139

[27] S. Yuzbasi, An efficient algorithm for solving multi-pantograph equation systems, Comput. Math. Appl. **64** (2012) 589603

[28] S. Yuzbasi, N. Sahin, M. Sezer, A Bessel collocation method for numerical solution of generalized pantograph equations, Numerical Methods for Partial Differential Equations, 28 (2012) 1105-1123

[29] E. Yusufoglu, An efficient algorithm for solving generalized pantograph equations with linear functional argument, Appl. Math. Comput. **217** (2010) 3591-3595

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Some properties of intuitionstic fuzzy metric spaces

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In this paper we show that a topological space is (completely) metrizable if and only if it admits a compatible (complete) intuitionistic fuzzy metric. we also prove that the topological space induced by a complete intuitionistic fuzzy metric space is completely metrizable. Finally, we consider the intuitionistic fuzzy product metric space of two intuitionistic fuzzy metric spaces and explore precompactness, completeness and compactness of the intuitionistic fuzzy product metric space.

Keywords: Intuitionistic fuzzy metric, Cauchy, Complete,Precompact, Completely metrizable.

AMS Subject Classifications: 54A40, 54E70

1 Introduction

Fuzzy metric is an important notion in Fuzzy Topology. Many author have introduced the concept of fuzzy metric from different point of view [5, 7, 8, 9]. In particular, George and Veeramani in [7] gave a definition of fuzzy metric with the help of continuous t-norms and proved that the topology induced by this fuzzy metric is first countable and Hausdorff. On the other hand, the theory of intuitionistic fuzzy set was first studied by Atanassov [2]. Using the idea of intuitionistic fuzzy set due to Atanassov, Park [10] presented the conception of intuitionistic fuzzy metric space, which is a generalization of fuzzy metric space gave by George and Veeramani, with the help of continuous t-norms and continuous t-conorms. Also, some known results of metric spaces including Uniform limit theorem and Baire's theorem for intuitionstic fuzzy metric spaces are proven in [10]. Saadati and Park explored some properties of intuitionistic fuzzy metric spaces as completeness, precompactness and compactness in [12]. Other more contributions to the study of intuitionstic fuzzy metric spaces can

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be found in [1, 3, 6, 11]. In this paper, we show that a topological space is metrizable if and only if it admits a compatible intuitionistic fuzzy metric. Using this result, we prove that a metrizable topological space is compact if and only if every compatible intuitionistic fuzzy metric is complete. Moreover, we prove that the topological space induced by a complete intuitionistic fuzzy metric space is completely metrizable. Finally, we construct the intuitionistic fuzzy product metric space of two intuitionistic fuzzy metric spaces. We also prove that if two intuitionistic fuzzy metric spaces are both compact then so is the intuitionistic fuzzy product metric space.

2 Preliminaries

We recall some concepts and auxiliary results in the section. Our basic reference for general topology is [4].

Definition 2.1 [7] A *continuous t-norm* is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ which satisfies the following conditions:
(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a * b = a \cdot b; a * b = \min\{a, b\}$.

Definition 2.2 [13] A *continuous t-conorm* is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:
(a) $\diamond$ is associative and commutative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-conorms: $a \diamond b = \min\{a + b, 1\}; a \diamond b = \max\{a, b\}$.

Definition 2.3 [10] An *intuitionstic fuzzy metric space* is 5-tuple $(X, M, N, *, \diamondsuit)$ such that X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:
(a) $M(x, y, t) + N(x, y, t) \leq 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ if and only if $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(f) the function $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(g) $N(x, y, t) < 1$;
(h) $N(x, y, t) = 0$ if and only if $x = y$;
(i) $N(x, y, t) = N(y, x, t)$;
(j) $N(x, y, t)\diamondsuit N(y, z, s) \geq N(x, z, t + s)$;

(k) the function $N(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.4 [10] In intuitionistic fuzzy metric space X, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

Definition 2.5 [10] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and let $r \in (0, 1), t > 0$ and $x \in X$. The set

$$B_{(M,N)}(x, r, t) = \{y \in X | M(x, y, t) > 1 - r, N(x, y, t) < r\}$$

is called *the open ball with center x and radius r with respect to t.*

Obviously, $\{B_{(M,N)}(x, r, t)\}$ forms a base of a topology on X. The topology is denoted by $\tau_{(M,N)}$.

Example 2.6 [10] Let $X = \mathbf{N}$. Denote $a * b = \max\{a + b - 1, 0\}$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M(x, y, t) = \begin{cases} \frac{x}{y} & x \leq y, \\ \frac{y}{x} & y \leq x, \end{cases} \quad N(x, y, t) = \begin{cases} \frac{y-x}{y} & x \leq y, \\ \frac{x-y}{x} & y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Example 2.7 [10] Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamondsuit b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by the metric d *the standard intuitionistic fuzzy metric.*

Remark 2.8 The metric space (X, d) is compatible with $(X, \tau_{(M_d,N_d)})$.

Definition 2.9 [10] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$ in X is said to be *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$.

(b) $(X, M, N, *, \diamondsuit)$ is called *complete* if every Cauchy sequence is convergent with respect to $\tau_{(M,N)}$.

Remark 2.10 It is easy to prove that a metric space (X, d) is complete if and only if the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \diamondsuit)$ is complete.

Definition 2.11 [12] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $A \subset X$. We say A is *precompact* if for each $0 < r < 1$ and $t > 0$ there exists a finite subset S of A such that $A \subset \bigcup_{x \in S} B_{(M,N)}(x, r, t)$.

3 Properties of intuitionistic fuzzy metric spaces

In the section we study some properties of intuitionistic fuzzy metric spaces and intuitionistic fuzzy product metric spaces.

Lemma 3.1 [12] *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then $(X, \tau_{(M,N)})$ is a metrizable topological space.*

Theorem 3.2 *Let (X, τ) be a topological space. Then (X, τ) is metrizable if and only if it admits a compatible intuitionistic fuzzy metric.*

Proof Suppose that (X, τ) is metrizable. Let d be a metric on X compatible with τ. Then, by Remark 2.8, we immediately deduce that the intuitionistic fuzzy metric(M_d, N_d) induced by d is compatible with τ. The converse is straightforward. We are done.

Theorem 3.3 *Let $(X, M, N, *, \diamondsuit)$ be a complete intuitionistic fuzzy metric space. Then $(X, \tau_{(M,N)})$ is completely metrizable.*

Proof According to Lemma 2.6 in [12], it is easy to see that $\{U_n | n \in \mathbf{N}\}$ is a base for a uniformity $\mathcal{U}$ on X whose induced topology coincides with $\tau_{(M,N)}$, where $U_n = \{(x, y) \in X \times X | M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}, N(x, y, \frac{1}{n}) < \frac{1}{n}\}$ for every $n \in \mathbf{N}$. Then we can find a metric d on X compatible with $\mathcal{U}$. Now we are going to proof that d is complete on X. Let $\{x_n\}$ be a Cauchy sequence in (X, d). To complete the proof, it suffices to show that $\{x_n\}$ is also a Cauchy sequence in $(X, M, N, *, \diamondsuit)$. Let $r \in (0, 1)$ and $t > 0$. Choose an $i \in \mathbf{N}$ such that $\frac{1}{i} < r$ and $\frac{1}{i} < t$. Then we can find an $n_0 \in \mathbf{N}$ such that $(x_n, x_m) \in U_i$ for all $n, m \geq n_0$. Therefore, $M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{i}) > 1 - \frac{1}{i} > 1 - r$ and $N(x_n, x_m, t) \leq N(x_n, x_m, \frac{1}{i}) < 1 - \frac{1}{i} < 1 - r$ whenever $n, m \geq n_0$. This show that $\{x_n\}$ is a Cauchy sequence in the complete intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$. We finish the proof.

Theorem 3.4 *Let (X, τ) be a topological space. Then (X, τ) is completely metrizable if and only if it admits a compatible complete intuitionistic fuzzy metric.*

Proof Assume that (X, τ) is completely metrizable. Let d be a complete metric on X compatible with τ. From Remark 2.8 and Remark 2.10, it follows that the intuitionistic fuzzy metric (M_d, N_d) induced by d is complete and it is compatible with τ. The converse follows immediately from Lemma 3.3. We are done.

Lemma 3.5 [12] *An intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is compact if and only if $(X, \tau_{(M,N)})$ is compact.*

Lemma 3.6 [12] *A subset of an intuitionistic fuzzy metric space* $(X, M, N, *, \diamondsuit)$ *is compact if and only if it is precompact and complete.*

Definition 3.7 [12] A topological space is called a *topologically complete intuitionistic fuzzy metrizable space* if there exists a complete intuitionistic fuzzy metric inducing the given topology on it.

Lemma 3.8 [12] *An open subspace of a complete intuitionistic fuzzy metrizable space is a topologically complete intuitionistic fuzzy metrizable space.*

Theorem 3.9 *Let* (X, τ) *be a metrizable topological space. Then* (X, τ) *is compact if and only if every compatible intuitionistic fuzzy metric is complete.*

Proof Suppose that (X, τ) is compact. Then, by Theorem 3.2, Lemma 3.5 and Lemma 3.6, we deduce that every compatible intuitionistic fuzzy metric is complete.

Conversely, suppose that each intuitionistic fuzzy metric on X compatible with τ is complete. Let d be a metric on X compatible with τ. The standard intuitionistic fuzzy metric (M_d, N_d) is complete. It follows from Remark 2.10 that d is complete. By Niemytzki-Tychonoff theorem (see [4]), we immediately conclude that (X, τ) is compact. The proof is finished.

Theorem 3.10 *Let* (X, τ) *be a second countable topological space. If* $\{K_n\}$ *is an increasing compact subset sequence in* X, $X = \bigcup_{n=1}^{\infty} K_n$ *and* $K_n \subset intK_{n+1}(n \in \mathbf{N})$, *then* X *is a topologically complete intuitionistic fuzzy metrizable space.*

Proof Let $\{U_n | n \in \mathbf{N}\}$ be a countable base on X. Since X is locally compact, we can observe that X has one-point compactification $X \cup \{\infty\}$. It is trivial to verify that $\{U_n | n \in \mathbf{N}\} \cup \{\{\infty\} \cup (X - K_n) : n \in \mathbf{N}\}$ is a countable base on the compact space $X \cup \{\infty\}$. Note that $X \cup \{\infty\}$ is Hausdorff, so it is regular. It follows from Urysohn metrization theorem that $X \cup \{\infty\}$ is a compact metrizable space. Take a metric d on $X \cup \{\infty\}$. By Remark 2.8 and Theorem 3.9, we deduce that $(X \cup \{\infty\}, M_d, N_d, *, \diamondsuit)$ is a complete intuitionistic fuzzy metric space. Observe that X is an open subspace in $X \cup \{\infty\}$. According to Lemma 3.8, we immediately conclude that X is a topologically complete intuitionistic fuzzy metrizable space.

Let $(X_i, M_i, N_i, *, \diamondsuit)(i = 1, 2)$ be two intuitionistic fuzzy metric spaces. We define two maps $M, N : (X_1 \times X_2) \times (X_1 \times X_2) \times (0, \infty) \to [0, 1]$ by

$$M((x_1, x_2), (y_1, y_2), t) = \min\{M_1(x_1, y_1, t), M_2(x_2, y_2, t)\}$$

and

$$N((x_1, x_2), (y_1, y_2), t) = \max\{N_1(x_1, y_1, t), N_2(x_2, y_2, t)\}$$

for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $t > 0$.

Theorem 3.11 $(X_1 \times X_2, M, N, *, \diamondsuit)$ *is an intuitionistic fuzzy metric space.*

Proof It is straightforward to verify that conditions (b)-(d), (f),(g)-(i) and (k) in Definition 2.3 are satisfied.

Let us verify condition (a) in Definition 2.3. Let $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $t > 0$. Suppose, without loss of generality, that

$$N((x_1, x_2), (y_1, y_2), t) = \max\{N_1(x_1, y_1, t), N_2(x_2, y_2, t)\} = N_1(x_1, y_1, t).$$

If $M_1(x_1, y_1, t) \leq M_2(x_2, y_2, t)$, then

$$M((x_1, x_2), (y_1, y_2), t) + N((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) + N_1(x_1, y_1, t) \leq 1.$$

If $M_2(x_2, y_2, t) \leq M_1(x_1, y_1, t)$, then

$$\begin{aligned} M((x_1, x_2), (y_1, y_2), t) + N((x_1, x_2), (y_1, y_2), t) &= M_2(x_2, y_2, t) + N_1(x_1, y_1, t) \\ &\leq M_1(x_1, y_1, t) + N_1(x_1, y_1, t) \\ &\leq 1. \end{aligned}$$

Thus, in any case, we obtain

$$M((x_1, x_2), (y_1, y_2), t) + N((x_1, x_2), (y_1, y_2), t) \leq 1.$$

We are now going to verify condition (e) in Definition 2.3. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$ and $t, s > 0$. Without loss of generality, one may assume that $M((x_1, x_2), (y_1, y_2), t) = \min\{M_1(x_1, y_1, t), M_2(x_2, y_2, t)\} = M_1(x_1, y_1, t)$. If $M_1(y_1, z_1, s) \leq M_2(y_2, z_2, s)$, then

$$M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s) = M_1(x_1, y_1, t) * M_1(y_1, z_1, s) \leq M_1(x_1, z_1, t+s).$$

If $M_2(y_2, z_2, s) \leq M_1(y_1, z_1, s)$, then

$$\begin{aligned} M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s) &= M_1(x_1, y_1, t) * M_2(y_2, z_2, s) \\ &\leq M_2(x_2, y_2, t) * M_2(y_2, z_2, s) \\ &\leq M_2(x_2, z_2, t + s). \end{aligned}$$

So, we have

$$\begin{aligned} M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s) &\leq \min\{M_1(x_1, z_1, t+s), M_2(x_2, z_2, t+s)\} \\ &= M((x_1, x_2), (z_1, z_2), t + s). \end{aligned}$$

Using the same method as above, we may verify condition (j) in Definition 2.3.

Theorem 3.12 *Let $(X_i, M_i, N_i, *, \diamondsuit)(i = 1, 2)$ be two precompact intuitionistic fuzzy metric spaces. Then $(X_1 \times X_2, M, N, *, \diamondsuit)$ is a precompact intuitionistic fuzzy metric space.*

Proof Let $0 < r < 1$ and $t > 0$. Then, by hypothesis, there exist a finite subset A of X_1 and a finite subset B of X_2 such that $X_1 = \bigcup_{a \in A} B_{(M_1, N_1)}(a, r, t)$ and $X_2 = \bigcup_{b \in B} B_{(M_2, N_2)}(b, r, t)$, respectively. So

$$X_1 \times X_2 = \bigcup_{(a,b) \in A \times B} B_{(M,N)}((a, b), r, t).$$

In fact, let $(x,y) \in X_1 \times X_2$ and $(a,b) \in A \times B$. Then

$M_1(x,a,t) > 1-r$ and $M_2(y,b,t) > 1-r$ and $N_1(x,a,t) < r$ and

$N_2(y,b,t) < r$.
Hence

$$M((x,y),(a,b),t) > 1-r \qquad \text{and} \qquad N((x,y),(a,b),t) < r.$$

We are done.

According to Definition 2.9, we get

Theorem 3.13 *Let $(X_i, M_i, N_i, *, \diamondsuit)(i=1,2)$ be two complete intuitionistic fuzzy metric spaces. Then $(X_1 \times X_2, M, N, *, \diamondsuit)$ is a complete intuitionistic fuzzy metric space.*

The proof of Theorem 3.13 is straightforward and so it is omitted.

From Lemma 3.6 and Theorem 3.12,3.13, we obtain immediately the following corollary.

Corollary 3.14 *Let $(X_i, M_i, N_i, *, \diamondsuit)(i=1,2)$ be two compact intuitionistic fuzzy metric spaces. Then $(X_1 \times X_2, M, N, *, \diamondsuit)$ is a compact intuitionistic fuzzy metric space.*

4 Conclusion

We have shown that a topological space is (completely) metrizable if and only if it admits a compatible (complete) intuitionistic fuzzy metric. A topologically complete intuitionistic fuzzy metrizable space was explored. At last, we have established the intuitionistic fuzzy product metric space of two intuitionistic fuzzy metric spaces and studied several properties of the intuitionistic fuzzy product metric space, as precompactness ,completeness and compactness.

References

[1] C. Alaca, H. Efe and C. Yildiz, On the completion of intuitionistic fuzzy metric spaces, Chaos,Solitons & Fractals, To Appear.

[2] K. Atanassov, In intuitionstic fuzzy sets, In: V. Sgurev, editor, VII ITKR's Session, Sofia June, 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[3] H. Efe and C. Yildiz, On the Hausdorff intuitionistic fuzzy metric on compact sets, International Journal of Pure and Applied Mathematics 31 (2) (2006) 143-155.

[4] R. Engelking, General Topology, PWN-Polish Science Publishers,warsaw, 1977.

[5] M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69 (1979) 205-230.

[6] V. Gregori, S. Romaguera, and P. Veeramani, A note on intuitionistic fuzzy metric spaces, Chaos,Solitons & Fractals 28 (2006) 902-905.

[7] A. George and P. Veeramani, On some resules in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395-399.

[8] O. Kaleva and S. Seikkala, on fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215-229.

[9] I.Kramosil and J.Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 326-334.

[10] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos,Solitons & Fractals 22 (2004) 1039-1046.

[11] J. H. Park ,Y. B. Park and R. Saadati, Some results in intuitionistic fuzzy metric spaces, Journal of Computational Analysis and Applications 10 (4) (2008) 441-451.

[12] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos,Solitons & Fractals 27 (2006) 331-344.

[13] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J Math 10 (1960) 314-334.

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A Certain Class of Harmonic Mappings Related to Functions of Bounded Boundary Rotation

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Abstract

Let $V(k)$ be the class of functions with bounded boundary rotation and let S_H be the class of sense-preserving harmonic mappings. In the present paper we investigate a certain class of harmonic mappings related to the function of bounded boundary rotation.

1 Introduction

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Moreover, let A be the class of functions in the open unit disc $\mathbb{D}$ that are normalized with $h(0) = h'(0) - 1 = 0$, then a function $h(z) \in A$ is called convex on starlike if it maps $\mathbb{D}$ into a convex or starlike region, respectively. Corresponding classes are denoted by $\mathbb{C}$ and S^*. It is well known that $\mathbb{C} \subset S^*$,

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that both are subclasses of the univalent functions and have the following analytical representations

$$h(z) \in \mathbb{C} \text{ if and only if } \operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > 0,\ z \in \mathbb{D}, \tag{1.2}$$

and

$$h(z) \in S^* \text{ if and only if } \operatorname{Re}\left(z\frac{h'(z)}{h(z)}\right) > 0,\ z \in \mathbb{D}. \tag{1.3}$$

More on these classes can be found in [1]. Let $h(z)$ be an element of A. If there is a function $s(z)$ in $\mathbb{C}$ and a real β such that

$$\operatorname{Re}\left(\frac{h'(z)}{e^{i\beta}s'(z)}\right) > 0,\ z \in \mathbb{D} \tag{1.4}$$

then $h(z)$ is called a close-to-convex function in $\mathbb{D}$, and the class of such functions is denoted by CC.

Further, let $h(z)$, $g(z) \in A$. Then we say that $h(z)$ is subordinate to $g(z)$ and we write $h(z) \prec g(z)$. If there exists a function $\phi(z) \in \Omega$ such that $h(z) = g(\phi(z))$ for all $z \in \mathbb{D}$. Specially if $g(z)$ is univalent in $\mathbb{D}$, then $h(z) \prec g(z)$ if and only if $h(0) = g(0)$, $h(\mathbb{D}) \subset g(\mathbb{D})$, implies $h(\mathbb{D}_r) \subset g(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z||z| < r, 0 < r < 1\}$ (Subordination and Lindelof Principle [1]).

In the terms of subordination we have

$$\mathcal{P} = \Big\{p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n | p(z) \text{ regular in } \mathbb{D}, p(z) \prec \frac{1+z}{1-z}\Big\}, \tag{1.5}$$

$$S^* = \Big\{h(z) \in A | z\frac{h'(z)}{h(z)} \prec \frac{1+z}{1-z}\Big\}, \tag{1.6}$$

$$C = \Big\{h(z) \in A | \left(1 + z\frac{h''(z)}{h'(z)}\right) \prec \frac{1+z}{1-z}\Big\}, \tag{1.7}$$

and

$$CC = \Big\{h(z), s(z) \in A | \frac{h'(z)}{e^{i\beta}s'(z)} \prec \frac{1+z}{1-z}, s(z) \in C\Big\}. \tag{1.8}$$

Finally, a function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit.

Let $V(k)$ denote the class of functions $h(z) \in A$ which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$. The class of functions of bounded boundary rotation was introduced by Loewner [4] in 1917, and was developed by Paatero [6], [7] who systematically developed their properties and made an exhaustive study of the class $V(k)$. Paatero has shown that $h(z) \in V(k)$ if and only if

$$h'(z) = \exp\Big\{ - \int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t) \Big\} \tag{1.9}$$

where $\mu(t)$ is real valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leq k. \tag{1.10}$$

For fixed $k \geq 2$, it can also be expressed as

$$\int_0^{2\pi} \left| Re \frac{(zh'(z))'}{h'(z)} \right| d\theta \leq k\pi, z = re^{i\theta}. \tag{1.11}$$

Clearly if $k_1 \leq k_2$, then $V(k_1) \subset V(k_2)$, that is the class $V(k)$ obviously expands on k increases. $V(2)$ is the class of C of convex univalent functions. Paatero showed that $V(4) \subset S$, where S is the class of normalized univalent functions [1]. Later Pinchuk [8] proved that functions in $V(k)$ are close-to-convex in $\mathbb{D}$ if $2 \leq k \leq 4$. More details on the functions with bounded boundary rotation can be found in [5].

A planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n, \ z \in \mathbb{D}.$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \cdots$. As in usual we call $h(z)$ is analytic part of f and $g(z)$ is co-analytic part of f. An elegant and complete account of the theory of the theory of harmonic mappings is given in Duren's monograph [2].

Lewy [2] proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its jacobien $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc $\mathbb{D}$, and the second complex dilatation $w(z) = \left(\frac{g'(z)}{h'(z)}\right)$ has the property $|w(z)| < 1$ in $\mathbb{D}$.

The class of all sense-preserving harmonic mappings of the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ and will be denoted by S_H. Thus S_H contains the standard class S of analytic univalent functions. The family of all mappings $f \in S_H$ with the additional property that $g'(0) = 0$, i.e., $b_1 = 0$ is denoted by S_H^0. Thus it is clear that $S \subset S_H^0 \subset S_H$ [2].

Now, we consider the following class of harmonic mappings

$$S_{HV(k)} = \left\{ f = h(z) + \overline{g(z)} \mid \frac{g'(z)}{h'(z)} \prec e^{i\beta} b_1 \left(\frac{1+z}{1-z}\right)^{\alpha}, \alpha = \frac{k}{2} - 1, \beta \in \mathbb{R}, h(z) \in C \right\} \tag{1.12}$$

the aim of this paper is to investigate the class $S_{HV(k)}$. For this aim we need the following lemma and theorems.

Lemma 1.1. *[3] Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$. Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.2. *[1] If $h(z)$ is in $V(k)$, then there is a $p(z)$ such that*

$$h'(z) = e^{i\beta}(p(z))^{\alpha}.g'(z) \tag{1.13}$$

where β is real, $g(z)$ is in C and $p(z) = \sum_{n=0}^{\infty} p_n z^n$ has positive real part in $\mathbb{D}$. Here $e^{i\beta}p_0^{\alpha} = 1$, $\alpha = \frac{k}{2} - 1$.

Theorem 1.3. *[1] Let $h(z)$ be an element of C, then*

$$\frac{r}{1+r} \leq |h(z)| \leq \frac{r}{1-r}$$

and

$$\frac{r}{(1+r)^2} \leq |h'(z)| \leq \frac{r}{(1-r)^2}$$

for all $|z| = r < 1$.

2 Main Results

Theorem 2.1. *Let $f = h(z) + \overline{g(z)}$ be an elemet of $S_{HV(k)}$, then*

$$\frac{g(z)}{h(z)} \prec e^{i\beta} b_1 \left(\frac{1+z}{1-z}\right)^{\alpha},\ z \in \mathbb{D}$$

where $\beta \in R$ and $\alpha = \frac{k}{2} - 1$.

Proof. Since $f = h(z) + \overline{g(z)} \in S_{HV(k)}$, then we have

$$\frac{g'(z)}{h'(z)} \prec e^{i\beta} b_1 \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{2.1}$$

and

$$\mathrm{Re}\left(z\frac{g'(z)}{h'(z)}\right) > \frac{1}{2} \Rightarrow z\frac{h'(z)}{h(z)} = \frac{1}{1-\phi(z)} \Rightarrow \frac{h(z)}{zh'(z)} = (1-\phi(z)). \tag{2.2}$$

On the other hand, if we investigate the properties of the linear transformation $w(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $\alpha = \frac{k}{2} - 1$, $k \geq 2$ and using the subordination and Lindelof Principle with $0 < r < 1$, $\frac{1+r}{1-r} > 1$, $0 < \frac{1-r}{1+r} < 1$, $p(z) \in P \Rightarrow (p(z))^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$, we get

$$\left(\frac{1-r}{1+r}\right)^{\alpha} \leq \left(\frac{1-r}{1+r}\right) \leq |p(z)|^{\alpha} \leq \left(\frac{1+r}{1-r}\right) \leq \left(\frac{1+r}{1-r}\right)^{\alpha}. \tag{2.3}$$

See Figure 1.

Now, we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = e^{i\beta} b_1 \left(p_0 \frac{1+\phi(z)}{1-\phi(z)}\right)^{\frac{k}{2}-1}$$

where $f = h(z) + \overline{g(z)} \in S_{HV(k)}$, $\beta \in R$, $e^{i\beta} p_0^{\frac{k}{2}-1} = 1$. Therefore we have $\frac{g'(z)}{h'(z)}|_{z=0} = b_1$, $1 = \frac{1+\phi(0)}{1-\phi(0)} \Rightarrow \phi(0) = 0$, $\phi(z)$ analytic, and

$$\begin{aligned} w(z) = \frac{g'(z)}{h'(z)} &= \frac{g(z)}{h(z)}\left(1 + \frac{2z\phi'(z)}{1+\phi(z)}\right) \\ &= e^{i\beta} b_0 \left(\frac{1+\phi(z)}{1-\phi(z)}\right)^{\frac{k}{2}-1} \left(1 + \frac{2z\phi'(z)}{1+\phi(z)}\right). \end{aligned} \tag{2.4}$$

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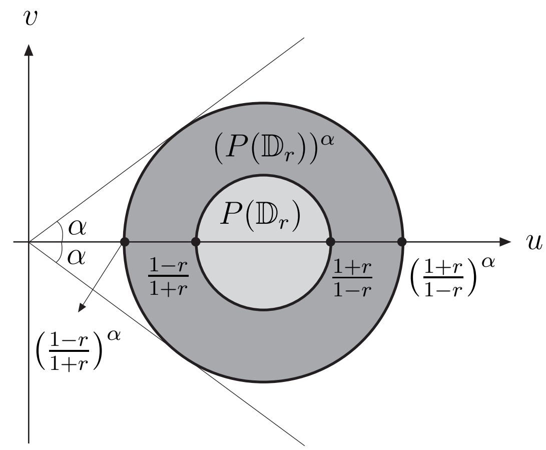

Figure 1:

Now, it is easy to realize that the subordination

$$\frac{g'(z)}{h'(z)} \prec e^{i\beta} b_1 (\frac{1+z}{1-z})^{\frac{k}{2}-1}$$

(from the definition of $S_{HV(k)}$) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed assume the contrary that there exists a $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. Then by I. S. Jack's lemma (Lemma 1.1) $z_1\phi'(z_1) = k\phi(z_1)$, $k \geq 1$ such z_1 we have

$$\begin{aligned} w(z_1) = \frac{g'(z_1)}{h'(z_1)} &= \frac{g(z_1)}{h(z_1)} \left(1 + \frac{2k\phi(z_1)}{1+\phi(z_1)}\right) \\ &= e^{i\beta} b_1 \left(p_0 \frac{1+\phi(z_1)}{1-\phi(z_1)}\right)^{\frac{k}{2}-1} \left(1 + \frac{2k\phi(z_1)}{1+\phi(z_1)}\right) \\ &= e^{i\beta} b_1 w(\phi(z_1)) \notin w(\mathbb{D}) \end{aligned}$$

because $|\phi(z_1)| = 1$, $k \geq 1$ and the relations (2.3). But this is a contradiction to the condition of the definition of $S_{HV(k)}$ and so assumption is wrong, i.e., $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

Corollary 2.2. *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HV(k)}$, then*

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \leq |g'(z)| \leq \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}, \tag{2.5}$$

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and

$$\frac{r(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}}} \leq |g(z)| \leq \frac{r(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}}} \tag{2.6}$$

for all $|z| = r < 1$.

Proof. Since $f = h(z) + \overline{g(z)} is \in S_{HV(k)}$, thus using Theorem 2.1 then we can write

$$\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}-1} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \left(\frac{1+r}{1-r}\right)^{\frac{k}{2}-1},$$

and

$$\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}-1} \leq \left|\frac{g(z)}{h(z)}\right| \leq \left(\frac{1+r}{1-r}\right)^{\frac{k}{2}-1}$$

for all $|z| = r < 1$. In this step, if we use Theorem 1.3 we get (2.5) and (2.6). We also note that the inequality (2.5) well known which was proved by Paatero [6]. $\square$

Corollary 2.3. *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $S_{HV(k)}$, *then*

$$F(-r) \leq J_f \leq F(r), \tag{2.7}$$

and

$$\frac{1}{4-k}\left[1-\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}(k-4)}\right]-\frac{1}{1+r} \leq |f| \leq \frac{1}{k-4}\left[\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}(k-4)}-1\right]+\frac{1}{1-r} \tag{2.8}$$

where

$$F(r) = \frac{1}{(1-r)^4}\left[1-\left(\frac{1-r}{1+r}\right)^{k-2}\right]$$

for all $|z| = r < 1$.

Proof. Since

$$\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}-1} \leq |w(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \left(\frac{1+r}{1-r}\right)^{\frac{k}{2}-1},$$

then

$$|h'(z)|^2\left[1-\left(\frac{1+r}{1-r}\right)^{k-2}\right] \leq |h'(z)|^2(1-|w(z)|^2) \leq |h'(z)|^2\left[1-\left(\frac{1-r}{1+r}\right)^{k-2}\right]. \tag{2.9}$$

Using Theorem 1.3 in the equality (2.9) we get (2.7). Similarly

$$(|h'(z)| - |g'(z)|)\,|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)\,|dz| \Rightarrow$$

$$|h'(z)|\,(1 - |w(z)|)\,|dz| \leq |df| \leq |h'(z)|\,(1 + |w(z)|)\,|dz| \Rightarrow$$

$$\frac{1}{(1+r)^2}\left[1 - \left(\frac{1+r}{1-r}\right)\right]^{\frac{k}{2}-1} dr \leq |df|\,\frac{1}{(1-r)^2}\left[1 + \left(\frac{1+r}{1-r}\right)\right]^{\frac{k}{2}-1} dr$$

which gives (2.8). □

References

[1] A. W. Goodman, *Univalent Functions*, Vol. I, II, Polygonal publishing Hours, Washington, New-Jersey, (1983).

[2] P. Duren, *Harmonic Mappings in the plane*, Vol. 156, of Cambridge Tract in Mathematics, Cambridge University press, Cambridge, UK, 2004.

[3] I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. 34(1959),215-216.

[4] C. Loewner, *Untersuchungen Uberdie Verzerrung bie Konformen Abbildungen des Einheitskeisez, $|z| < 1$, die durch Funktionen mit nicht verschwinden der Ableitung geliefert Werden*,Ber. Verh. Sachs. Ges. Wiss. Leipzig, 69 (1917), 89-106.

[5] K. I. Noor, B. Malik and S. Mustafa, *A survey on functions of Bounded Boundary and Bounded Radius Rotation*, Applied Mathematics, E-Notes 12, (2012), 136-152.

[6] V. Paatero, *Uber die Konforme Abbildungen Von Gebietten, deren Rander von Beschrankter Drehung* , Sind. Ann. Acad. Sci. Feen. ser. A ,33 (1931).

[7] V. Paatero, *Uber Gebiete von beschrankter randdrehung*, Ann. Acad. Sci. Fenn. Ser A. , 37 (1933).

[8] B. Pinchuk, *Functions with bounded boundary rotation* , Isr. J. Math 10, (1971), 7-16.

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Fuzzy norms on BCK-algebras and non-negativity of norms in algebras†

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Abstract. In this paper, we discuss some fuzzy norms on BCK-algebras, and we find several conditions for norms to be non-negative in algebras. Finally we discuss fuzzy stable norms on several algebras.

1. **Introduction**

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7, 8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: A. Dvurečenskij and M. G. Graziano ([2]), C. S. Hoo ([6]), J. M. Font, A. J. Rodrígez and A. Torrens ([3]) discussed BCK-algebras in connection with the areas of lattice ordered groups, MV-algebras and Wajsberg algebras. D. Mundici ([12]) proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng ([11]) proved that implicative commutative semigroups are equivalent to a class of BCK-algebras.

J. Neggers and H. S. Kim ([15]) introduced a new notion which appears to be of some interest, i.e., that of a B-algebra, and studied some of its properties. J. Neggers and H. S. Kim ([14]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. After that some further aspects were studied (see [9, 10, 13]). P. J. Allen et al. ([1]) developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type.

J. S. Han et al. ([4]) introduced several triangular norms in an arbitrary algebra, and investigated some conditions for the kernel $Ker\bigtriangledown$ of a triangular norm $\bigtriangledown$ to be a d^*-ideal, and as an application they constructed a quotient d-algebra. J. S. Han et al. ([5]) introduced the notion of an action Y_X as a generalization of the notion of a module, and obtained that the set of all fuzzy norms on Y_X forms a commutative monoid.

In this paper, we discuss some fuzzy norms on BCK-algebras, and we find several conditions for norms to be non-negative in algebras. Finally we discuss fuzzy stable norms on several algebras.

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2. **Preliminaries**

An *algebra* (or a *groupoid*) $(X, *)$ is a non-empty set X equipped with a binary operation $*$ on X. An algebra $(X, *)$ is called a *d-algebra* ([14]) if there is a constant 0 satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

We denote it by $(X, *, 0)$ or X for brevity. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $((x * (x * y)) * y = 0$ for all $x, y, z \in X$.

J. S. Han et al. ([4]) introduced several (triangular) norms, and obtained some properties. Given an algebra $(X, *)$, we consider mappings $\bigtriangledown : X \to \mathbf{R}$ such that one of the following identities holds:

$(T1)$ $\bigtriangledown(x) \leq \bigtriangledown(y) + \bigtriangledown(x * y)$,

$(T2)$ $\bigtriangledown(x * y) \leq \bigtriangledown(x) + \bigtriangledown(y)$,

$(T3)$ $\bigtriangledown(x * z) \leq \bigtriangledown(x * y) + \bigtriangledown(y * z)$,

$(T4)$ $\bigtriangledown((x * z) * (y * z)) + \bigtriangledown((z * x) * (z * y)) \leq \bigtriangledown(x * y) + \bigtriangledown(y * x)$,

for any $x, y, z \in X$. All of $(T1), (T2), (T3), (T4)$ are versions of the "usual" triangle inequality, and thus we shall refer to a mapping $\bigtriangledown : X \to \mathbf{R}$ which satisfies an inequality (Ti) above as a *triangular norm of type* (Ti) when $Ker\bigtriangledown := \{x \in X \mid \bigtriangledown(x) = 0\} \neq \emptyset$ $(i = 1, 2, 3, 4)$. If we don't require that $Ker\bigtriangledown \neq \emptyset$, then we shall refer to it as a *norm* of type (Ti) (or a (Ti)-*norm*). A mapping $\bigtriangledown : X \to \mathbf{R}$ is said to be *stable* if $\bigtriangledown(x * y) \leq \bigtriangledown(x)$ for any $x, y \in X$. In particular, if we replace $\mathbf{R}$ by $[0, 1]$, we may consider $\bigtriangledown$ to be a *fuzzy (triangular) norm of type* (Ti) $(i = 1, 2, 3, 4)$.

Example 2.1. ([4]) Let X be the power set of a finite set F, i.e., $x \in X$ means $x \subseteq F$. If $\bigtriangledown(x) = |\,x\,|$, the cardinality of x, and $x * y := x - y$, the collection of all elements of x not in y, then $\bigtriangledown(\emptyset) = 0$, i.e., $Ker\bigtriangledown \neq \emptyset$. Also, $\bigtriangledown(x * y) \leq \bigtriangledown(x)$, i.e., $\bigtriangledown$ is stable, whence certainly $\bigtriangledown(x * y) \leq \bigtriangledown(x) + \bigtriangledown(y)$, i.e., it is a triangular norm of type $(T2)$. Note that $x = (x * y) \cup (x \cap y)$, $\bigtriangledown(x) = \bigtriangledown(x * y) + \bigtriangledown(x \cap y)$ and $\bigtriangledown(x \cap y) = \bigtriangledown(y * x^C) \leq \bigtriangledown(y)$, so that $\bigtriangledown(x) \leq \bigtriangledown(y) + \bigtriangledown(x * y)$ as well, and $\bigtriangledown$ is a triangular norm of type $(T1)$. Finally, $x * z \subseteq (x * y) \cup (y * z)$ means that $\bigtriangledown(x * z) \leq \bigtriangledown(x * y) + \bigtriangledown(y * z)$, i.e., $\bigtriangledown$ is a triangular norm of type $(T3)$. Since $[(x * z) * (y * z)] \cup [(z * x) * (z * y)] \subseteq (x * y) \cup (y * x)$, we have $\bigtriangledown((x * z) * (y * z)) + \bigtriangledown((z * x) * (z * y)) \leq \bigtriangledown(x * y) + \bigtriangledown(y * x)$, i.e., $\bigtriangledown : X \to \mathbf{R}$ is a stable and triangular norm of types $(T1) \sim (T4)$.

Example 2.2. ([4]) If $X = \mathbf{R}$ and $x * y := \max\{0, x - y\}$, $\forall x, y \in X$, then $\bigtriangledown(x) = |\,x\,|$ satisfies $(T2)$. Indeed, if $x \leq y$, then $x * y = 0$ and $\bigtriangledown(x * y) = 0 \leq \bigtriangledown(x) + \bigtriangledown(y)$. If $x > y$, then $\bigtriangledown(x * y) = x - y \leq |\,x - y\,| \leq |x| + |y| = \bigtriangledown(x) + \bigtriangledown(y)$. On the other hand, if $x = -5, y = -1, x - y = -4$ and $x * y = 0, \bigtriangledown(x) = 5 \not\leq \bigtriangledown(y) + \bigtriangledown(x * y)$, so that $(T1)$ fails to hold. Also, if $z = 0$ and if $(T3)$ holds, then $\bigtriangledown(x) \leq \bigtriangledown(x * y) + \bigtriangledown(y)$, which means $(T1)$ holds. Thus $(T3)$ fails, since $(T1)$ fails. Finally, if $x = 5, y = -1, x * y = 6$ and $\bigtriangledown(x * y) > \bigtriangledown(x)$, i.e., the mapping $\bigtriangledown$ is not stable.

Example 2.3. ([4]) Let $X := \{0, 1, 2, 3\}$ be an algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then it is a d-algebra. If we define a map $\triangledown : X \to [0, 1]$ by $0 = \triangledown(0) < \triangledown(1) < \triangledown(2) < \triangledown(3) \le 1$, then it is easy to show that $\triangledown$ is a fuzzy triangular norm of types $(T1) \sim (T3)$, but not of type $(T4)$, since $\triangledown((3 * 1) * (2 * 1)) + \triangledown((1 * 3) * (1 * 2)) > \triangledown(3 * 2) + \triangledown(2 * 3)$.

3. **Fuzzy norms on BCK-algebras**

Proposition 3.1. *Let $(X, *)$ be an algebra and let $z_0 \in X$ such that $x * z_0 = x$ for all $x \in X$. Then every fuzzy $(T3)$-norm on $(X, *)$ is a fuzzy $(T1)$-norm.*

Proof. If $\triangledown$ is a fuzzy $(T3)$-norm on X, then for any $x, y \in X$, we have

$$\triangledown(x * z_0) \le \triangledown(x * y) + \triangledown(y * z_0)$$

It follows that $\triangledown(x) \le \triangledown(x * y) + \triangledown(y)$, i.e., $\triangledown$ is a fuzzy $(T1)$-norm on $(X, *)$. □

Corollary 3.2. *If $(X, *, 0)$ is a BCK-algebra, then every fuzzy $(T3)$-norm on X is a fuzzy $(T1)$-norm on X.*

Proof. In a $BCK/BCI/BF$-algebra $(X, *, 0)$, $x * 0 = x$ holds for any $x \in X$. □

Proposition 3.3. *Let $(X, *, 0)$ be a BCK-algebra and let $\triangledown$ be a fuzzy $(T1)$-norm with $\triangledown(0) = 0$. If $x \le y$ in X, then $\triangledown(x) \le \triangledown(y)$.*

Proof. Let $\triangledown$ be a fuzzy $(T1)$-norm with $\triangledown(0) = 0$. If $x \le y$ in X, then $\triangledown(x) \le \triangledown(y) + \triangledown(x*y) = \triangledown(y) + \triangledown(0) = \triangledown(y)$. □

Proposition 3.4. *Let $(X, *, 0)$ be a BCK-algebra and let $\triangledown$ be a fuzzy $(T3)$-norm on X with $\triangledown(0) = 0$. If $x \le y$ in X, then $\triangledown(x * z) \le \triangledown(y * z)$ for any $z \in X$.*

Proof. Given $z \in X$, we have $\triangledown(x * z) \le \triangledown(x * y) + \triangledown(y * z) = \triangledown(0) + \triangledown(y * z) = \triangledown(y * z)$, proving the proposition. □

4. **Non-negative norms**

A map $\triangledown : X \to \mathbf{R}$ is said to be *non-negative* if $\triangledown(x) \ge 0$ for all $x \in X$. Fortunately as we shall see, it is easy for norms to be non-negative.

Proposition 4.1. *Let $(X, *)$ be an algebra. If $\triangledown$ is a norm of types $(T1)$ and $(T2)$, then $\triangledown$ is non-negative.*

Proof. Assume $\triangledown$ is a norm of types $(T1)$ and $(T2)$. If we let $y := x$ in $(T1)$, then $\triangledown(x) \le \triangledown(x) + \triangledown(x * x)$ for any $x \in X$. It follows that $0 \le \triangledown(x*x)$ for any $x \in X$. Since $\triangledown$ is a $(T2)$-norm, we obtain $\triangledown(x*x) \le \triangledown(x)+\triangledown(x)$. Hence $0 \le \triangledown(x * x) \le 2 \triangledown(x)$, proving that $\triangledown(x) \ge 0$. □

Note that the converse of Proposition 4.1 need not be true in general. In Example 3.2, the mapping $\bigtriangledown(x) = |x|$ is non-negative, but it is not a $(T1)$-norm on X.

Proposition 4.2. *Let $(X, *)$ be an algebra. If $\bigtriangledown$ is a norm of types $(T2)$ and $(T3)$, then $\bigtriangledown$ is non-negative.*

Proof. Let $\bigtriangledown$ be a norm of types $(T2)$ and $(T3)$. If we let $z := x, y := x$ in $(T3)$, then $\bigtriangledown(x*x) \le \bigtriangledown(x*x)+\bigtriangledown(x*x)$ for any $x \in X$, which means that $0 \le \bigtriangledown(x * x)$ for any $x \in X$. If we let $y := x$ in $(T2)$, then $\bigtriangledown(x * x) \le \bigtriangledown(x) + \bigtriangledown(x) = 2 \bigtriangledown (x)$. Hence $0 \le \frac{1}{2} \bigtriangledown (x * x) \le \bigtriangledown(x)$, proving that $\bigtriangledown$ is non-negative. □

In Proposition 4.1, it was necessary to introduce two types of norms in a groupoid so that the norm is non-negative. If we add some condition(s) in a groupoid, then we can find each types of norms can be non-negative.

Proposition 4.3. *Let $(X, *, 0)$ be an algebra with $0 * x = 0$ for all $x \in X$.*

(i). *If $\bigtriangledown$ is either a $(T1)$-norm or a $(T2)$-norm on X, then $\bigtriangledown$ is non-negative,*
(ii). *If $\bigtriangledown$ is a $(T3)$-norm on X, then $\bigtriangledown(y * z) \ge 0$ for all $y, z \in X$.*

Proof. (i). If we let $x := 0$ in $(T1)$, then $\bigtriangledown(0) \le \bigtriangledown(y) + \bigtriangledown(0 * y) = \bigtriangledown(y) + \bigtriangledown(0)$ and hence $0 \le \bigtriangledown(y)$ for all $y \in X$.

If we let $x := 0$ in $(T2)$, then $\bigtriangledown(0) = \bigtriangledown(0 * y) \le \bigtriangledown(0) + \bigtriangledown(y)$ and hence $0 \le \bigtriangledown(y)$ for all $y \in X$.

(ii). If we let $x := 0$ in $(T3)$, then $\bigtriangledown(0 * z) \le \bigtriangledown(0 * y) + \bigtriangledown(y * z)$ for any $y, z \in X$. Since $0 * x = 0$ for all $x \in X$, we obtain $\bigtriangledown(0) \le \bigtriangledown(0) + \bigtriangledown(y * z)$, proving that $0 \le \bigtriangledown(y * z)$ for all $y, z \in X$. □

Example 4.4. Every $(T1)$-norm or $(T2)$-norm on a BCK-algebra is non-negative. Moreover, every $(T3)$-norm on a BCK-algebra is also non-negative, since $x * 0 = x$ for all $x \in X$.

Corollary 4.5. *Let $(X, *, 0)$ be an algebra with $0 * x = 0$ for all $x \in X$. Assume that there exist $y, z \in X$ such that $x = y * z$ for any $x \in X$. Then every $(T3)$-norm is non-negative.*

Proof. It follows immediately from Proposition 4.3-(ii). □

Proposition 4.6. *Let $(X, *, 0)$ be an algebra with $x * x = x$ for all $x \in X$. If $\bigtriangledown$ is a $(T2)$-norm on X, then it is non-negative.*

Proof. If $\bigtriangledown$ is a $(T3)$-norm on X, then $\bigtriangledown(x) = \bigtriangledown(x * x) \le \bigtriangledown(x * x) + \bigtriangledown(x * x) = 2 \bigtriangledown (x)$ and hence $0 \le \bigtriangledown(x)$ for all $x \in X$. □

Theorem 4.7. *Let $(X, \cdot, e)$ be a group all of whose elements are of finite order. Define a binary operation "$*$" on X by $x * y := x \cdot y^{-1}$ for all $x, y \in X$. Then every $(T1)$-norm on $(X, *)$ is non-negative.*

Proof. If $\bigtriangledown$ is a $(T1)$-norm on $(X, *)$, then $\bigtriangledown(x) \le \bigtriangledown(y) + \bigtriangledown(x * y)$ for all $x, y \in X$. Since $x * y = x \cdot y^{-1}$, we obtain

$$\bigtriangledown(x) \le \bigtriangledown(y) + \bigtriangledown(x \cdot y^{-1}) \tag{1}$$

If we let $y := e$, the identity of X in (1), then we have $\bigtriangledown(x) \le \bigtriangledown(e) + \bigtriangledown(x \cdot e^{-1}) = \bigtriangledown(e) + \bigtriangledown(x)$, which proves that $0 \le \bigtriangledown(e)$.

Given $x \in X$, we let the order of x be finite, i.e., $o(x) := n < \infty$. Then we have

$$\begin{aligned}\triangledown(x^n) &\leq \triangledown(x) + \triangledown(x^n * x)\\ &= \triangledown(x) + \triangledown(x^n \cdot x^{-1})\\ &= \triangledown(x) + \triangledown(x^{n-1})\end{aligned}$$

By induction, we obtain $0 \leq \triangledown(e) = \triangledown(x^n) \leq n \triangledown (x)$ for any $x \in X$. This proves the theorem. □

The non-negativity of a $(T2)$-norm on an arbitrary group holds in general as follows:

Theorem 4.8. *Let $(X, \cdot, e)$ be a group. Define a binary operation "$*$" on X by $x * y := x \cdot y^{-1}$ for all $x, y \in X$. Then every $(T2)$-norm on $(X, *)$ is non-negative.*

Proof. Since $\triangledown$ is a $(T2)$-norm on $(X, *)$, we have $\triangledown(x * y) \leq \triangledown(x) + \triangledown(y)$ for any $x, y \in X$. It follows that

(2)
$$\triangledown(x \cdot y^{-1}) \leq \triangledown(x) + \triangledown(y)$$

If we let $Y := x$ in (2), then $\triangledown(x \cdot x^{-1}) \leq \triangledown(x) + \triangledown(x)$, i.e., $\triangledown(e) \leq 2 \triangledown (x)$ for all $x \in X$. If we let $y := e$ in (2), then $\triangledown(x) = \triangledown(x \cdot e^{-1}) \leq \triangledown(x) + \triangledown(e)$ and hence $0 \leq \triangledown(e)$. Hence $0 \leq \triangledown(x)$ for all $x \in X$, proving the theorem. □

J. Neggers and H. S. Kim introduced the notion of B-algebras. A *B-algebra* ([15]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (i) $x * x = 0$, (ii) $x * 0 = x$, (iii) $(x * y) * z = x * (z * (0 * y))$ for all x, y, z in X.

Proposition 4.9. *Let $(X, *, 0)$ be a B-algebra. If $\triangledown$ is a $(T1)$-norm on X with $\triangledown(x) = \triangledown(0 * x)$ for any $x \in X$, then it is non-negative.*

Proof. Let $\triangledown$ be a $(T1)$-norm on X with $\triangledown(x) = \triangledown(0 * x)$ for any $x \in X$. Then

(3)
$$\triangledown(x) \leq \triangledown(y) + \triangledown(x * y)$$

for any $x, y \in X$. If we let $x := 0$, then $\triangledown(0) \leq \triangledown(y) + \triangledown(0 * y) = 2 \triangledown (y)$ for any $y \in X$. If we let $x := y$ in (3), then $\triangledown(y) \leq \triangledown(y) + \triangledown(y * y) = \triangledown(y) + \triangledown(0)$, i.e., $0 \leq \triangledown(0)$. This means that $0 \leq\leq \triangledown(0) \leq 2 \triangledown (y)$, proving the proposition. □

Proposition 4.10. *If $(X, *, 0)$ is a B-algebra, then every $(T2)$-norm of X is non-negative.*

Proof. If $\triangledown$ is a $(T2)$-norm on X, then for any $x, y \in X$, we have

(4)
$$\triangledown(x * y) \leq \triangledown(x) + \triangledown(y)$$

If we let $x := y$ in (4), then $\triangledown(0) = \triangledown(y * y) \leq \triangledown(y) + \triangledown(y) = 2 \triangledown (y)$, i.e., $\triangledown(0) \leq 2 \triangledown (y)$ for all $y \in X$. If we let $y := 0$ in (4), then $\triangledown(x) = \triangledown(x * 0) \leq \triangledown(x) + \triangledown(0)$, which shows that $0 \leq \triangledown(0)$. Hence $\triangledown$ is non-negative. □

5. **Fuzzy stable norms**

In this section we discuss the notion of fuzzy stable norms in groupoids.

Proposition 5.1. *Let $(X,*)$ be a groupoid. If $\bigtriangledown$ is a fuzzy stable $(T1)$-norm on $(X,*)$, then it is a fuzzy $(T2)$-norm on $(X,*)$.*

Proof. Since $\bigtriangledown$ is stable, we have $\bigtriangledown(x*y) \le \bigtriangledown(x) \le \bigtriangledown(x)+\bigtriangledown(y)$ for all $x,y \in X$, i.e., $\bigtriangledown$ is a fuzzy $(T2)$-norm on $(X,*)$. □

Proposition 5.2. *Let $(X,\cdot,e)$ be a group. If $\bigtriangledown$ is a fuzzy stable norm on $(X,\cdot)$, then $\bigtriangledown$ is a constant mapping on X.*

Proof. Since $\bigtriangledown$ is a fuzzy stable norm on $(X,\cdot)$, we have

$$\bigtriangledown(x\cdot y) \le \bigtriangledown(x) \tag{5}$$

for all $x,y\in X$. If we let $x:=e$ in (5), then $\bigtriangledown(e\cdot y)\le\bigtriangledown(e)$ for all $y\in X$, i.e., $\bigtriangledown(y)\le\bigtriangledown(e)$. If we let $y:=x^{-1}$ in (5), then $\bigtriangledown(e)=\bigtriangledown(x\cdot x^{-1})\le\bigtriangledown(x)$ for all $x\in X$. Hence $\bigtriangledown(x)=\bigtriangledown(e)$, i.e., $\bigtriangledown$ is a constant mapping on $(X,\cdot)$. □

Theorem 5.3. *Let $(X,*,0)$ be a BCK-algebra. If $\bigtriangledown$ is a fuzzy $(T1)$-norm on $(X,*,0)$ with $\bigtriangledown(0)=0$, then it is a fuzzy stable $(T3)$-norm on $(X,*,0)$.*

Proof. If $(X,*,0)$ is a BCK-algebra, then $(x*y)*x=0$ for all $x,y\in X$. Since $\bigtriangledown$ is a fuzzy $(T1)$-norm on $(X,*,0)$, we have

$$\begin{aligned}\bigtriangledown(x*y) &\le \bigtriangledown(x) + \bigtriangledown((x*y)*x)\\ &= \bigtriangledown(x) + \bigtriangledown(0)\\ &= \bigtriangledown(x)\end{aligned}$$

for all $x,y\in X$, i.e., $\bigtriangledown$ is stable.

Since $((x*y)*(x*z))*(z*y)=0$ holds always in any BCK-algebra and $\bigtriangledown$ is a fuzzy $(T1)$-norm on X, we obtain

$$\begin{aligned}\bigtriangledown((x*y)*(x*z)) &\le \bigtriangledown(z*y) + \bigtriangledown(((x*y)*(x*z))*(z*y))\\ &= \bigtriangledown(z*y) + \bigtriangledown(0)\\ &= \bigtriangledown(z*y)\end{aligned}$$

for any $x,y,z\in X$. It follows from $\bigtriangledown$ is a fuzzy $(T1)$-norm on X that

$$\begin{aligned}\bigtriangledown(x*y) &\le \bigtriangledown(x*z) + \bigtriangledown((x*y)*(x*z))\\ &\le \bigtriangledown(x*z) + \bigtriangledown(z*y)\\ &= \bigtriangledown(z*y)\end{aligned}$$

for all $x,y,z\in X$, proving that $\bigtriangledown$ is a fuzzy $(T3)$-norm on $(X,*,0)$. □

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References

[1] P. J. Allen, H. S. Kim and J. Neggers, Companion d-algebras, *Math. Slovaca* 57 (2007), 93–106.

[2] A. Dvurečenskij and M. G. Graziano, Commutative BCK-algebras and lattice ordered groups, *Math. Japonica* 32 (1999), 227–246.

[3] J. M. Font, A. J. Rodrígez and A. Torrens, Wajsberg algebras, *Stochastica* 8 (1984), 5–31.

[4] J. S. Han, H. S. Kim and J. Neggers, Constructions of quotient algebras via triangular norms, *Math. Slovaca* 60 (2010), 1-10.

[5] J. S. Han, H. S. Kim and J. Neggers, Actions, norms, subactions and kernels of (fuzzy) norms, *Iranian J. Fuzzy Systems* 7 (2010), 141-147.

[6] C. S. Hoo, Unitary extensions of MV and BCK-algebras, *Math. Japonica* 37 (19xx), 585–590.

[7] K. Iséki, On BCI-algebras, *Math. Seminar Notes* 8 (1980), 125–130.

[8] K. Iséki and S. Tanaka, An introduction to theory of BCK-algebras, *Math. Japonica* 23 (1978), 1–26.

[9] Y. B. Jun, J. Neggers and H. S. Kim, Fuzzy d-ideals of d-algebras, *J. Fuzzy Math.* 8 (2000), 123-130.

[10] Y. C. Lee and H. S. Kim, On d-subalgebras of d-transitive d^*-algebras, *Math. Slovaca* 49 (1999), 27–33.

[11] J. Meng, Implicative commutative semigroups are equivalent to a class of BCK-algebras, *Semigroup Forum* 50 (1995), 89–96.

[12] D. Mundici, MV-algebras are categorically equivalent to bounded commutative BCK-algebras, *Math. Japonica* 31 (1986), 889–894.

[13] J. Neggers, Y. B. Jun and H. S. Kim, On d-ideals in d-algebras, *Math. Slovaca* 49 (1999), 243–251.

[14] J. Neggers and H. S. Kim, On d-algebras, *Math. Slovaca* 49 (1999), 19–26.

[15] J. Neggers and H. S. Kim, On B-algebras, *Mate. Vesnik* 54 (2002), 21–29.

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EXTENDED CESÁRO OPERATOR FROM HARDY SPACE TO ZYGMUND-TYPE SPACE ON THE UNIT BALL

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ABSTRACT. In this paper, we characterize the boundedness and compactness of extended Cesáro operator from Hardy space to Zygmund-type space on the unit ball of $\mathbb{C}^n$.

1. INTRODUCTION

Let $H(B_n)$ be the class of all holomorphic functions on B_n, where B_n is the unit ball in the n-dimensional complex space $\mathbb{C}^n$. Let dv denote the Lebesegue measure on B_n normalized so that $v(B_n) = 1$, and $d\sigma$ the normalized rotation invariant measure on the boundary $S_n(= \partial B_n)$ of B_n.

For $f \in H(B_n)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of f.

The Bloch space $\mathcal{B}(= \mathcal{B}(B_n))$ is defined as the space of holomorphic functions such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup\{(1 - |z|^2)|\Re f(z)| : z \in B_n\} < \infty.$$

Let $\mathcal{B}_0(= \mathcal{B}_0(B_n))$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which

$$(1 - |z|^2)\,|\Re f(z)| \to 0, |z| \to 1.$$

This space is called the little Bloch space. Moreover, for $\alpha > 0$, we say $f \in \mathcal{B}_\alpha$ if $f \in H(B_n)$ and

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup\{(1 - |z|^2)^\alpha|\Re f(z)| : z \in B_n\} < \infty.$$

For $0 < p < \infty$, Hardy space $H^p(B_n)$ consists of all holomorphic functions $f \in H(B_n)$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} M^p_p(f, r) = \sup_{0<r<1} \int_{S_n} |f(r\zeta)|^p d\sigma(\zeta) < \infty. \tag{1}$$

Hardy space $H^p(B_n)$ is the most well-known and widely studied space of holomorphic functions. When $1 \le p < \infty, H^p(B_n)$ is a Banach space with norm $\|.\|_{H^p}$. If $0 < p < 1, H^p(B_n)$ is a Fréchet space with the metric $\|.\|^p_{H^p}$. For more details about the spaces described above, we recommend the readers refer to [2, 19, 20].

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The Zygmund space $\mathcal{Z}$ in the unit ball B_n consists of those functions whose first order partial derivatives are in the Bloch space. It is well known that $f \in \mathcal{Z}$ if and only if $\|f\|_{\mathcal{Z}} = |f(0)| + \sup_{z\in B_n}(1-|z|^2)|\Re^2 f(z)| < \infty$.

The little Zygmund space on B_n, denoted by $\mathcal{Z}_0$, is the closed subspace of $\mathcal{Z}$ consisting of functions f satisfying the following condition:

$$\lim_{|z|\to 1}(1-|z|^2)|\Re^2 f(z)| = 0. \tag{2}$$

A positive continuous function μ on [0,1) is called normal (see e.g.[16]), if there exist three positive constants $0 \le \delta < 1$, and $0 < s < t < \infty$, such that for $r \in [\delta, 1)$

$$\frac{\mu(r)}{(1-r)^s} \downarrow 0, \frac{\mu(r)}{(1-r)^t} \uparrow \infty \tag{3}$$

as $r \to 1$. In the rest of this paper we always assume that μ is normal on $[0,1)$.

From now on if we say that a function $\mu : B_n \to [0,\infty)$ is normal we will also assume that it is radial on B_n, that is, $\mu(z) = \mu(|z|), z \in B_n$.

For a normal function μ on $[0,1)$, the Zygmund-type space on the unit ball of $\mathbb{C}^n$ denoted by $\mathcal{Z}_\mu$, is defined as follows, for all $f \in H(B_n)$ satisfy the norm:

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + \sup_{z\in B_n}\mu(z)|\Re^2 f(z)| < \infty. \tag{4}$$

Moreover all $f \in H(B_n)$ such that

$$\lim_{|z|\to 1}\mu(z)|\Re^2 f(z)| = 0 \tag{5}$$

consist of the little Zygmund-type space on the unit ball in $\mathbb{C}^n$. It is well known that the little Zygmund-type space is a closed subspace of the Zygmund-type space and we denote it by $\mathcal{Z}_{\mu,0}$. The Zygmund-type space and the little Zygmund-type space are both Banach space under the norm in (4). When $\mu(r) = 1 - r^2$, the (little) Zygmund-type space is the (little) Zygmund space. For some results on the Zygmund space and Zygmund-type space ,we can refer to [5, 8, 9, 21, 22] and their related references therein.

Let $f(z)$ be a holomorphic function on the unit disc $\mathbb{D}$ with Taylor expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty}\Big(\frac{1}{j+1}\sum_{k=0}^{j} a_k\Big)z^j.$$

Until now, many authors pay attention to the properties of Cesáro operator between several spaces of holomorphic functions . It is well known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces on the unit disc $(H^p(\mathbb{D}))$ for $0 < p < \infty$ and Bergman space as well as the Dirichlet space, for the interested readers, we refer to see the papers [3, 5, 10, 12, 13, 14] and so on. But the operator $\mathcal{C}$ is not always bounded, in [15], Shi and Ren gave a sufficient and necessary condition for the operator $\mathcal{C}$ to be bounded on mixed norm spaces on the unit disc.

A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z}\int_0^z f(t)(log\frac{1}{1-t})'dt$. From this point of view, if $g \in H(B_n)$, it is natural to consider the extended Cesáro operator (also called Volterra-type operator or Riemann-Stieltijes type operator) T_g on $H(B_n)$

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defined by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}. \tag{6}$$

In the past few years, some scholars gave some sufficient and necessary conditions for the extended $\mathcal{C}$ to be bounded and compact on Hardy spaces, BMOA spaces, mixed norm spaces, Bergman spaces, Bloch space as well as Dirichlet space in the unit ball[1, 4, 6, 7, 18].

Building on those foundations, the present paper focuses on the boundedness and compactness of extended cesáro operator from Hardy space to the Zygmund-type space on the unit ball of $\mathbb{C}^n$. The paper is organized as follows: in section 1, we give some lemmas which will be used in the proof of the main results, and then prove the boundedness in section 2 and compactness in section 3, respectively.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Some Lemmas

Lemma 1. *Let $g \in H(B_n)$, then*

$$\Re(T_g f)(z) = f(z)\Re g(z)$$

for any $f \in H(B_n)$ and $z \in B_n$.

Proof. Let $\sum_{|\alpha|\geq 1} a_\alpha z^\alpha$ be the $Taylor$ expansion of the holomorphic function $f(z)\Re g(z)$. Then

$$\begin{aligned}\Re[T_g(f)](z) &= \Re\int_0^1 f(tz)\Re g(tz)\frac{dt}{t} = \Re\int_0^1 \sum_{|\alpha|\geq 1} a_\alpha (tz)^\alpha \frac{dt}{t} \\ &= \Re\Big(\sum_{|\alpha|\geq 1}\frac{a_\alpha}{|\alpha|}z^\alpha\Big) = \sum_{|\alpha|\geq 1} a_\alpha z^\alpha = (f\Re g)(z).\end{aligned}$$

The proof of this Lemma is finished. □

Lemma 2. *(Lemma 2.1,* [17]*) Suppose $0 < p < \infty$, then we have*

(1) If $f \in H^p(B_n)$, then $|f(z)| \leq \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{n}{p}}}$ for all $z \in B_n$;

(2)If $f \in H^p(B_n)$, then $f \in \mathcal{B}_{\frac{n+p}{p}}$ and

$$\|f\|_{\mathcal{B}_{\frac{n+p}{p}}} \leq C\|f\|_{H^p}.$$

The following criterion for compactness follows from Lemma 2, Montel theorem and the definition of compact operator. Its proof is similar to the Proposition 3.11 of [2], so we omit the details here.

Lemma 3. *Assume that $0 < p < \infty$, μ is normal and $g \in H(B_n)$. Then $T_g : H^p \to \mathcal{Z}_\mu$ is compact if and only if $T_g : H^p \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in H^p which converges to zero uniformly on compact subsets of B_n as $k \to \infty$, then we have $\|T_g f_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$.*

The proof of next lemma follows the same idea as the proof of Lemma 1 in [11] with minor modifications, we omit the details.

Lemma 4. *A close set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies $\lim\limits_{|z|\to 1}\sup\limits_{f\in K}\mu(z)|\Re^2 f(z)| = 0$.*

3. The boundedness of $T_g : H^p \to \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$

Theorem 1. *Suppose that $0 < p < \infty$ and μ is normal and that $g \in H(B_n)$. Then $T_g : H^p \to \mathcal{Z}_\mu$ is bounded if and only if*

$$M_1 := \sup_{z\in B_n} \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}} < \infty \tag{7}$$

and

$$M_2 := \sup_{z\in B_n} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} < \infty. \tag{8}$$

Proof. First suppose that (7) and (8) hold. Notice that $T_g f(0) = 0$, then for any $f \in H^p$ and by Lemma 1 and Lemma 2 we get

$$\begin{aligned}
\|T_g f\|_{\mathcal{Z}_\mu} &= \sup_{z\in B_n} \mu(z)|\Re^2(T_g f)(z)| = \sup_{z\in B_n} \mu(z)|\Re(f\Re g)(z)| \\
&= \sup_{z\in B_n} \mu(z)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\
&\le \sup_{z\in B_n} \mu(z)|\Re f(z)\Re g(z)| + \sup_{z\in B_n} \mu(z)|f(z)\Re^2 g(z)| \\
&\le C \sup_{z\in B_n} \frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} \|f\|_{\mathcal{B}_{\frac{n}{p}+1}} + \sup_{z\in B_n} \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}} \|f\|_{H^p} \\
&= C(M_2 + M_1)\|f\|_{H^p} < \infty. \qquad (9)
\end{aligned}$$

From (9) we obtain the boundedness of $T_g : H^p \to \mathcal{Z}_\mu$.

Conversely, assume that $T_g : H^p \to \mathcal{Z}_\mu$ is bounded. Set

$$f_w(z) = \frac{(1-|w|^2)^a}{(1-\langle z, w\rangle)^{\frac{n}{p}+a}}, w \in B_n, \tag{10}$$

where $a > 0$. By Theorem 1.12 in [20] we know

$$M_p(f_w, r) = \Big(\int_{S_n} \frac{(1-|w|^2)^{pa}}{|1-\langle r\zeta, w\rangle|^{n+pa}} d\sigma(\zeta)\Big)^{\frac{1}{p}} \le \frac{(1-|w|^2)^a}{(1-r|w|^2)^a} \le 1.$$

Therefore $f_w \in H^p$ and $\sup_{w\in B_n} \|f_w\|_{H^p} \le 1$. We can easily obtain that

$$f_w(w) = \frac{1}{(1-|w|^2)^{\frac{n}{p}}}$$

and

$$|(\Re f_w)(w)| = (\frac{n}{p} + a)\frac{|w|^2}{(1-|w|^2)^{\frac{n}{p}+1}}.$$

Using Lemma 1 and the above equalities we have

$$\begin{aligned}
\|T_g f_w\|_{\mathcal{Z}_\mu} &= \sup_{z\in B_n} \mu(z)|\Re^2(T_g f_w)(z)| = \sup_{z\in B_n} \mu(z)|\Re(f_w\Re g)(z)| \\
&= \sup_{z\in B_n} \mu(z)|(\Re f_w)(z)\Re g(z) + f_w(z)\Re^2 g(z)| \\
&\ge \mu(w)|(\Re f_w)(w)||\Re g(w)| - \mu(w)|f_w(w)||\Re^2 g(w)| \\
&= (\frac{n}{p} + a)\frac{\mu(w)|\Re g(w)||w|^2}{(1-|w|^2)^{\frac{n}{p}+1}} - \frac{\mu(w)|\Re^2 g(w)|}{(1-|w|^2)^{\frac{n}{p}}} \qquad (11)
\end{aligned}$$

Fix $w \in B_n$, taking the function

$$h_w(z) = 2\frac{(1-|w|^2)^a}{(1-\langle z,w\rangle)^{\frac{n}{p}+a}} - (1-|w|^2)^{\frac{n}{p}}\Big(\frac{(1-|w|^2)^a}{(1-\langle z,w\rangle)^{\frac{n}{p}+a}}\Big)^2 \tag{12}$$

By Theorem 1.12 in [20] we know the second part of $h_w(z)$ satisfy the following condition

$$\Big(\int_{S_n}\frac{(1-|w|^2)^{2pa+n}}{|1-\langle r\zeta,w\rangle|^{2(n+pa)}}d\sigma(\zeta)\Big)^{\frac{1}{p}} \quad\leq\quad \frac{(1-|w|^2)^{2a+\frac{n}{p}}}{(1-r|w|^2)^{2a+\frac{n}{p}}} \leq 1.$$

Therefore $h_w \in H^p$ and $\sup\limits_{w\in B_n}\|h_w(z)\| < C$. Moreover, $(\Re h_w)(w) = 0$ and

$$h_w(w) = \frac{1}{(1-|w|^2)^{\frac{n}{p}}}.$$

Hence

$$\begin{aligned}
\|T_g h_w\|_{\mathcal{Z}_\mu} &= \sup_{z\in B_n}\mu(z)|\Re^2(T_g h_w)(z)|\\
&= \sup_{z\in B_n}\mu(z)|(\Re h_w)(z)\Re g(z) + h_w(z)\Re^2 g(z)|\\
&\geq \mu(w)|(\Re h_w)(w)\Re g(w) + h_w(w)\Re^2 g(w)|\\
&= \frac{\mu(w)|\Re^2 g(w)|}{(1-|w|^2)^{\frac{n}{p}}}
\end{aligned}$$

From the above inequality and since w is an arbitrary element in B_n, we obtain (7).

Combining (7) and (11) we obtain

$$\sup_{z\in B_n}\frac{\mu(z)|\Re g(z)||z|^2}{(1-|z|^2)^{\frac{n}{p}+1}} < \infty. \tag{13}$$

By (13) we obtain

$$\sup_{|z|>1/2}\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} \leq \sup_{z\in B_n}\frac{\mu(z)|\Re g(z)|(2|z|)^2}{(1-|z|^2)^{\frac{n}{p}+1}} < \infty. \tag{14}$$

and

$$\sup_{|z|\leq 1/2}\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} \leq C\sup_{|z|\leq 1/2}\mu(z)|\Re g(z)| < \infty. \tag{15}$$

Combining (14) and (15), we get (8). The proof of this Theorem is completed. □

Theorem 2. *Suppose that $0 < p < \infty$ and μ is normal and that $g \in H(B_n)$. Then $T_g : H^p \to \mathcal{Z}_{\mu,0}$ is bounded if and only if $T_g : H^p \to \mathcal{Z}_\mu$ is bounded and*

$$\lim_{|z|\to 1}\mu(z)|\Re g(z)| = 0 \tag{16}$$

and

$$\lim_{|z|\to 1}\mu(z)|\Re^2 g(z)| = 0. \tag{17}$$

Proof. Suppose that $T_g : H^p \to \mathcal{Z}_\mu$ is bounded, (16)and (17) hold. For any polynomial P we have that

$$\begin{aligned}\mu(z)|\Re^2(T_gP)(z)| &= \mu(z)|\Re P(z)\Re g(z) + P(z)\Re^2 g(z)| \\ &\leq \mu(z)|\Re g(z)|\|\Re P\|_\infty + \mu(z)|\Re^2 g(z)|\|P\|_\infty \\ &\to 0, |z| \to 1.\end{aligned}$$

Thus for any polynomial P we have $T_gP \in \mathcal{Z}_{\mu,0}$. As we all know that the set of all polynomials is dense in Hardy space H^p. Thus, for every $f \in H^p$, there is a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ such that $\|f - P_n\|_{H^p} \to 0, n \to \infty$. Hence

$$\|T_gf - T_gP_n\|_{\mathcal{Z}_\mu} \leq \|T_g\|_{H^p\to\mathcal{Z}_\mu}\|f - P_n\|_{H^p} \to 0, n \to \infty.$$

Since the operator $T_g : H^p \to \mathcal{Z}_\mu$ is bounded and $\mathcal{Z}_{\mu,0}$ is a closed subset of $\mathcal{Z}_\mu$, we obtain $T_g(H^p) \subseteq \mathcal{Z}_{\mu,0}$. Therefore, the operator $T_g : H^p \to \mathcal{Z}_{\mu,0}$ is bounded.

Conversely, we suppose that $T_g : H^p \to \mathcal{Z}_{\mu,0}$ is bounded. The boundedness of $T_g : H^p \to \mathcal{Z}_\mu$ follows. Taking the function $f(z) = 1 \in H^p$, we obtain that

$$\begin{aligned}\mu(z)|\Re^2(T_gf)(z)| &= \mu(z)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\ &= \mu(z)|\Re^2 g(z)| \to 0, |z| \to 1.\end{aligned}$$

which implies (17).

On the other hand, choosing the function $f_j(z) = z_j \in H^p, j \in \{1, ..., n\}$. We have that

$$\begin{aligned}\mu(z)|\Re^2(T_gf)(z)| &= \mu(z)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\ &= \mu(z)|z_j||\Re g(z) + \Re^2 g(z)| \to 0, |z| \to 1.\end{aligned}$$

Thus

$$\mu(z)|z||\Re g(z) + \Re^2 g(z)| \leq \mu(z)(\sum_{j=1}^{n}|z_j|)|\Re g(z) + \Re^2 g(z)| \to 0, |z| \to 1.$$

The above inequality and (17) imply (16). The proof of this Theorem is completed. □

4. The compactness of $T_g : H^p \to \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$.

Theorem 3. *Suppose that $0 < p < \infty$, and μ is normal and that $g \in H(B_n)$. Then the following statements are equivalent.*

(a) $T_g : H^p \to \mathcal{Z}_\mu$ is compact;

(b) $T_g : H^p \to \mathcal{Z}_{\mu,0}$ is compact;

(c) $T_g : H^p \to \mathcal{Z}_\mu$ is bounded and

$$\lim_{|z|\to 1}\frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}} = 0 \tag{18}$$

and

$$\lim_{|z|\to 1}\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} = 0. \tag{19}$$

Proof. (b)⇒(a). This implication is obvious.

(a)⇒(c). Suppose $T_g : H^p \to \mathcal{Z}_\mu$ is compact. Then the boundedness of $T_g : H^p \to \mathcal{Z}_\mu$ follows. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B_n such that $\lim_{k\to\infty} |z_k| = 1$. Setting the function sequence

$$f_k(z) = \frac{(1-|z_k|^2)^a}{(1-\langle z, z_k\rangle)^{\frac{n}{p}+a}}$$

where $a > 0$. As in the proof of Theorem 1(where $f_w(z)$ defined in (10)) it is easy to get that $f_k \in H^p$ and $\sup_{k\in N} \|f_k\|_{H^p} \le 1$. It is obvious that that $f_k \to 0$ uniformly on compact subsets of B_n as $k\to\infty$. By lemma 3 we get $\lim_{k\to\infty} \|T_g f_k\|_{\mathcal{Z}_\mu} = 0$.
By the similar proof of (11) in Theorem 1 we obtain

$$\begin{aligned}
\|T_g f_k\|_{\mathcal{Z}_\mu} &= \sup_{z\in B_n} \mu(z)|\Re f_k(z)\Re g(z) + f_k(z)\Re^2 g(z)| \\
&\ge \mu(z_k)|\Re f_k(z_k)||\Re g(z_k)| - \mu(z_k)|f_k(z_k)||\Re^2 g(z_k)| \\
&= \Big(\frac{n}{p}+a\Big)\frac{\mu(z_k)|\Re g(z_k)||z_k|^2}{(1-|z_k|^2)^{\frac{n}{p}+1}} - \frac{\mu(z_k)|\Re^2 g(z_k)|}{(1-|z_k|^2)^{\frac{n}{p}}} \qquad (20)
\end{aligned}$$

On the other hand, taking the function sequence

$$h_k(z) = 2\frac{(1-|z_k|^2)^a}{(1-\langle z, z_k\rangle)^{\frac{n}{p}+a}} - (1-|z_k|^2)^{\frac{n}{p}}\Big(\frac{(1-|z_k|^2)^a}{(1-\langle z, z_k\rangle)^{\frac{n}{p}+a}}\Big)^2$$

As in the proof of Theorem 1 we get that $h_k \in H^p$ and $\sup_{k\in\mathbb{N}} \|h_k\|_{H^p} \le C$. It is obvious that $h_k \to 0$ uniformly on compact subsets of B_n as $k\to\infty$. By lemma 3 we get $\lim_{k\to\infty} \|T_g h_k\|_{\mathcal{Z}_\mu} = 0$.
By (12) in the proof of Theorem 1 we obtain

$$\begin{aligned}
\|T_g h_k\|_{\mathcal{Z}_\mu} &= \sup_{z\in B_n} \mu(z)|\Re h_k(z)\Re g(z) + h_k(z)\Re^2 g(z)| \\
&\ge \mu(z_k)|\Re h_k(z_k)\Re g(z_k) + h_k(z_k)\Re^2 g(z_k)| \\
&= \frac{\mu(z_k)|\Re^2 g(z_k)|}{(1-|z_k|^2)^{\frac{n}{p}}} \qquad (21)
\end{aligned}$$

Thus from (21) we obtain (18). And from (20) and (21) we obtain (19).

(c)⇒(b). Suppose $T_g : H^p \to \mathcal{Z}_\mu$ is bounded, (18) and (19) hold. For any $f \in H^p$, by Lemma 1 and Lemma 2 we get

$$\mu(z)|\Re^2(T_g f)(z)| \le C\Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}}\Big)\|f\|_{H^p}.$$

Taking the supremum in the above inequality for all $f \in H^p$ such that $\|f\|_{H^p} \le 1$, then letting $|z| \to 1$, by (18) and (19) we have

$$\begin{aligned}
&\lim_{|z|\to 1} \sup_{\|f\|_{H^p}\le 1} \mu(z)|\Re^2(T_g f)(z)| \\
&\le C \lim_{|z|\to 1} \sup_{\|f\|_{H^p}\le 1} \Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}}\Big)\|f\|_{H^p} \\
&\le \lim_{|z|\to 1} C\Big(\frac{\mu(z)|\Re g(z)|}{(1-|z|^2)^{\frac{n}{p}+1}} + \frac{\mu(z)|\Re^2 g(z)|}{(1-|z|^2)^{\frac{n}{p}}}\Big) = 0
\end{aligned}$$

From which and Lemma 4 we obtain the compactness of $T_g : H^p \to \mathcal{Z}_{\mu,0}$. So, the proof of this Theorem is completed. □

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References

[1] K. Avetisyan, S. Stević, Extended Cesàro operators between different Hardy spaces, Appl. Math. Comput. 207, (2009), 346-350.

[2] C. C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.

[3] N. Danikas and A. Siskakis, The cesáro operator on bounded analytic functions, Analysis, 13(1993), 195-199.

[4] Z. S. Fang, Z. H. Zhou, Extended Cesàro operators on BMOA spaces in the unit ball, J. Math. Inequal. , 4(1),(2010), 27-36.

[5] Z. S. Fang, Z. H. Zhou, Extended Cesàro operators on Zygmund spaces in the unit balL, J. Comput. Anal. Appl. 11(3), (2009), 406-413.

[6] Z. J. Hu, Extended Cesàro operators on mixed norm spaces, Proc. Amer. Math. Soc. 131(7), (2003), 2171-2179.

[7] Z. J. Hu, Extended Cesàro operators on Bergman spaces, J. Math. Anal. Appl. 296, (2004), 435-454.

[8] S. Li, S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space, J. Math. Anal. Appl. 345, (2008), 40-52.

[9] S. Li, S. Stević, On an integral-type operator from ω-Bloch spaces to μ-Zygmund spaces, Appl. Math. Comput. 215(12), (2010), 4385-4391.

[10] J. Miao, The cesáro operator is bounded on H_p for $0 < p < 1$, Proc. Amer. Math. Soc., 116(1992), 1077-1079.

[11] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, Trans. Amer. Math. Soc. 347(7), (1995), 2679-2687.

[12] A. Siskakis, Composition semigroups and the cesáro operator on $H_p(D)$, J. London Math. Soc. 36(1987),153-164.

[13] A. Siskakis, The cesáro operator is bounded on H^1, Proc. Amer. Math. Soc., 110(1990), 461-462.

[14] A. Siskakis, On the Bergman space norm of the cesáro operator, Arch. Math. 67, (1996), 312-318.

[15] J. H. Shi and G. B. Ren, Boundedness of the cesáro operator on mixed norm space, Proc. Amer. Math. Soc. , 126, (1998), 3553-3560.

[16] A. L. Shields and D. L. Williams, Bounded projection, duality,and multipliers in spaces of analytic functions, Trans. Amer. Math. Soc. 162, (1971), 287-302.

[17] L. X. Wang, S. T. Liu, Composition type operators from Hardy spaces to μ-Bloch spaces on the unit ball, Acta Mathematica Scientia 2009, 29B(5): 1430 C 1438.

[18] X. J. Zhang, Extended cesáro operators on Dirichlet type spaces and Bloch spaces of Cn, Chin. ANN. of Math., 26A(2005), 139-150. (in Chinese)

[19] K. H. Zhu, Operator Theory in Function Spaces, Marcel Dekker. Inc, New York, 1990.

[20] K. H. Zhu, Spaces of holomorphic functions in the unit ball, Graduate Texts in Mathematics 226, Springer, New York, 2005.

[21] X. L. Zhu, Generalized composition operators on Zygmund spaces and Bloch type spaces, J. Math. Anal. Appl., 338(2), (2008), 1282-1295.

[22] X. L. Zhu, Integral-type operators from iterated logarithmic Bloch spaces to Zygmund-type spaces, Appl. Math. Comput. 215(3), (2009), 1170-1175.

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SOME SPECIAL POLYNOMIALS AND SHEFFER SEQUENCES

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ABSTRACT. In this paper we give some identities of several special polynomials arising from umbral calculus.

1. INTRODUCTION

As is well known, the higher-order Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\ (r\in\mathbb{R}). \tag{1.1}$$

In the special case, $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$ are called the *Bernoulli numbers* of order r.

By (1.1), we easily see that

$$B_n^{(r)}(x)=\left(B^{(r)}+x\right)^r=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}^{(r)}x^l, \tag{1.2}$$

with the usual convention about replacing $\left(B^{(r)}\right)^n$ by $B_n^{(r)}$ (see [1-15]).

The higher-order Euler polynomials are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\frac{E_n^{(r)}(x)}{n!}t^n,\ (r\in\mathbb{R}). \tag{1.3}$$

In the special case, $x=0$, $E_n^{(r)}(0)=E_n^{(r)}$ are called the *Euler numbers* of order r. From (1.3), we have

$$E_n^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{n-l}^{(r)}x^l,\ \text{(see [1-15])}. \tag{1.4}$$

The *Stirling number of the second kind* is defined by

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}, \tag{1.5}$$

and the *Stirling number of the first kind* is given by

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\ \text{(see [6,8,12])}, \tag{1.6}$$

where $(x)_n=x(x-1)\cdots(x-n+1)$.

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Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.7}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L \mid p(x) \rangle$ denotes the action of the linear functional L on a polynomials $p(x)$ and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\begin{aligned} \langle L + M | p(x) \rangle &= \langle L | p(x) \rangle + \langle M | p(x) \rangle \\ \langle cL | p(x) \rangle &= c \langle L | p(x) \rangle, \end{aligned}$$

where c is a complex constant (see [6, 8, 12]).

The formal power series $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t) | x^n \rangle = a_n, \text{ for all } n \geq 0. \tag{1.8}$$

By (1.7) and (1.8), we get

$$\left\langle t^k | x^n \right\rangle = n! \delta_{n,k}, \ (n, k \geq 0) \tag{1.9}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [12]).

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k$, we have

$$\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle, \ n \geq 0. \tag{1.10}$$

By (1.10), we see that $f_L(t) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [6, 8, 12]).

The order $o(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ has $o(f(t)) = 1$ is called a *delta series* and a series $f(t)$ has $o(f(t)) = 0$ is called an *invertible series*. Let $f(t)$ be a delta series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t) f(t)^k | S_n(x) \right\rangle = n! \delta_{n,k}$, where $n, k \geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. By (1.8) and (1.9), we easily see that $\langle e^{yt} | p(x) \rangle = p(y)$.

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\left\langle f(t) | x^k \right\rangle}{k!} t^k, \ p(x) = \sum_{k=0}^{\infty} \frac{\left\langle t^k | p(x) \right\rangle}{k!} x^k, \tag{1.11}$$

and

$$\langle f_1(t) f_2(t) \cdots f_m(t) | x^n \rangle = \sum_{i_1 + \cdots + i_m = n} \binom{n}{i_1, \ldots, i_m} \left\langle f_1(t) | x^{i_1} \right\rangle \cdots \left\langle f_m(t) | x^{i_m} \right\rangle \tag{1.12}$$

where $f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F}$ (see [6, 8, 12]).

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. By (1.11), we easily see that

$$p^{(k)}(0) = \left\langle t^k | p(x) \right\rangle, \ \left\langle 1 | p^{(k)}(x) \right\rangle = p^{(k)}(0). \tag{1.13}$$

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Thus, from (1.13), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \ (k \geq 0), \quad \text{(see [12])}. \tag{1.14}$$

For $S_n(x) \sim (g(t), f(t))$, we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \ \text{for all } y \in \mathbb{C}, \tag{1.15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$,

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) S_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(y) S_k(x), \tag{1.16}$$

where $p_k(y) = g(t)S_k(y) \sim (1, f(t))$. Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then we have

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \ \text{(see [6,8,12])}. \tag{1.17}$$

The Abel sequences are given by

$$A_n(x; b) = x(x - bn)^{n-1} \sim \left(1, te^{bt}\right), \ (b \neq 0). \tag{1.18}$$

Now we introduce several important sequences which are used to derive our results:

(Mittag-Leffler sequences)

$$M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-1)_{n-k} 2^k (x)_k \sim \left(1, \frac{e^t - 1}{e^t + 1}\right), \tag{1.19}$$

(The exponential sequences)

$$\phi_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k \sim (1, \log(1+t)), \tag{1.20}$$

(The Laguerre sequence)

$$L_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} (-x)^k \sim \left(1, \frac{t}{t-1}\right), \tag{1.21}$$

(The Poisson-Charlier sequences)

$$C_n(x; a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim \left(e^{a(e^t - 1)}, a(e^t - 1)\right), \tag{1.22}$$

where $a \neq 0$. From (1.22), we note that

$$e^{a(e^t-1)} C_n(x; a) \sim \left(1, a(e^t - 1)\right). \tag{1.23}$$

Recently, several people have studied the umbral calculus related to special polynomials. In this paper, we derive some interesting identities associated with several special polynomials arising from umbral calculus.

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2. Some identities of special polynomials

It is not difficult to show that

$$xE_{n-1}^{(n)}(x) \sim \left(1, \frac{t(e^t+1)}{2}\right). \tag{2.1}$$

For $n \geq 1$, by (1.17), (1.19) and (2.1), we get

$$\begin{aligned} M_n(x) &= 2^n x \left(\frac{t}{e^t-1}\right)^n \left(\frac{e^t+1}{2}\right)^{2n} E_{n-1}^{(n)}(x) \\ &= 2^n x \left(\frac{t}{e^t-1}\right)^n \left(\frac{e^t+1}{2}\right)^{n} x^{n-1} \\ &= x(e^t+1)^n B_{n-1}^{(n)}(x) = x\sum_{j=0}^{n}\binom{n}{j} B_{n-1}^{(n)}(x+j). \end{aligned} \tag{2.2}$$

Therefore, by (1.19) and (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k = x\sum_{j=0}^{n}\binom{n}{j}B_{n-1}^{(n)}(x+j).$$

On the other hand, from (1.17), (1.19) and (2.1), we have

$$\begin{aligned} xE_{n-1}^{(n)}(x) &= x\left(\frac{\frac{e^t-1}{e^t+1}}{\frac{t(e^t+1)}{2}}\right)^n x^{-1}M_n(x) \\ &= x\left(\frac{e^t-1}{t}\right)^n \frac{2^n}{(e^t+1)^{2n}} x^{-1}M_n(x). \end{aligned} \tag{2.3}$$

From (1.6) and (1.19), we note that

$$x^{-1}M_n(x) = \sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k\sum_{l=0}^{k-1}S_1(k-1,l)(x-1)^l. \tag{2.4}$$

For $n \geq 1$, by (2.3) and (2.4), we get

$$\begin{aligned} &xE_{n-1}^{(n)}(x) = \frac{x}{2^n}\left(\frac{e^t-1}{t}\right)^n\left(\frac{2}{e^t+1}\right)^{2n}x^{-1}M_n(x) \\ =&\frac{x}{2^n}\left(\frac{e^t-1}{t}\right)^n\sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^kS_1(k-1,l)\left(\frac{2}{e^t+1}\right)^{2n}(x-1)^l \\ =&\frac{x}{2^n}\left(\frac{e^t-1}{t}\right)^n\sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^kS_1(k-1,l)E_l^{(2n)}(x-1). \end{aligned} \tag{2.5}$$

From (1.5), we can derive the following equation (2.6):

$$\left(\frac{e^t-1}{t}\right)^n = \sum_{m=0}^{\infty}\frac{n!}{(m+n)!}S_2(m+n,n)t^m. \tag{2.6}$$

By (2.5) and (2.6), we see that

$$
\begin{aligned}
&xE_{n-1}^{(n)}(x)\\
=&\frac{x}{2^n}\sum_{m=0}^{n-1}\frac{n!}{(m+n)!}S_2(m+n,n)\sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^kS_1(k-1,l)t^mE_l^{(2n)}(x-1)\\
=&\frac{x}{2^n}\sum_{m=0}^{n-1}\frac{n!}{(m+n)!}S_2(m+n,n)\sum_{l=0}^{n-1}\sum_{k=l+1}^{n}\binom{n}{k}(n-1)_{n-k}2^kS_1(k-1,l)t^mE_l^{(2n)}(x-1)\\
=&x\sum_{m=0}^{n-1}\sum_{l=m}^{n-1}\sum_{k=l+1}^{n}2^{k-n}\frac{n!}{(m+n)!}\binom{n}{k}(n-1)_{n-k}(l)_mS_2(m+n,n)S_1(k-1,l)E_{l-m}^{(2n)}(x-1).
\end{aligned}
\tag{2.7}
$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \geq 1$, we have*

$$
\begin{aligned}
&xE_{n-1}^{(n)}(x)\\
=&x\sum_{m=0}^{n-1}\sum_{l=m}^{n-1}\sum_{k=l+1}^{n}2^{k-n}\frac{n!}{(m+n)!}\binom{n}{k}(n-1)_{n-k}(l)_mS_2(m+n,n)S_1(k-1,l)E_{l-m}^{(2n)}(x-1).
\end{aligned}
$$

By (1.9), we easily see that

$$x^n \sim (1,t), \ (n \geq 0). \tag{2.8}$$

From (1.17), (1.21) and (2.8), we can derive the following equation (2.9):

$$
\begin{aligned}
x^n &= x(t-1)^{-n}x^{-1}L_n(x) = (-1)^n x\sum_{k=0}^{n-1}\binom{n+k-1}{n-1}t^kx^{-1}L_n(x)\\
&= (-1)^n x\sum_{k=0}^{n-1}\binom{n+k-1}{n-1}t^k\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}(-1)^lx^{l-1}\\
&= (-1)^n x\sum_{k=0}^{n-1}\binom{n+k-1}{n-1}\sum_{l=k+1}^{n}\binom{n-1}{l-1}\frac{n!}{l!}(-1)^l(l-1)_kx^{l-1-k}.
\end{aligned}
\tag{2.9}
$$

Thus, by (2.9), we get

$$
\begin{aligned}
\frac{(-1)^nx^{n-1}}{(n-1)!} &= \sum_{k=0}^{n-1}\binom{n+k-1}{n-1}\sum_{l=k+1}^{n}(-1)^l\frac{\binom{n}{l}}{(l-1-k)!}x^{l-1-k}\\
&= \sum_{k=0}^{n-1}\binom{n+k-1}{n-1}\sum_{l=0}^{n-k-1}(-1)^{l+k+1}\frac{\binom{n}{l+1+k}}{l!}x^l\\
&= \sum_{l=0}^{n-1}\left\{\sum_{k=0}^{n-1-l}(-1)^{l+k+1}\frac{\binom{n+k-1}{n-1}\binom{n}{l+1+k}}{l!}\right\}x^l.
\end{aligned}
\tag{2.10}
$$

Therefore, by (2.10), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, we have*

$$\frac{(-1)^nx^{n-1}}{(n-1)!} = \sum_{l=0}^{n-1}\left\{\sum_{k=0}^{n-1-l}(-1)^{l+k+1}\frac{\binom{n+k-1}{n-1}\binom{n}{l+1+k}}{l!}\right\}x^l.$$

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Remark. It is easy to show that

$$(-1)^n x^{n-1} = (1-t)^{-n} x^{-1} L_n(x) = \underbrace{(1-t)^{-1} \times \cdots \times (1-t)^{-1}}_{n-\text{times}} x^{-1} L_n(x)$$
$$= \int_0^\infty \cdots \int_0^\infty \frac{1}{x+u_1+\cdots+u_n} L_n(x+u_1+\cdots+u_n) e^{-(u_1+\cdots+u_n)} du_1 \cdots du_n.$$

By (1.9) and (1.21), we easily get

$$L_n(-x) \sim \left(1, \frac{t}{1+t}\right). \tag{2.11}$$

For $n \geq 1$, by (1.17), (1.21) and (2.11), we get

$$L_n(-x) = x \left(\frac{t+1}{t-1}\right)^n x^{-1} L_n(x). \tag{2.12}$$

From (1.15) and (1.19), we can derive the generating function of the Mittag-Leffler polynomials as follows:

$$\sum_{k=0}^{\infty} \frac{M_k(x)}{k!} t^k = \left(\frac{1+t}{1-t}\right)^x. \tag{2.13}$$

By (2.12) and (2.13), we see that

$$\begin{aligned} L_n(-x) &= (-1)^n x \left(\sum_{k=0}^{n-1} \frac{M_k(n)}{k!} t^k\right) x^{-1} L_n(x) \\ &= (-1)^n x \sum_{k=0}^{n-1} \frac{M_k(n)}{k!} t^k \sum_{l=0}^{n} \binom{n-1}{l-1} \frac{n!}{l!} (-1)^l x^{l-1} \\ &= (-1)^n x \sum_{k=0}^{n-1} \frac{M_k(n)}{k!} \sum_{l=k+1}^{n} \binom{n-1}{l-1} \frac{n!}{l!} (-1)^l (l-1)_k x^{l-1-k}. \end{aligned} \tag{2.14}$$

Thus, from (2.14), we have

$$\begin{aligned} &\frac{(-1)^n x^{-1} L_n(-x)}{(n-1)!} \\ =&\frac{1}{(n-1)!} \sum_{k=0}^{n-1} \frac{M_k(n)}{k!} \sum_{l=k+1}^{n} \binom{n-1}{l-1} \frac{n!}{l!} (-1)^l (l-1)_k x^{l-1-k} \\ =&\sum_{k=0}^{n-1} \sum_{l=k+1}^{n} M_k(n) \frac{(-1)^l \binom{n}{l}}{k!(l-1-k)!} x^{l-1-k} \\ =&\sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} M_k(n) \frac{\binom{n}{l+k+1} (-1)^{l+k+1}}{k!l!} x^l \\ =&\sum_{l=0}^{n-1} \left\{ \sum_{k=0}^{n-1-l} M_k(n) \frac{\binom{n}{l+k+1} (-1)^{l+k+1}}{k!l!} \right\} x^l. \end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

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Theorem 2.4. *For $n \geq 1$, we have*

$$\frac{(-1)^n x^{-1} L_n(-x)}{(n-1)!} = \sum_{l=0}^{n-1} \left\{ \sum_{k=0}^{n-1-l} M_k(n) \frac{\binom{n}{l+k+1}(-1)^{l+k+1}}{k!l!} \right\} x^l.$$

It is easy to see that

$$(x)_n \sim (1, e^t - 1), \ (n \geq 0). \tag{2.16}$$

For $n \geq 1$, by (1.17), (1.23) and (2.16), we get

$$e^{a(e^t-1)} C_n(x;a) = x \left(\frac{e^t - 1}{a(e^t - 1)} \right)^n x^{-1} (x)_n = a^{-n} (x)_n. \tag{2.17}$$

Thus, from (2.17), we have

$$C_n(x;a) = e^{-a(e^t-1)} a^{-n} (x)_n. \tag{2.18}$$

From (1.15) and (1.20), we can derive the generating function of exponential polynomials as follows:

$$\sum_{k=0}^{\infty} \phi_k(x) \frac{t^k}{k!} = e^{x(e^t-1)}. \tag{2.19}$$

By (2.18) and (2.19), we get

$$\begin{aligned} C_n(x;a) &= a^{-n} \sum_{k=0}^{n} \frac{\phi_k(-a)}{k!} t^k (x)_n \\ &= a^{-n} \sum_{k=0}^{n} \frac{\phi_k(-a)}{k!} \sum_{l=0}^{n} S_1(n,l) t^k x^l \\ &= a^{-n} \sum_{k=0}^{n} \frac{\phi_k(-a)}{k!} \sum_{l=k}^{n} S_1(n,l) (l)_k x^{l-k} \\ &= a^{-n} \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{l}{k} \phi_k(-a) S_1(n,l) x^{l-k}. \end{aligned} \tag{2.20}$$

Therefore, by (1.22) and (2.20), we obtain the following theorem.

Theorem 2.5. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} (x)_k = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{l}{k} \phi_k(-a) S_1(n,l) x^{l-k}.$$

Remark. For $n \geq 1$, by (1.22) and (2.20), we get

$$\sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} S_1(k,m) = \sum_{k=0}^{n-m} \binom{k+m}{k} \phi_k(-a) S_1(n,k+m).$$

From (1.17), (1.23) and (2.8), we have

$$\begin{aligned} e^{a(e^t-1)} C_n(x;a) &= a^{-n} x \left(\frac{t}{e^t - 1} \right)^n x^{n-1} \\ &= a^{-n} x B_{n-1}^{(n)}(x), \ (n \geq 1). \end{aligned} \tag{2.21}$$

By using (2.19) and (2.21), we obtain

$$\begin{aligned} &C_n(x;a)\\ =&a^{-n}\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}\phi_k(-a)xB_{n-1-k}^{(n)}(x)+\sum_{k=1}^{n}\binom{n-1}{k-1}\phi_k(-a)B_{n-k}^{(n)}(x)\right\}. \end{aligned} \tag{2.22}$$

Therefore, by (1.22) and (2.22), we obtain the following theorem.

Theorem 2.6. *For $n \geq 1$, $a \neq 0$, we have*

$$\begin{aligned} &\sum_{k=0}^{n-1}\binom{n-1}{k}\phi_k(-a)xB_{n-1-k}^{(n)}(x)+\sum_{k=1}^{n}\binom{n-1}{k-1}\phi_k(-a)B_{n-k}^{(n)}(x)\\ =&\sum_{k=0}^{n}\binom{n}{k}(-a)^{n-k}(x)_k. \end{aligned}$$

Remark. By (1.17), (1.19), (1.20) and (1.23), we get

$$\begin{aligned} C_n(x;a)&=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a^{-k}(x)_k\\ &=(2a)^{-n}\sum_{l=0}^{n}\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n}{k}\binom{j}{l}(n-1)_{n-k}2^k\phi_l(-a)S_1(k-1,j)xE_{j-l}^{(n)}(x-1)\\ &+(2a)^{-n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n}{k}\binom{j}{l-1}(n-1)_{n-k}2^k\phi_l(-a)S_1(k-1,j)E_{j-l+1}^{(n)}(x-1). \end{aligned}$$

From (1.22), we can derive the generating function of the Poisson-Charlier polynomials as follows:

$$\sum_{k=0}^{\infty}\frac{C_k(x;a)}{k!}t^k=e^{-t}\left(1+\frac{t}{a}\right)^x,\ (a\neq 0). \tag{2.23}$$

Replacing t by at and x by b, we get

$$\sum_{k=0}^{\infty}\frac{C_k(b;a)}{k!}a^kt^k=\frac{t(1+t)^b}{te^{at}}. \tag{2.24}$$

Let us consider the following Sheffer sequence:

$$P_n(x)\sim\left(1,t(1+t)^b\right). \tag{2.25}$$

Then, by the transfer formula, we get

$$\begin{aligned} P_n(x)&=x(1+t)^{-nb}x^{n-1}=x\sum_{k=0}^{\infty}\binom{-nb}{k}t^kx^{n-1}\\ &=x\sum_{k=0}^{n-1}\binom{-nb}{k}(n-1)_kx^{n-1-k}\\ &=\sum_{k=0}^{n-1}\binom{n-1}{k}(-nb)_kx^{n-k}\\ &=\sum_{k=1}^{n}\binom{n-1}{k-1}(-nb)_{n-k}x^k. \end{aligned} \tag{2.26}$$

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For $n \geq 1$, by (1.17), (1.18), (2.25) and (2.26), we get

$$\begin{aligned} A_n(x;a) &= x\left(\frac{t(1+t)^b}{te^{at}}\right)^n x^{-1}P_n(x) \\ &= x\left(\sum_{k=0}^{\infty}\frac{C_k(b;a)}{k!}a^k t^k\right)^n x^{-1}P_n(x) \\ &= x\sum_{k=0}^{\infty}\frac{a^k}{k!}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n}C_{l_1}(b;a)\cdots C_{l_n}(b;a)\right)t^k x^{-1}P_n(x). \end{aligned} \tag{2.27}$$

From (2.26), we have

$$\begin{aligned} t^k x^{-1}P_n(x) &= t^k\sum_{l=1}^{n}\binom{n-1}{l-1}(-nb)_{n-l}x^{l-1} \\ &= \sum_{l=1}^{n}\binom{n-1}{l-1}(-nb)_{n-l}(l-1)_k x^{l-1-k} \\ &= \sum_{l=k+1}^{n}\binom{n-1}{l-1}(-nb)_{n-l}(l-1)_k x^{l-1-k}. \end{aligned} \tag{2.28}$$

By (2.27) and (2.28), we get

$$\begin{aligned} &A_n(x;a) \\ =&\sum_{k=0}^{n-1}\frac{a^k}{k!}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n}C_{l_1}(b;a)\cdots C_{l_n}(b;a)\right) \\ &\times\sum_{l=k+1}^{n}\binom{n-1}{l-1}(-nb)_{n-k}(l-1)_k x^{l-k} \\ =&\sum_{k=0}^{n-1}\frac{a^k}{k!}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n}C_{l_1}(b;a)\cdots C_{l_n}(b;a)\right) \\ &\times\sum_{m=1}^{n-k}\binom{n-1}{m+k-1}(-nb)_{n-m-k}(k+m-1)_k x^m. \end{aligned} \tag{2.29}$$

For $n \geq 1$, by (2.29), we get

$$\begin{aligned} &A_n(x;a) \\ =&\sum_{m=1}^{n}\left\{\sum_{k=0}^{n-m}\sum_{l_1+\cdots+l_n=k}a^k\binom{n-m}{k}\binom{n-1}{m-1}\binom{k}{l_1,\ldots,l_n}(-nb)_{n-k-m}C_{l_1}(b;a)\cdots C_{l_n}(b;a)\right\}x^m. \end{aligned} \tag{2.30}$$

By (1.18), we see that

$$\begin{aligned} A_n(x;a) = x(x-an)^{n-1} &= x\sum_{m=0}^{n-1}\binom{n-1}{m}(-an)^{n-1-m}x^m \\ &= \sum_{m=0}^{n-1}\binom{n-1}{m}(-an)^{n-1-m}x^{m+1} \\ &= \sum_{m=1}^{n}\binom{n-1}{m-1}(-an)^{n-m}x^m. \end{aligned} \tag{2.31}$$

Therefore, by (2.30) and (2.31), we obtain the following proposition.

Proposition 2.7. *For $n \geq 1$ and $1 \leq m \leq n$, we have*

$$\begin{aligned} &\binom{n-1}{m-1}(-an)^{n-m} \\ &= \sum_{k=0}^{n-m}\sum_{l_1+\cdots+l_n=k} a^k \binom{n-m}{k}\binom{n-1}{m-1}\binom{k}{l_1,\ldots,l_n}(-nb)_{n-k-m}C_{l_1}(b;a)\cdots C_{l_n}(b;a) \end{aligned}$$

References

[1] S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. **22** (2012), no. 3, 399-406. no. 3, 311-320.

[2] S. Araci, M. Acikgoz, H. Jolany, J. J. Seo, *A unified generating function of the q-Genocchi polynomials with their interpolation functions*, Proc. Jangjeon Math. Soc. **15** (2012), no. 2, 227-233.

[3] A. Bayad, T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys. **18** (2011), no. 2, 133-143.

[4] L.Carlitz, *The product of two Eulerian polynomials*, Math. Mag. **368** (1963) 37-41.

[5] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$* , Russ. J. Math. Phys. 16 (2009), no. 4, 484-491.

[6] D. S. Kim, T. Kim, S.-H. Rim, S.-H. Lee, *Some identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus*, Advances in Difference Equations 2013, 2013:15.

[7] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$*, Rocky Mountain J. Math. 41 (2011), no. 1, 239-247.

[8] D. S. Kim and T. Kim, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Difference Equ. 2012, 2012:196 doi:10.1186/1687-1847-2012-196.

[9] T. Kim, *New approach to q-Euler, Genocchi numbers and their interpolation functions*, Adv. Stud. Contemp. Math. **18** (2009), no. 2, 105-112.

[10] T. Kim, *A note on q-Bernstein polynomials*, uss. J. Math. Phys. 18 (2011), no. 1, 73-82.

[11] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **16** (2009), no. 4, 484-491

[12] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[13] T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys. 19 (2012), no. 1, 1-10.

[14] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$* , Russ. J. Math. Phys. **16** (2009), no. 1, 93-96.

[15] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*,J. Difference Equ. Appl. 14 (2008), no. 12, 1267-1277.

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Pseudo-differentiability, Pseudo-integrability and Nonlinear Differential Equations†

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Abstract: Based on the concepts of the pseudo-differentiability and the pseudo-integrability proposed in this paper, the transformation theorems for them are given. Newton-Leibniz formula is also obtained. As applications, the results can be applied directly to the discussion of nonlinear differential equations.
Keywords: seudo-analysis; pseudo-differentiability; Pseudo-integrability
AMS subject classifications. 26E50, 28E10.

1 Introduction

Pseudo-analysis was introduced and applied in different fields, e.g., measure theory, integration, integral operators, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. by E. Pap [11-18]. In fact, in many problems with uncertainty as in the theory of probabilistic metric spaces, fuzzy logics, fuzzy sets, fuzzy measures often we work with many operations different from the usual addition and multiplication of reals, e.g., triangular norms, triangular conorms, pseudo-additions, pseudo-multiplications, etc. Triangular conorm decomposable measures were first introduced by Dubois and Prade [3] as special important class of fuzzy measures [2]. Furthermore, it could be transferred into the corresponding results of reals [8-10, 12, 14, 19] such as the addition operator, multiplication operator, differentiability and integrability by using Aczel's representation [1, 7, 14]. However, we find that the definition of g-integrability does not coincide with the definition of pseudo-integrability with respect to a decomposable measure in different papers[8, 9, 10, 19]. One [4, 11, 14-15, 18, 21] is defined by first taking the integrability of an elementary function and the limit, and another [8-10, 12, 19] is defined using the usual Riemann, Stieltjes or Lebesgue integal of reals by Aczel's representation. In this paper, first, the definitions of the pseudo-differentiability and the pseudo-integrability are given. In addition, the transformation theorems for them are discussed. The Newton-Leibniz formula is also obtained. Finally, the results can be applied directly to the discussion of nonlinear differential equations.

To make our analysis possible, first we will recall some basic results of pseudo-additions in section 2. Section 3 contains the concepts of the pseudo-differentiability and the pseudo-integrability and the transformation theorems for them. Newton-Leibniz formula is also obtained in the section 4. We apply the obtained results directly to the discussion of nonlinear differential equations in section 5.

2 Notations and preliminaries

Let $[a, b]$ be a closed real interval of $[-\infty, +\infty]$.

According to [4, 14], a 2-place function $\oplus : [a, b] \times [a, b] \to [a, b]$ is said to be a pseudo-addition, if it is commutative, nondecreasing in each place, associative and has a zero element, denoted by $\mathbf{0}$.

Obviously, the usual addition and the triangular conorm, for instance, $S(x, y) = x \vee y = \max(x, y)$ and $S(x, y) = x + y - xy$, are typical pseudo-additions.

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Lemma 2.1 (Aczel's theorem [1, 7]) If $\oplus$ is continuous and strictly increasing in $(a,b)\times(a,b)$, then there exists a monotone function $g:[a,b]\to[-\infty,+\infty]$ such that $g(\mathbf{0})=0$ and

$$x\oplus y=g^{-1}(g(x)+g(y)),$$

where g is called a generator of $\oplus$.

3 The definitions of the pseudo-substraction and the pseudo-division

Definition 3.1 Let $x,\ y\in[a,b]$ and $\oplus$ be continuous and strictly increasing. If there exists $z\in[a,b]$ such that $x=y\oplus z$, then z is said to be a pseudo-difference of x and y, denoted by $x\ominus y$. The operator $\ominus$ is the pseudo-substraction.

Obviously, the following statements hold.

Corollary 3.1 Let $\ominus$ be a pseudo-substraction. Then $x\ominus y=g^{-1}(g(x)-g(y))$, where g is a generator of $\oplus$.

Corollary 3.2 Let $\ominus$ be a pseudo-substraction. Then $a\oplus(-b)=a\ominus b$.

Definition 3.2 Let $x,\ y\in[a,b]$ and $\oplus$ be continuous and strictly increasing. If $x\odot y=g^{-1}(g(x)\cdot g(y))$, where g is a generator of $\oplus$, then $\odot$ is said to be a pseudo-multiplication.

Definition 3.3 Let $x,\ y\in[a,b]$ and $y\neq\mathbf{0}$, and $\odot$ be a pseudo-multiplication. If there exists $z\in[a,b]$ such that $x=y\odot z$, then z is said to be a pseudo-quotient of x and y, denoted by $x\oslash y$. The $\oslash$ is the pseudo-division.

Corollary 3.3 If $\oslash$ is a pseudo-division and $y\neq\mathbf{0}$, then $x\oslash y=g^{-1}(g(x)/g(y))$.

Proof Since $\oslash$ is a pseudo-division, there exists z such that $x=y\odot z$, i.e., $z=x\oslash y$. On the other hand, since $\odot$ is a pseudo-multiplication, there exists a continuous and monotone function g such that $g^{-1}(g(y)\cdot g(z))=x$, thus $g(y)\cdot g(z)=g(x)$. For $y\neq\mathbf{0}$, $z=g^{-1}(g(x)/g(y))$. Therefore $x\oslash y=g^{-1}(g(x)/g(y))$.

4 The definitions of the pseudo-differentiability and the pseudo-integrability

Let $\oplus$ be continuous and strictly increasing. The pseudo-metric on $[a,b]$ is a function $d:[a,b]\times[a,b]\to[0,\infty]$ and defined by

$$d(x,y)=|x\ominus y|_\oplus,$$

where $|\cdot|_\oplus$ is a pseudo-absolute value. Obviously, operator d fulfills all conditions for being a metric. Furthermore, we have the following representant.

Remark 4.1 The pseudo-absolute value $|\cdot|_\oplus$ on $[a,b]$ is defined by

$$|x|_\oplus=|g(x)|$$

for $x\in[a,b]$, and where g is a generator of $\oplus$.

Remark 4.2 Let $d:[a,b]\times[a,b]\to[0,+\infty]$ be the metric on $[a,b]$. Then

$$d(x,y)=|g(x)-g(y)|$$

for $x,\ y\in[a,b]$, and where g is a generator of $\oplus$.

Definition 4.1 Let $\oplus$ be continuous and strictly increasing. A function $f:[c,d]\to[a,b]$ is said to be pseudo-differentiable at the point $x\in[c,d]$, if there exists $\frac{d^\oplus f(x)}{dx}\in[a,b]$ such that

$$\lim_{h\to0}[(f(x+h)\ominus f(x))\oslash((x+h)\ominus x)]$$

exists and equals to $\frac{d^\oplus f(x)}{dx}$. The $\frac{d^\oplus f(x)}{dx}$ is the pseudo-derivative of $f(x)$ at the point x. For $x=c,\ x=d$, only consider the single pseudo-derivative: $\lim_{h\to0}[(f(x+h)\ominus f(x))\oslash((x+h)\ominus x)]$ or $\lim_{h\to0}[(f(x)\ominus f(x-h))\oslash(x\ominus(x-h))]$.

Here the limit is taken in the metric space $([a,b],d)$.

Remark 4.3 Let f be a pseudo-differentiable function on $[c,d]$ and with the values in $[a,b]$, and the generator g of $\oplus$ be continuous on $[c,d]$. Then f is pseudo-continuous on $[c,d]$, i.e., $\lim_{h\to0}f(x+h)=f(x)$ for any $x\in[c,d]$.

Proof Fixed $x \in [c,d]$. Let f is pseudo-differentiable at x, and

$$\lim_{h\to 0}[(f(x+h)\ominus f(x))\oslash((x+h)\ominus x)] = \frac{d^{\oplus}f(x)}{dx}.$$

For the generator g of $\oplus$, we have

$$\lim_{h\to 0}\left|g[(f(x+h)\ominus f(x))\oslash((x+h)\ominus x)] - g(\frac{d^{\oplus}f(x)}{dx})\right| = 0.$$

It follows that

$$\lim_{h\to 0}\left|\frac{g(f(x+h))-g(f(x))}{g(x+h)-g(x)} - g(\frac{d^{\oplus}f(x)}{dx})\right| = 0.$$

That is to say,

$$\lim_{h\to 0}\left|g(f(x+h))-g(f(x))-(g(x+h)-g(x))g(\frac{d^{\oplus}f(x)}{dx})\right| = 0.$$

By the continuity of g, we have

$$\lim_{h\to 0}|g(f(x+h))-g(f(x))| = 0.$$

It implies

$$\lim_{h\to 0} f(x+h) = f(x).$$

Thus, f is pseudo-continuous on $[c,d]$.

Theorem 4.1 Let f_1 and f_2 be two pseudo-differentiable functions on $[c,d]$ and with the values in $[a,b]$. Then the following statements hold for any $\lambda,\ \lambda_1,\ \lambda_2 \in [a,b]$.

(1) $\lambda_1\odot f_1\oplus\lambda_2\odot f_2$ is pseudo-differentiable on $[c,d]$ and

$$\frac{d^{\oplus}(\lambda_1\odot f_1\oplus\lambda_2\odot f_2)}{dx} = \lambda_1\odot\frac{d^{\oplus}f_1}{dx}\oplus\lambda_2\odot\frac{d^{\oplus}f_2}{dx};$$

(2) $f_1\odot f_2$ is pseudo-differentiable on $[c,d]$ and

$$\frac{d^{\oplus}(f_1\odot f_2)}{dx} = \frac{d^{\oplus}f_1}{dx}\odot f_2\oplus f_1\odot\frac{d^{\oplus}f_2}{dx};$$

(3) $\frac{d^{\oplus}\lambda}{dx} = \mathbf{0}$.

Proof (1) Since f_1 and f_2 are pseudo-differentiable, we have

$$\begin{aligned}
&\frac{d^{\oplus}(\lambda_1\odot f_1\oplus\lambda_2\odot f_2)}{dx}\\
=&\lim_{h\to 0}\{[(\lambda_1\odot f_1(x+h)\oplus\lambda_2\odot f_2(x+h))\ominus\big(\lambda_1\odot f_1(x)\oplus\lambda_2\odot f_2(x)\big)]\oslash[(x+h)\ominus x]\}\\
=&\lim_{h\to 0}\{[\lambda_1\odot(f_1(x+h)\ominus f_1(x))\oplus\lambda_2\odot(f_2(x+h)\ominus f_2(x))]\oslash[(x+h)\ominus x]\}\\
=&\lambda_1\odot\lim_{h\to 0}[(f_1(x+h)\ominus f_1(x))\oslash((x+h)\ominus x)]\oplus\lambda_2\odot\lim_{h\to 0}[(f_2(x+h)\ominus f_2(x))\oslash((x+h)\ominus x)]\\
=&\lambda_1\odot\frac{d^{\oplus}f_1}{dx}\oplus\lambda_2\odot\frac{d^{\oplus}f_2}{dx}.
\end{aligned}$$

According to Definition 4.1, $\lambda_1\odot f_1\oplus\lambda_2\odot f_2$ is pseudo-differentiable on $[c,d]$ and

$$\frac{d^{\oplus}(\lambda_1\odot f_1\oplus\lambda_2\odot f_2)}{dx} = \lambda_1\odot\frac{d^{\oplus}f_1}{dx}\oplus\lambda_2\odot\frac{d^{\oplus}f_2}{dx}.$$

(2) Since f_1 and f_2 are pseudo-differentiable, we have

$$\begin{aligned}
&\frac{d^{\oplus}(f_1 \odot f_2)}{dx}\\
&= \lim_{h\to 0}\{[f_1(x+h)\odot f_2(x+h)\ominus f_1(x)\odot f_2(x)]\oslash[(x+h)\ominus x]\}\\
&= \lim_{h\to 0}\{[f_1(x+h)\odot f_2(x+h)\ominus f_1(x)\odot f_2(x)\ominus f_1(x)\odot f_2(x+h)\oplus f_1(x)\odot f_2(x+h)]\oslash[(x+h)\ominus x]\}\\
&= \lim_{h\to 0}\{[(f_1(x+h)\ominus f_1(x))\odot f_2(x+h)\oplus f_1(x)\odot (f_2(x+h)\ominus f_2(x))]\oslash[(x+h)\ominus x]\}\\
&= \lim_{h\to 0}[(f_1(x+h)\ominus f_1(x))\oslash((x+h)\ominus x)]\odot f_2(x)\oplus f_1(x)\odot \lim_{h\to 0}[(f_2(x+h)\ominus f_2(x))\oslash((x+h)\ominus x)]\\
&=\frac{d^{\oplus}f_1}{dx}\odot f_2\oplus f_1\odot\frac{d^{\oplus}f_2}{dx}.
\end{aligned}$$

According to Definition 4.1, $f_1 \odot f_2$ is pseudo-differentiable on $[c,d]$ and

$$\frac{d^{\oplus}(f_1 \odot f_2)}{dx}=\frac{d^{\oplus}f_1}{dx}\odot f_2\oplus f_1\odot\frac{d^{\oplus}f_2}{dx}.$$

(3) $\frac{d^{\oplus}\lambda}{dx}=\lim\limits_{h\to 0}(\lambda\ominus\lambda)\oslash((x+h)\ominus x)=\mathbf{0}$.

Theorem 4.2 Let f be a pseudo-differentiable function on $[c,d]$ and with the values in $[a,b]$, and the generator g of $\oplus$ be differentiable on $[c,d]$. Then

$$\frac{d^{\oplus}f(x)}{dx}=g^{-1}\left(\frac{dg(f(x))}{dg(x)}\right).$$

Proof

$$\begin{aligned}
\frac{d^{\oplus}f(x)}{dx} &= \lim_{h\to 0}[(f(x+h)\ominus f(x))\oslash((x+h)\ominus x)]\\
&= \lim_{h\to 0} g^{-1}\left(\frac{g(f(x+h))-g(f(x))}{g(x+h)-g(x)}\right)\\
&= g^{-1}\left(\lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{h}\cdot\frac{h}{g(x+h)-g(x)}\right)\\
&= g^{-1}\left([g(f)]'/g'(x)\right)\\
&= g^{-1}\left(\frac{dg(f(x))}{dg(x)}\right).
\end{aligned}$$

Remark 4.4 In [8, 9, 10, 19], the authors directly defined the g-derivative as follows.

$$\frac{d^{\oplus}f(x)}{dx}=g^{-1}\left(\frac{dg(f(x))}{dx}\right).$$

However, it may be more natural to define as the way proposed in this paper by Definition 4.1 and obtain Theorem 4.2.

Definition 4.2 Suppose that a function $f:[c,d]\to[a,b]$ has the $(n-1)$-th pseudo-derivative, then the (n)-th pseudo-derivative of f (if it exists) is defined as

$$\frac{d^{(n)\oplus}f}{dx^n}=\frac{d^{\oplus}}{dx}\left(\frac{d^{(n-1)\oplus}f}{dx^{n-1}}\right),\ n\geqslant 1.$$

Theorem 4.3 If there exists an (n)-th pseudo-derivative of f, then

$$\frac{d^{(n)\oplus}f(x)}{x^n}=g^{-1}\left(\frac{d^n g(f(x))}{d[g(x)]^n}\right),\ n\geqslant 0.$$

Proof For $n=0$, the theorem is obviously true.

Suppose that the theorem is true for $n-1$, i.e.,

$$\frac{d^{(n-1)\oplus} f(x)}{x^{n-1}} = g^{-1}\left(\frac{d^{n-1}g(f(x))}{d[g(x)]^{n-1}}\right).$$

Then

$$\begin{aligned}\frac{d^{(n)\oplus} f(x)}{x^{n}} &= \frac{d^{\oplus}}{dx}\left(\frac{d^{(n-1)\oplus} f}{dx^{n-1}}\right)\\ &= \frac{d^{\oplus}}{dx}\left(g^{-1}\left(\frac{d^{n-1}g(f(x))}{d[g(x)]^{n-1}}\right)\right)\\ &= g^{-1}\left(\frac{d}{dg(x)}\left(\frac{d^{n-1}g(f(x))}{d[g(x)]^{n-1}}\right)\right)\\ &= g^{-1}\left(\frac{d^{n}g(f(x))}{d[g(x)]^{n}}\right).\end{aligned}$$

By mathematical induction, the theorem is proved.

Definition 4.3 Let $\oplus$ be continuous and strictly increasing. Let $f(x)$ be a bounded function defined on $[c, d]$. If for any partition of $[c, d]$

$$P : c = x_0 < x_1 < x_2 < \cdots < x_n = d,$$

denote $\lambda = \max\limits_{1\leqslant i\leqslant n}(x_i \ominus x_{i-1})$, if for any $\xi_i \in [x_{i-1}, x_i]$, the limit

$$\lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f(\xi_i) \odot (x_i \ominus x_{i-1})$$

exists, then $f(x)$ is said to be pseudo-integrable on $[c, d]$, and its pseudo-integral value equals to the limit value, denoted by $\int_{[c,d]}^{(\oplus,\odot)} f(x)dx$.

Theorem 4.4 Let f_1 and f_2 be two pseudo-integrable functions on $[c, d]$ and with the values in $[a, b]$. Then for $\lambda_1,\ \lambda_2 \in [a, b],\ \lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2$ is also generalized integrable on $[c, d]$ and

$$\int_{[c,d]}^{(\oplus,\odot)} (\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2)dx = \lambda_1 \odot \int_{[c,d]}^{(\oplus,\odot)} f_1dx \oplus \lambda_2 \odot \int_{[c,d]}^{(\oplus,\odot)} f_2dx.$$

Proof For any partition of $[c, d]$

$$P : c = x_0 < x_1 < x_2 < \cdots < x_n = d$$

and for any $\xi_i \in [x_{i-1}, x_i]$, we have

$$\begin{aligned}&\bigoplus_{i=1}^{n} \big(\lambda_1 \odot f_1(\xi_i) \oplus \lambda_2 \odot f_2(\xi_i)\big) \odot (x_i \ominus x_{i-1})\\ =&\bigoplus_{i=1}^{n} \big(\lambda_1 \odot f_1(\xi_i) \odot (x_i \ominus x_{i-1}) \oplus \lambda_2 \odot f_2(\xi_i) \odot (x_i \ominus x_{i-1})\big)\\ =&\lambda_1 \odot \left(\bigoplus_{i=1}^{n} f_1(\xi_i) \odot (x_i \ominus x_{i-1})\right) \oplus \lambda_2 \odot \left(\bigoplus_{i=1}^{n} f_2(\xi_i) \odot (x_i \ominus x_{i-1})\right).\end{aligned}$$

Let $\lambda = \max\limits_{1\leqslant i\leqslant n}(x_i \ominus x_{i-1}) \to \mathbf{0}$, since f_1 and f_2 are pseudo-integrable on $[c, d]$, we have

$$\begin{aligned}&\lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} \big(\lambda_1 \odot f_1(\xi_i) \oplus \lambda_2 \odot f_2(\xi_i)\big) \odot (x_i \ominus x_{i-1})\\ =&\lambda_1 \odot \left(\lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f_1(\xi_i) \odot (x_i \ominus x_{i-1})\right) \oplus \lambda_2 \odot \left(\lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f_2(\xi_i) \odot (x_i \ominus x_{i-1})\right)\\ =&\lambda_1 \odot \int_{[c,d]}^{(\oplus,\odot)} f_1dx \oplus \lambda_2 \odot \int_{[c,d]}^{(\oplus,\odot)} f_2dx.\end{aligned}$$

According to Definition 4.3, $\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2$ is pseudo-integrable on $[c, d]$ and

$$\int_{[c,d]}^{(\oplus,\odot)} (\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2)dx = \lambda_1 \odot \int_{[c,d]}^{(\oplus,\odot)} f_1 dx \oplus \lambda_2 \odot \int_{[c,d]}^{(\oplus,\odot)} f_2 dx.$$

Theorem 4.5 Let $f(x)$ be pseudo-integrable on $[c, d]$. Then

$$\int_{[c,d]}^{(\oplus,\odot)} f dx = g^{-1}\left(\int_c^d g(f(x))dg(x)\right),$$

when the right part is meaningful.

Proof

$$\begin{aligned}\int_{[c,d]}^{(\oplus,\odot)} f dx &= \lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f(\xi_i) \odot (x_i \ominus x_{i-1}) \\ &= \lim_{\lambda\to\mathbf{0}} [f(\xi_1) \odot (x_1 \ominus x_0) \oplus f(\xi_2) \odot (x_2 \ominus x_1) \oplus \cdots \oplus f(\xi_n) \odot (x_n \ominus x_{n-1})] \\ &= \lim_{\lambda\to\mathbf{0}} \{g^{-1}[g(f(\xi_1)) \cdot g(x_1 \ominus x_0)] \oplus g^{-1}[g(f(\xi_2)) \cdot g(x_2 \ominus x_1)] \oplus \cdots \\ &\quad \oplus g^{-1}[g(f(\xi_n)) \cdot g(x_n \ominus x_{n-1})]\} \\ &= \lim_{\lambda\to\mathbf{0}} g^{-1}[g(f(\xi_1)) \cdot g(x_1 \ominus x_0) + g(f(\xi_2)) \cdot g(x_2 \ominus x_1) + \cdots + g(f(\xi_n)) \cdot g(x_n \ominus x_{n-1})] \\ &= g^{-1}[\lim_{\lambda'\to 0} \big(g(f(\xi_1)) \cdot (g(x_1) - g(x_0)) + g(f(\xi_2)) \cdot (g(x_2) - g(x_1)) + \cdots \\ &\quad + g(f(\xi_n)) \cdot (g(x_n) - g(x_{n-1}))\big)] \\ &= g^{-1}\left(\int_c^d g(f(x))dg(x)\right),\end{aligned}$$

where $\lambda' = \max_{1\leqslant i\leqslant n} |g(x_i) - g(x_{i-1})|$.

Remark 4.5 For $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}x_i \ominus x_{i-1} \to \mathbf{0} &\Longleftrightarrow d(x_i, x_{i-1}) \to 0 \\ &\Longleftrightarrow |g(x_i) - g(x_{i-1})| \to 0,\end{aligned}$$

therefore

$$\max_{1\leqslant i\leqslant n} (x_i \ominus x_{i-1}) \to \mathbf{0} \Longleftrightarrow \max_{1\leqslant i\leqslant n} |g(x_i) - g(x_{i-1})| \to 0,$$

namely,

$$\lambda \to \mathbf{0} \Longleftrightarrow \lambda' \to 0.$$

Remark 4.6 In [8, 9, 10, 19], the authors directly defined the g-integral as follows.

$$\int_{[c,d]}^{(\oplus,\odot)} f dx = g^{-1}\left(\int_c^d g(f)dx\right).$$

However, it may be more natural to define as the way proposed in this paper by Definition 4.3 and obtain Theorem 4.5. Additionally, the definition proposed in this paper coincides with the definition of integral with respect to a decomposable measure m proposed in [11-18], i.e., $\int_{[c,d]}^{(\oplus,\odot)} f dm = g^{-1}\left(\int_c^d g(f)dg \circ m\right)$.

Theorem 4.6 Suppose that f is continuous on $[c, d]$. Then we have

$$\frac{d^{\oplus}}{dt}\left(\int_{[c,x]}^{(\oplus,\odot)} f(t)dt\right) = f(x)$$

for any $x \in [c, d]$.

Proof From Theorem 4.2 and 4.5, we have for any $x \in [c,d]$

$$\begin{aligned}\frac{d^{\oplus}}{dt}\left(\int_{[c,x]}^{(\oplus,\odot)} f(t)dt\right) &= g^{-1}\left(\frac{dg\left(\int_{[c,x]}^{(\oplus,\odot)} f(t)dt\right)}{dg(t)}\right)\\ &= g^{-1}\left(\frac{d(\int_c^x g(f(t))dg(t))}{dg(t)}\right)\\ &= g^{-1}(g(f(x)))\\ &= f(x),\end{aligned}$$

where we have used the fundamental theorem of the usual calculus.

Theorem 4.7 (Newton-Leibniz formula) Suppose that f has continuous pseudo-derivative on $[c,d]$. Then we have for any $x \in [c,d]$

$$\int_{[c,x]}^{(\oplus,\odot)} \frac{d^{\oplus} f}{dt} dt = f(x) \ominus f(c)$$

for any $x \in [c,d]$.

Proof From Theorem 4.2 and 4.5, we have for any $x \in [c,d]$

$$\begin{aligned}\int_{[c,x]}^{(\oplus,\odot)} \frac{d^{\oplus} f(t)}{dt} dt &= g^{-1}\left(\int_c^x g\left(\frac{d^{\oplus} f(t)}{dt}\right) dg(t)\right)\\ &= g^{-1}\left(\int_c^x \frac{dg(f(t))}{dg(t)} dg(t)\right)\\ &= g^{-1}\left(\int_c^x dg(f(t))\right)\\ &= g^{-1}\big(g(f(x)) - g(f(c))\big)\\ &= f(x) \ominus f(c).\end{aligned}$$

5 Applications

Example 5.1 Consider the following ordinary differential equation of the first order for $s \in [0,+\infty)$

$$\ln y' - y + x^s + x = 0. \tag{1}$$

Let $g(x) = e^{-x}$, then Equation (1) can be represented as the following pseudo-differential equation

$$\frac{d^{\oplus} y}{dx} = x^s.$$

Pseudo-integrating the preceding equation correspondingly, we have

$$\begin{aligned}y &= \int^{(\oplus,\odot)} x^s dx \oplus C_1\\ &= g^{-1}\left(\int g(x^s) dg(x) + g(C_1)\right)\\ &= -ln\left(\frac{e^{-x-x^s}}{1+sx^{s-1}} + C\right),\end{aligned}$$

where $C = C_2 + e^{-C_1}$.

Example 5.2 Consider the following equation for $p > 0$ and $s \in [0,+\infty)$

$$(y')^{1/p} y^{1-1/p} - x^{s+(1-1/p)} = 0. \tag{2}$$

Let $g(x) = x^p$, then Equation (2) can be represented as the following pseudo-differential equation

$$\frac{d^{\oplus} y}{dx} = x^s.$$

Pseudo-integrating the preceding equation correspondingly, we have

$$\begin{aligned} y &= \int^{(\oplus,\odot)} x^s dx \oplus C_1 \\ &= g^{-1}\left(\int g(x^s)dg(x) + g(C_1)\right) \\ &= \left(\frac{x^{p(s+1)}}{s+1} + C\right)^{1/p}, \end{aligned}$$

where $C = C_2 + C_1^p$.

Example 5.3 Consider the following equation

$$(x-1)^3(1-y)y' + xy^2 + \big((x-1)^3 - 2x\big)y + x = 0. \tag{3}$$

Let $g(x) = \frac{x}{1-x}$, then Equation (3) can be represented as the following pseudo-differential equation

$$\frac{d^{\oplus} y}{dx} = x.$$

Pseudo-integrating the preceding equation correspondingly, we have

$$\begin{aligned} y &= \int^{(\oplus,\odot)} x dx \oplus C_1 \\ &= g^{-1}\left(\int g(x)dg(x) + g(C_1)\right) \\ &= \frac{2C(1-x)^2 + x^2}{2(1-x)^2 + 2C(1-x)^2 + x^2}, \end{aligned}$$

where $C = C_2 + \frac{C_1}{1-C_1}$.

Example 5.4 Solve the following equation for $\lambda > 0$

$$\lambda(1+\lambda)^y y' - (1+\lambda)^{2x} + (1+\lambda)^x = 0. \tag{4}$$

Let $g(x) = \frac{(1+\lambda)^x - 1}{\lambda}$, $\lambda > 0$, then Equation (4) can be represented as the following pseudo-differential equation

$$\frac{d^{\oplus} y}{dx} = x.$$

Pseudo-integrating the preceding equation correspondingly, we have

$$\begin{aligned} y &= \int^{(\oplus,\odot)} x dx \oplus C_1 \\ &= g^{-1}\left(\int g(x)dg(x) + g(C_1)\right) \\ &= \frac{\ln\big(((1+\lambda)^x - 1)^2/2\lambda + C\big)}{\ln(1+\lambda)}, \end{aligned}$$

where $C = C_2\lambda + (1+\lambda)^{C_1}$.

Example 5.5 Let $\Omega = [-L/2, L/2]$, $L > 0$ given. We consider as an important nonlinear partial differential equation the Kuramoto-Sivashinsky equation (see [5, 6, 20])

$$u_t + \lambda u_{xxxx} + u_{xx} + \frac{1}{2}u_x^2 = 0$$

for $u = u(x,t)$ and $x \in \Omega$, where λ is the given positive constant.

Let $g(x) = e^{-x/c}$, then $k \odot u = k + u, u \oplus u = u - c\ln 2$. If u is a solution of the Kuramoto-Sivashinsky equation, then put $k \odot u$ and $u \oplus u$ in the Kuramoto-Sivashinsky equation, respectively, we obtain an identity, thus $k \odot u$ and $u \oplus u$ are also solutions of the Kuramoto-Sivashinsky equation. Therefore $(k \odot u) \oplus (k \odot u)$ is a solution of the Kuramoto-Sivashinsky equation. We find new solutions of the Kuramoto-Sivashinsky equation.

References

[1] J. Aczel, Lectures on Functional Equations and their Applications, Academic Press, New York, 1966.

[2] D. Dubois, M. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.

[3] D. Dubois, M. Prade, A class of fuzzy measures based on triangular norms, Int.J. Gen. System 8 (1982) 43-61.

[4] H. Ichihashi, H. Tanaka, K. Asai, Fuzzy integrals based on pseudo-additions and multiplications, Journal of Mathematical Analysis and Applications 130 (1988) 354-364.

[5] Y. Kuramoto, Diffusion induced chaos in reactions systems, Progr. Theoret. Phys. Suppl. 64 (1978) 346-367.

[6] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, Prog. Theor. Phys. 55 (1976) 356-369.

[7] C. H. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189-212.

[8] A. Markov*á*, A note on g-integral, Tatra Mountains Math. Publ. 8 (1996) 71-76.

[9] A. Markov*á*, B. Riečan, On the double g-integral, Novi Sad J. Math. (1996) 67-70.

[10] R. Mesiar, Pseudo-linear integrals and derivatives based on a generator g, Tatra Mountains Math. Publ. 8 (1996) 67-70.

[11] E. Pap, Integral generated by decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20 (1990) 135-144.

[12] E. Pap, g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23 (1993) 145-150.

[13] E. Pap, Nonlinear difference equations and neural nets, BUSEFAL 60 (1994) 7-14.

[14] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995.

[15] E. Pap, Decomposable measures and nonlinear equations, Fuzzy Sets Syst. 92 (1997) 205-221.

[16] E. Pap, N. Ralević, Pseudo-Laplace Transform, Nonlinear Anal. 33 (1998) 533-550.

[17] E. Pap, I. Š*tajner, Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory, Fuzzy Sets Syst. 102 (1999) 393-415.

[18] E. Pap, Pseudo-additive measures and their applications, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, North-Holland, Amsterdam, 2002, pp. 1237-1260.

[19] N. Ralević, Some new properties of g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 24 (1994) 139-157.

[20] G. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flame. I . Derivation of basic equations, Acta Astron. 4 (1977) 1117-1206.

[21] M. Sugeno, T. Murofushi, Pseudo-additive measures and intgerals, J. Math. Anal. Appl. 122 (1987) 197-222.

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A GENERAL THEOREM ASSOCIATED WITH THE BRIOT–BOUQUET DIFFERENTIAL SUBORDINATION

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Abstract. Recently, many authors used the techniques of differential subordination to find the best dominant of different subordination relation for certain subclasses of analytic functions, associated with the Briot–Bouquet differential subordination. In this paper we find some general results, which determine the best dominant of a general form of differential subordination relation for the above mentioned Briot–Bouquet differential subordination.

1. Basic definitions and preliminaries

We denote by $\mathcal{H}(\mathrm{U})$ the class of all analytic functions in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the class of functions $\varphi \in \mathcal{H}(\mathrm{U})$ which satisfy $\varphi(0) = 1$.

To prove our main results, we need the following definitions and the lemmas.

Definition 1.1. [15, p. 36] *Let f and F be two analytic functions. The function f is said to be subordinate to F, written as $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$.*

It is well-known that if the function F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathrm{U}) \subset F(\mathrm{U})$.

Definition 1.2. [7, p. 16] *Let $\Psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$, and let h be univalent in U. If p is an analytic function in U, and $\Psi(p(z), z\,p'(z); z)$ is also analytic in U, then we say that p satisfies a first order differential subordination if*

$$\Psi(p(z), z\,p'(z); z) \prec h(z). \tag{1.1}$$

The univalent function q is called to be a dominant of the differential subordination (1.1), if $p(z) \prec q(z)$ for all p satisfying (1.1). If $\tilde{q}(z) \prec q(z)$ for all the dominants q of (1.1), then we say that $\tilde{q}$ is the best dominant of (1.1).

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Lemma 1.1 (Wilken and Feng [17]; see also [7])**.** *Let μ be a positive measure on the unit interval $[0,1]$. Let $g(z,t)$ be a complex valued function defined on $\mathrm{U}\times[0,1]$ such that $g(\cdot,t)$ is analytic in U for each $t\in[0,1]$, and such that $g(z,\cdot)$ is μ-integrable on $[0,1]$, for all $z\in\mathrm{U}$. In addition, suppose that $\operatorname{Re} g(z,t)>0$, $g(-r,t)$ is real, and*

$$\operatorname{Re}\frac{1}{g(z,t)}\geq\frac{1}{g(-r,t)},\ \textit{for all}\ |z|\leq r<1,\ t\in[0,1].$$

If the function G is defined by

$$G(z)=\int_0^1 g(z,t)\,\mathrm{d}\,\mu(t),$$

then

$$\operatorname{Re}\frac{1}{G(z)}\geq\frac{1}{G(-r)},\ \textit{for all}\ |z|\leq r<1.$$

Each of the following identities asserted by Lemma 1.2 is well known:

Lemma 1.2. [5, Chapter 9] *For all real or complex numbers a, b and c, with $c\notin\mathbb{Z}_0^-$, the following equalities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}\,t=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z), \tag{1.2}$$
where $\operatorname{Re}c>\operatorname{Re}b>0$;

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\ {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right); \tag{1.3}$$

and

$${}_2F_1(a,b;c;z)=\ {}_2F_1(b,a;c;z). \tag{1.4}$$

The next lemma is a special case of a more general result [6, Corollary3.2.]:

Lemma 1.3. [6, Example 1.] *If $-1\leq A<B\leq1$, $\beta>0$, and the complex number γ satisfies*

$$\operatorname{Re}\gamma\geq-\frac{\beta(1-A)}{1-B},$$

then the following differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz},\quad \textit{with}\quad q(0)=1,$$

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has a univalent solution in U *given by*

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\,\mathrm{d}\,t}-\dfrac{\gamma}{\beta}, & \text{if} \quad B\neq 0\\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\displaystyle\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\,\mathrm{d}\,t}-\dfrac{\gamma}{\beta}, & \text{if} \quad B=0.\end{cases}$$

Moreover, if the function φ *is analytic in* U *and satisfies the following subordination*

$$\varphi(z)+\frac{z\varphi'(z)}{\beta\varphi(z)+\gamma}\prec\frac{1+Az}{1+Bz}, \tag{1.5}$$

then

$$\varphi(z)\prec q(z)\prec\frac{1+Az}{1+Bz},$$

and q *is the best dominant of* (1.5).

2. Main results

Theorem 2.1. *Let* A, B *and* β *be real numbers which satisfy the conditions*

$$-1\leq B<A\leq 1, \quad \text{and} \quad \beta>0, \tag{2.1}$$

and let γ *be a complex number with*

$$\operatorname{Re}\gamma\geq-\frac{\beta(1-A)}{1-B}. \tag{2.2}$$

1. *If* $\varphi\in\mathcal{A}$ *and satisfies* (1.5), *then*

$$\varphi(z)\prec q(z)=\frac{1}{\beta}\left[\frac{1}{M(z)}-\gamma\right]\prec\frac{1+Az}{1+Bz}, \tag{2.3}$$

where

$$M(z)=\begin{cases}\displaystyle\int_0^1 t^{\beta+\gamma-1}\left(\frac{1+Btz}{1+Bz}\right)^{\frac{\beta(A-B)}{B}}\mathrm{d}\,t, & \text{if} \quad B\neq 0,\\[2ex] \displaystyle\int_0^1 t^{\beta+\gamma-1}\exp\left(\beta(t-1)Az\right)\mathrm{d}\,t, & \text{if} \quad B=0,\end{cases} \tag{2.4}$$

and q *is the best dominant of* (2.3).

2. *Suppose, in addition, that* γ *is a real number, with*

$$\gamma>-1-\frac{\beta A}{B}, \quad \text{and} \quad -1\leq B<0. \tag{2.5}$$

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Then

(2.6)
$$\operatorname{Re}\varphi(z) > \frac{\beta+\gamma}{\beta}\left[{}_2F_1\left(1, \frac{\beta(B-A)}{B}; \beta+\gamma+1; \frac{B}{B-1}\right)\right]^{-1} - \frac{\gamma}{\beta},\ z \in \mathrm{U},$$

and the right hand side of (2.6) *cannot be replaced by a larger one.*

Proof. Since the function φ satisfies (1.5), from Lemma 1.3 we deduce that

$$\varphi(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where q is given in (2.3), and it is the best dominant of the given subordination. This proves the assertion (2.3) of the theorem.

In order to prove (2.6) it is sufficient to show that

$$\inf\{\operatorname{Re} q(z) : z \in \mathrm{U}\} = q(-1). \tag{2.7}$$

Putting $a = \dfrac{\beta(B-A)}{B}$, since $B \geq -1$, then from (2.4) by using (1.2), (1.3), and (1.4) for $B \neq 0$, we obtain that

$$\begin{aligned} M(z) &= \int_0^1 t^{\beta+\gamma-1}\left(\frac{1+Btz}{1+Bz}\right)^{\frac{\beta(A-B)}{B}} \mathrm{d}\,t \\ &= (1+Bz)^a \int_0^1 t^{\beta+\gamma-1}(1+Btz)^{-a}\,\mathrm{d}\,t \\ &= \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta+\gamma+1)}\,{}_2F_1\left(1, a; \beta+\gamma+1; \frac{Bz}{Bz+1}\right). \end{aligned} \tag{2.8}$$

In view of the formula (1.2), this last relation holds whenever $\operatorname{Re}(\beta+\gamma+1) > \operatorname{Re} a > 0$, and under our assumption these inequalities are equivalent to (2.5). We remark that the condition (2.2) implies $\beta+\gamma > 0$, and also the convergence of the integrals from the formula (2.8).

Now, to prove (2.7) we need to show that

$$\operatorname{Re}\frac{1}{M(z)} \geq \frac{1}{M(-1)},\ z \in \mathrm{U}.$$

By using the relations (1.2) and (2.8) we have

$$M(z) = \int_0^1 h(z,t)\,\mathrm{d}\,\nu(t),$$

where

$$h(z,t) = \frac{1+Bz}{1+(1-t)Bz} \qquad (z \in \mathrm{U},\ 0 \leq t \leq 1),$$

and

$$\mathrm{d}\,\nu(t) = \frac{\Gamma(\beta+\gamma)}{\Gamma(a)\Gamma(\beta+\gamma+1-a)}\,t^{a-1}(1-t)^{\beta+\gamma-a}\,\mathrm{d}\,t,$$

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is a positive measure on $[0,1]$.

We note that $\operatorname{Re} h(z,t) > 0$, $h(-r,t)$ is real whenever $r \in [0,1)$, and for $-1 \le B < 0$ we have that

$$\operatorname{Re} \frac{1}{h(z,t)} = \operatorname{Re} \frac{1+(1-t)Bz}{1+Bz} \ge \frac{1+(1-t)Br}{1+Br} = \frac{1}{h(-r,t)}.$$

Therefore, by using Lemma 1.1 we have

$$\operatorname{Re} \frac{1}{M(z)} \ge \frac{1}{M(-r)}, \ |z| \le r < 1,$$

which, upon letting $r \to 1^-$, yields

$$\operatorname{Re} \frac{1}{M(z)} > \frac{1}{M(-1)}, \ z \in \mathrm{U}.$$

Since q is the best dominant of (2.3), it follows that the constant from the right-hand side of (2.6) cannot be replaced by a larger one. □

Corollary 2.1. *Let A, B, β and γ be constrained by the conditions* (2.1), (2.2), *and* (2.5). *Let $\mathrm{H}_s : \mathcal{H}(\mathrm{U}) \to \mathcal{H}(\mathrm{U})$ be an operator such that $\dfrac{z\mathrm{H}'_{s+1}(f)(z)}{\mathrm{H}_{s+1}(f)(z)}$ is analytic in* U, *with*

$$\left.\frac{z\mathrm{H}'_{s+1}(f)(z)}{\mathrm{H}_{s+1}(f)(z)}\right|_{z=0} = \beta + k + \gamma,$$

and satisfies

$$z\mathrm{H}'_{s+1}(f)(z) = k\mathrm{H}_{s+1}(f)(z) + m\mathrm{H}_s(f)(z), \tag{2.9}$$

for some $k, m \in \mathbb{C}$ and for all $f \in \mathcal{H}(\mathrm{U})$. Also, let define the class $R_{s;k,\gamma}(A,B)$ by

$$R_{s;k,\gamma}(A,B) = \left\{ f \in \mathcal{H}(\mathrm{U}) : \frac{1}{\beta}\left(\frac{z\mathrm{H}'_s(f)(z)}{\mathrm{H}_s(f)(z)} - (k+\gamma)\right) \prec \frac{1+Az}{1+Bz} \right\}.$$

1. *If $f \in R_{s;k,\gamma}(A,B)$, then*

$$\frac{1}{\beta}\left(\frac{z\mathrm{H}'_{s+1}(f)(z)}{\mathrm{H}_{s+1}(f)(z)} - (k+\gamma)\right) \prec q(z) = \frac{1}{\beta}\left[\frac{1}{M(z)} - \gamma\right] \prec \frac{1+Az}{1+Bz}, \tag{2.10}$$

where M is given by (2.4), *and q is the best dominant of* (2.10), *hence*

$$R_{s;k,\gamma}(A,B) \subset R_{s+1;k,\gamma}(A,B).$$

2. *Moreover, if γ is a real number, $-1 \le B < 0$, and $\gamma > -1 - \dfrac{\beta A}{B}$, then*

$$\mathrm{H}_{s+1}\left(R_{s;k,\gamma}(A,B)\right) \subset S^*(\mu),$$

(where $S^*(\mu)$ represents the class of starlike functions of order μ) and

$$R_{s;k,\gamma}(A,B) \subset R_{s+1;k,\gamma}\left(1 - \frac{2\left(\mu - \gamma - \operatorname{Re} k\right)}{\beta}, -1\right),$$

where

$$\mu = (\beta+\gamma)\left[{}_2F_1\left(1, \frac{\beta(B-A)}{B}; \beta+\gamma+1; \frac{B}{B-1}\right)\right]^{-1} + \operatorname{Re} k.$$

The constant μ cannot be replaced by a larger one.

Proof. Putting

$$\varphi(z) = \frac{1}{\beta}\left(\frac{z\mathrm{H}'_{s+1}(f)(z)}{\mathrm{H}_{s+1}(f)(z)} - (k+\gamma)\right),$$

then $\varphi \in \mathcal{A}$, and using the identity (2.9), we have

$$\frac{m\mathrm{H}_s(f)(z)}{\mathrm{H}_{s+1}(f)(z)} = \beta\varphi(z) + \gamma.$$

Carrying out logarithmic differentiation in the above relation and using Theorem 2.1, we deduce the result of the corollary. □

Corollary 2.2. *Let A, B, β and γ be constrained by the conditions (2.1), (2.2), and (2.5). Let $\mathrm{H}_s : \mathcal{H}(\mathrm{U}) \to \mathcal{H}(\mathrm{U})$ be an operator such that $\dfrac{z\mathrm{H}'_s(f)(z)}{\mathrm{H}_s(f)(z)}$ is analytic in U, with*

$$\left.\frac{z\mathrm{H}'_s(f)(z)}{\mathrm{H}_s(f)(z)}\right|_{z=0} = \beta + k + \gamma,$$

and satisfies

$$z\mathrm{H}'_s(f)(z) = k\mathrm{H}_{s+1}(f)(z) + m\mathrm{H}_s(f)(z), \tag{2.11}$$

for some $k, m \in \mathbb{C}$ and for all $f \in \mathcal{H}(\mathrm{U})$. Also, let define the class $T_{s;m,\gamma}(A,B)$ by

$$T_{s;m,\gamma}(A,B) = \left\{f \in \mathcal{H}(\mathrm{U}) : \frac{1}{\beta}\left(\frac{z\mathrm{H}'_s(f)(z)}{\mathrm{H}_s(f)(z)} - (m+\gamma)\right) \prec \frac{1+Az}{1+Bz}\right\}.$$

1. *If $f \in T_{s+1;m,\gamma}(A,B)$, then*

$$\frac{1}{\beta}\left(\frac{z\mathrm{H}'_s(f)(z)}{\mathrm{H}_s(f)(z)} - (m+\gamma)\right) \prec q(z) = \frac{1}{\beta}\left[\frac{1}{M(z)} - \gamma\right] \prec \frac{1+Az}{1+Bz}, \tag{2.12}$$

where M is given by (2.4), and q is the best dominant of (2.12), hence

$$T_{s+1;m,\gamma}(A,B) \subset T_{s;m,\gamma}(A,B).$$

2. *Moreover, if γ is a real number, $-1 \le B < 0$, and $\gamma > -1 - \dfrac{\beta A}{B}$, then*

$$\mathrm{H}_s\left(T_{s+1;m,\gamma}(A,B)\right) \subset S^*(\mu),$$

(where $S^*(\mu)$ represents the class of starlike functions of order μ) and

$$T_{s+1;m,\gamma}(A,B) \subset T_{s;m,\gamma}\left(1 - \frac{2(\mu-\gamma-\operatorname{Re} m)}{\beta}, -1\right),$$

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where

$$\mu = (\beta+\gamma)\left[{}_2F_1\left(1, \frac{\beta(B-A)}{B}; \beta+\gamma+1; \frac{B}{B-1}\right)\right]^{-1} + \operatorname{Re} m.$$

The constant μ cannot be replaced by a larger one.

Proof. If we let

$$\varphi(z) = \frac{1}{\beta}\left(\frac{z\mathrm{H}_s'(f)(z)}{\mathrm{H}_s(f)(z)} - (m+\gamma)\right),$$

then $\varphi \in \mathcal{A}$, and by using the identity (2.11) we have

$$\frac{k\mathrm{H}_{s+1}(f)(z)}{\mathrm{H}_s(f)(z)} = \beta\varphi(z) + \gamma.$$

Calculating the logarithmic derivative of the above relation and using Theorem 2.1, we deduce immediately the above result. □

Remarks 2.1. i) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$, $k = -n$, $m = n+p$, $\beta = p - \eta$, and $\gamma = \eta + n$ in Corollary 2.1, we get the result due to Patel and Cho [10].

ii) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$, $\varphi(z) = \dfrac{zf'(z)}{pf(z)^{1-\mu}g(z)^{\mu}}$ $\left(g \in S_p^*\right)$, $\beta = \dfrac{p}{\lambda}$, and $\gamma = 0$ in Theorem 2.1, we get the result due to Patel [9].

iii) Putting $f(z) = z + \sum_{t=2}^{\infty} a_t z^t$, $\mathrm{H}_s = \mathrm{J}_{s,b}$ (Srivastava-Attiya operator see([16], [4])), $k = -b$, $m = 1+b$, $\beta = 1-\alpha$, and $\gamma = \alpha + b$ in Corollary 2.1, we get the result due to Kutbi and Attiya [3].

iv) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$, $\varphi(z) = \lambda + (1-\lambda)\dfrac{f'(z)}{pz^{p-1}}$, $\beta = \dfrac{p}{\lambda}$, and $\gamma = -p$ in Theorem 2.1, we get the result due to Patel and Cho [11].

v) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$ and $\mathrm{H}_s = \mathrm{T}_p^{\lambda}(a,c)$ (see [12]), $k = \lambda + p$, $m = -\lambda$, $\beta = p - \eta$, and $\gamma = \eta + \lambda$ in Corollary 2.2, we get the result due to Patel et al. [12].

vi) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$, $\mathrm{H}_s = \mathrm{H}_p^{l,m}(a_1)$ (Dziok-Srivastava operator), $k = a_1$, $m = p - a_1$, $\beta = p - \alpha$, and $\gamma = a_1 + \alpha - p$ in Corollary 2.2, we get the result due to Patel et al. [14].

vii) Putting $f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t$ $(p \in \mathbb{N})$ and $\mathrm{H}_s = \Omega_z^{\lambda,p}$ (see [13]), $k = \beta = p - \lambda$, and $m = \lambda$ in Corollary 2.2, we get the result due to Patel and Mishra [13].

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Example 2.1. i) Cho et al. [2] (see also [8]) introduced the operator $\mathrm{I}_p^\lambda(a,c)$ defined by $\mathrm{I}_p^\lambda(a,c)(f)(z) = z^p\,{}_2F_1(c,\lambda+p,a;z) * f(z)$ ($a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $p, n \in \mathbb{N}$, and $f(z) = z^p + \sum\limits_{t=p+n}^{\infty} a_t z^t$). In [2] the authors showed that the operator $\mathrm{I}_p^\lambda(a,c)(f)$ satisfies the identity (2.9) with $k = p - a$ and $m = a$. Then we may use the Corollary 2.1 to generalize their result, by replacing the operator H_s with $\mathrm{I}_p^\lambda(a,c)$.

ii) Cho et al. [1] introduced the operator $\mathrm{I}_\lambda^{k,p}$ (see[1]), where they proved that the operator $\mathrm{I}_\lambda^{k,p}$ satisfies the identity (2.11) with $k = \lambda + p$ and $m = -\lambda$. In order to generalize their result, we can apply Corollary 2.2 by replacing the operator H_s with $\mathrm{I}_\lambda^{k,p}$.

References

[1] N. E. Cho, I. H. Kim and H. M. Srivastava, Sandwich-type theorems for multivalent functions associated with the Srivastava–Attiya operator, Appl. Math. Comput., 217 (2010), no. 2, 918—928.

[2] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292 (2004), no. 2, 470–483.

[3] M. A. Kutbi and A. A. Attiya, Differential subordination result with the Srivastava–Attiya integral operator, J. Ineq. Appl., 2010 (2010), 1–10.

[4] M.A. Kutbi and A.A. Attiya, Differential subordination results for certain integrodifferential operator and it's applications, Abs. Appl. Anal., 2012(2012), 13 pp.

[5] N. N. Lebedev, Special Functions and Their Applications. Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication, Dover Publications Inc., New York, 1972.

[6] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot–Bouquet differential subordinations, J. Differ. Equations, 56 (1985), 297–309.

[7] S. S. Miller and P. T. Mocanu, Differential Subordinations. Theory and Applications, Series in Pure and Applied Mathematics, No. 225., Marcel Dekker, Inc., New York, 2000.

[8] A. O. Mostafa and M. K. Aouf, Some applications of differential subordination of p–valent functions associated with Cho–Kwon–Srivastava operator, Acta Math. Sinica, 25 (2009), no. 9, 1483–1496.

[9] J. Patel, On certain subclass of p–valently Bazilević functions., J. Inequal. Pure Appl. Math., 6 (2005), no. 1, Article 16, 1–13.

[10] J. Patel and N. E. Cho, Some classes of analytic functions involving Noor integral operator. J. Math. Anal. Appl., 312 (2005), no. 2, 564–575.

[11] J. Patel and N. E. Cho, On certain sufficient conditions for close-to-convexity, Appl. Math. Comput., 187 (2007), no. 1, 369–378.

[12] J. Patel, N. E. Cho and H. M. Srivastava, Certain subclasses of multivalent functions associated with a family of linear operators, Math. Comput. Modelling, 43 (2006), 320–338.

[13] J. Patel and K. Mishra, On certain subclasses of multivalent functions associated with the extended fractional differintegral operator, J. Math. Anal. Appl., 332 (2007),109–122.

[14] J. Patel, A. K. Mishra and H. M. Srivastava, Classes of multivalent analytic functions involving the Dziok–Srivastava operator, Comput. Math. Appl., 54 (2007), 599–616.

[15] Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht, Göttingen, 1975.

[16] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz–Lerch Zeta function and differential subordination, Integr. Transf. Spec. Funct., 18(2007), no. 3, 207–216.

[17] D. R. Wilken and J. Feng, A remark on convex and starlike functions, J. London Math. Soc., (2)21 (1980), 287–290.

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An SVD Free Construction of an Indicator Function as an Imaging Algorithm

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Abstract — A novel SVD free method is presented to reconstruct the shape of a sound-soft obstacle from the knowledge of the time harmonic incident electromagnetic wave and the far field pattern of the scattered wave. The approach presented is based on Kirsch's factorization method and constructs a simple indicator function which is used to visualize the scattering profile. The method is baptized as Singular Value Decomposition Free Indicator (SVDFI) and its performance is evaluated by comparing our reconstructions with those obtained via Morozov's discrepancy principle (MDP). Numerical results that illustrate the effectiveness of SVDFI on reconstruction problems involving both simulated and real data are reported and analyzed.

Keywords: factorization method, Helmholtz equation, inverse problems, singular value decomposition.

1 Introduction

The linear sampling method introduced by Colton and Kirsch [4] is one of the major visualization algorithms for solving inverse obstacle scattering problems in the resonance region. The method creates a binary criterion for points from the grid to be inside the scatterer and is very fast since it reconstructs the obstacle directly from the given data without requiring the solution of a forward problem. One of its disadvantages is that it requires lots of data available since one theoretically needs to know the far field for all incident and observation directions on the unit sphere Ω. The linear sampling method states that the norm of the solution g of the far field equation $Fg = e^{-ikx\cdot\cdot z}$ tends to infinity as the sampling point z moves away from the obstacle. However, since the far field operator F is compact, the right-hand side is almost never in the range of F and the far field equation is not solvable in general. Therefore one has to resort in finding regularized solutions

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g.

Kirsch in [5, 6], improved the method through a certain factorization of the far field operator F of the form $(F^{\star}F)^{1/4}$ which replaced the far field equation with $(F^{\star}F)^{1/4}g = e^{-ikx\cdots z}$. Then he was able to prove that a point z is located in the interior of the object if and only if the right-hand side belongs to the range of the operator $(F^{\star}F)^{1/4}$.

In this work we propose a new visualization technique, inspired in part from the Multiple Signal Classification (MUSIC) algorithm [2]. Our method constructs an indicator function $I(z)$ without using singular value decomposition. The profile of the object is then reconstructed by noting that the indicator function gets large values for z inside the object and it gets smaller for z outside. Our method is fast and simple and does not really solve the far field equation with respect to g for several right hand sides. In addition, no choice of cut-off for the indicator function is needed and *a priori* knowledge of the noise level in the data is not required.

We organize our paper as follows. Section 2 will be devoted to the formulation of the problem and a brief description of the factorization method. Subsequently, Section 3 will deal with the formulation of the SVDFI within the framework of the factorization method. In order to show the effectiveness of our method, in Section 4, we will present numerical examples for the case of impenetrable and penetrable scatterers and we will compare the reconstructions obtained via SVDFI with the ones obtained by means of the Morozov's discrepancy principle (MDP). In our experiments we will use simulated data obtained by means of the Nyström method [3] as well as real data. The real data are made available by the *Electromagnetics Technology Division, Sensors Directorate, Air Force Research Laboratory, Hanscom Air Force Base, Massachusetts* and are known by the name of *The Ipswich Data* [7].

2 Formulation of the problem and the linear sampling method

It is well known that the propagation of time-harmonic acoustic fields in a homogeneous medium, in the presence of a sound soft obstacle D, is modeled by the exterior boundary value problem (direct obstacle scattering problem)

$$\triangle_2 u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \bar{D} \tag{2.1}$$

$$u(x) + u^i(x) = 0, \quad x \in \partial D \tag{2.2}$$

where k is a real positive wave number and u^i is a given incident field, that in the presence of D will generate the scattered field u. In addition, the scattered field u

will satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial u}{\partial r} - iku) = 0 \tag{2.3}$$

$r = |x|$, $x \in \mathbb{R}^2 \setminus \bar{D}$, and the limit is taken uniformly for all directions $\hat{x} = x/|x|$.

The Green's formula implies that the solution u of the direct obstacle scattering problem above has the asymptotic behavior [3]

$$u(x) = u_\infty(\hat{x})\frac{e^{ikr}}{\sqrt{r}} + O(r^{-3/2}) \tag{2.4}$$

for some analytic function u_∞, called the far- field pattern of u, given by

$$u_\infty(\hat{x}) = \frac{-e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \frac{\partial u}{\partial n}(y) e^{-ik\hat{x}\cdot y}\, ds(y) \tag{2.5}$$

for $\hat{x} = x/|x|$ on the unit sphere Ω. In the case of the inverse problem, it represents the measured data. In particular, the inverse problem that will be considered here, is the problem of finding the shape of D from a complete knowledge of the far-field pattern.

We now define the far-field equation

$$(Fg_z)(\hat{x}) = \Phi^\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}, \quad z \in \mathbb{R}^2 \tag{2.6}$$

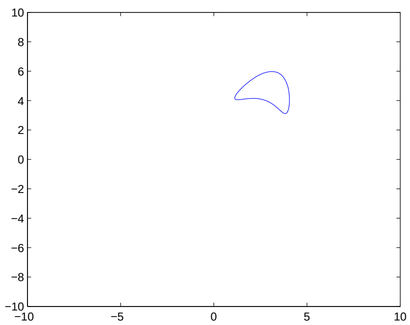
where $\Phi^\infty(\hat{x}, z)$ is the far-field pattern of the fundamental solution of the Helmholtz equation given by

$$\Phi(x, z) = \frac{i}{4} H_0^{(1)}(k|x - z|),\ x \neq z \tag{2.7}$$

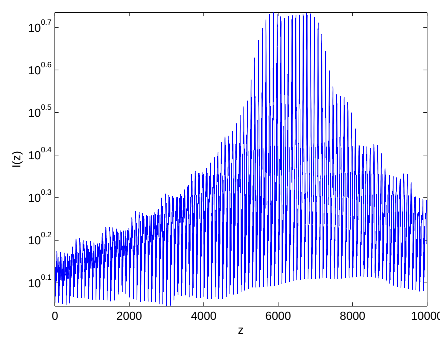
in which $H_0^{(1)}$ is the Hankel function of order zero and of the first kind.
Moreover $F : L^2(\Omega) \to L^2(\Omega)$ is given by

$$(Fg)(\hat{x}) = \int_\Omega u_\infty(\hat{x}; \hat{d})\, g(\hat{d})\ ds(\hat{d}), \quad \hat{d} \in \Omega \tag{2.8}$$

It is well known that the first version of the linear sampling method [4] solves the linear operator equation (2.6) based on the numerical observation that its solution will have a large L^2- norm outside and close to ∂D. Hence, reconstructions are obtained by plotting the norm of the solution. However, the problem is that the right-hand side does not in general belong to the range of the operator F. Kirsch [5] was able to overcome this difficulty with the introduction of a new version of the linear sampling method based on appropriate factorization of the



(a)



(b)

Figure 1: (a) Original profile of a kite, (b) Indicator function $I(z)$.

far-field operator F. In this method, Kirsch is elegantly using the spectral properties of the operator F to characterize the obstacle. In particular, the following linear operator equation is now used in place of equation (2.6)

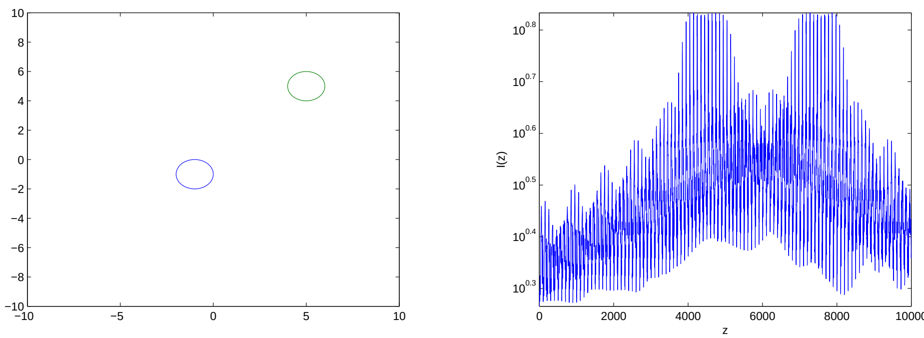
$$(F^{\star}F)^{1/4}g_z = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot z} \tag{2.9}$$

and the spectral properties of F are used for the reconstructions. To be more specific, since F is normal and compact, which guarantees the existence of a singular system $\{\sigma_j^c, u_j^c, v_j^c\}$, $j \in \mathbb{N}$, of F with $v_j^c = s_j u_j^c$ and $s_j \in \mathbb{C}$ with $|s_j| = 1$, then the characterization of the object depends on a range test as described in the following theorem due to Kirsch [5].

Theorem 2.1 *For any $z \in \mathbb{R}^2$ assume that k^2 is not a Dirichlet eigenvalue of $-\Delta_2$ in D i.e. the corresponding homogeneous problem has only the trivial solution. Then a point $z \in \mathbb{R}^2$ belongs to D if and only if the series*

$$\sum_{j=1}^{\infty} \frac{|(\Phi^{\infty}, v_j^c)|^2}{\sigma_j^c} \tag{2.10}$$

converges, or equivalently, if and only if $\Phi^{\infty}(\hat{x}, z)$ belongs to the range of the operator $(F^{*}F)^{1/4}$.



(a) (b)

Figure 2: (a) Original profile of two circles, (b) Indicator function $I(z)$

3 The new imaging algorithm

In this section we propose a new visualization algorithm that does not use the L^2-norm of $g_\epsilon(.,z)$ as an indicator of the domain D. Our approach makes use of the finite dimensionality present in any numerical computation and avoids the necessity of using the SVD of the discretized far field operator. Here, the direct scattering problem is solved for N directions of incidence distributed uniformly on the unit circle, $\hat{x}_j = (\cos(2\pi j/N), \sin(2\pi j/N)^T$, $j = 0, \cdots, N-1$, and the far field pattern of the scattered field is determined for those same N directions. We then use these data and a composite trapezoidal rule to approximate the far field operator F given by (2.8) by its discrete analogue $F_D : \mathbb{C}^N \to \mathbb{C}^N$ for $u \in \mathbb{C}^N$,

$$(F_D u)_j = \frac{2\pi}{N} \sum_{k=0}^{N-1} u_\infty(\hat{x_j}; \hat{x_k})\, u_k \quad j = 0 \cdots, N-1 \tag{3.11}$$

Arens et al proved in [1], through the use of a semi-discrete operator and some arguments from perturbation theory, that the first few eigenvalues and eigenspaces of F_D are good approximations to those of F.

Using the discrete operator above for each z we form the following $N \times N$ linear system of the form

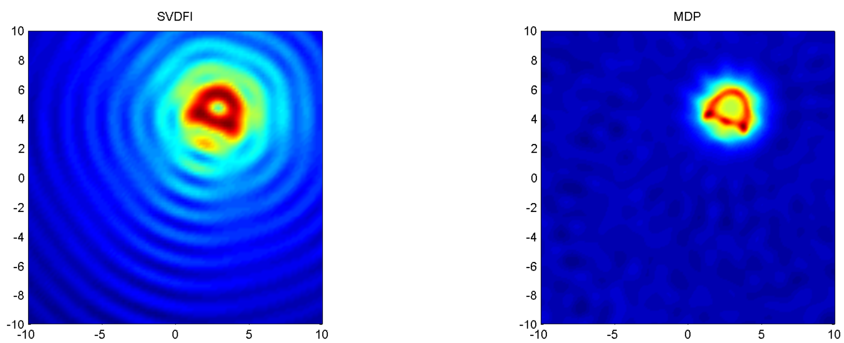
$$(F_D^\star F_D)^{1/4}\, g(z) = b^\infty(z) = (\Phi^\infty(\hat{x}_0, z), \cdots, \Phi^\infty(\hat{x}_{N-1}, z))^T \tag{3.12}$$

We need to emphasize here that we are not directly solving the system above for g, but our imaging algorithm is rather based on the following simple observation.

If $z \in D$, then $b^{\infty}(z) \in \mathcal{R}((F_D^{\star} F_D)^{1/4})$ and hence $b^{\infty}(z) \notin \mathcal{N}(((F_D^{\star} F_D)^{1/4})^{\star})$, therefore the indicator function

$$I(z) = \frac{\|((F_D^{\star} F_D)^{1/4})^{\star}\, b^{\infty}(z)\|_{L^2}}{\|b^{\infty}(z)\|_{L^2}} \tag{3.13}$$

is larger for points inside the object and smaller for points outside. Figure 1 below



(a) (b)

Figure 3: Reconstructions of a kite with 9% error, (a) via SVDFI, (b) via MDP.

shows an example of a kite excited by 60 incident waves, located in a grid of 100×100 points. The measurement error is 9%. Therefore the far-field matrix of equation (3.12) is 60×60 and the right hand side is $60 \times 10,000$. We observe that the indicator function $I(z)$ is large when z is located inside the object and rather small for z outside. Similarly 2 shows two circles excited by 100 incident waves, located in a grid of 100×100 points. The measurement error is 13%. Notice that the indicator function now exhibits two picks due to the presence of two objects.

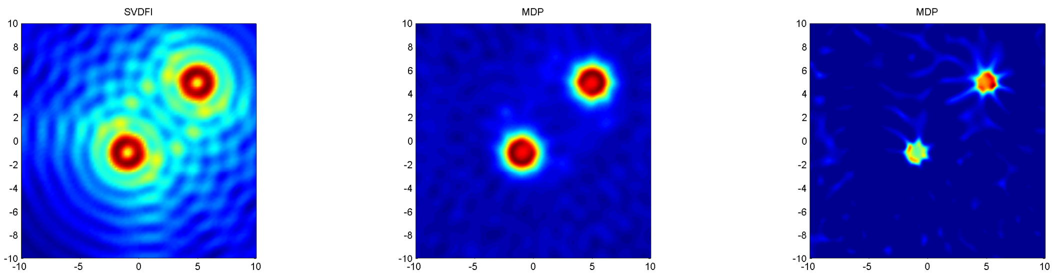
4 Numerical Results

In order to simulate perturbed data, we generate Gaussian random matrices E_1, E_2 and use a far-field matrix defined by

$$\widetilde{F}_D = F_D + \delta(E_1 + i\, E_2)\|F_q\|_2, \quad \delta > 0,$$

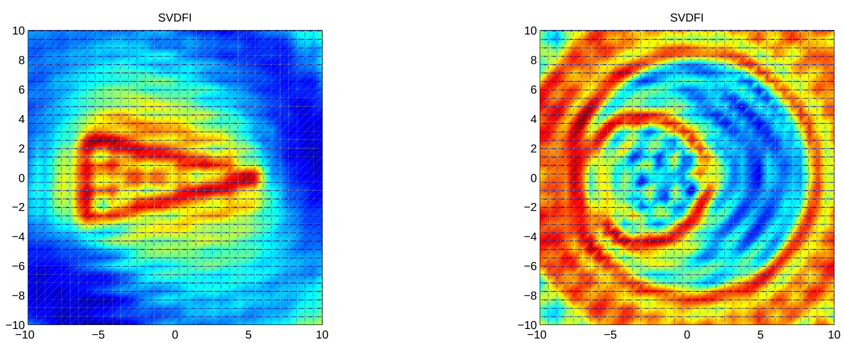
where δ is given and where F_D is constructed by using the Nyström method [3]. As indicated in [3], the Nyström method not only requires less computational effort

compared to collocation and Galerkin methods but it is also generically stable in the sense that it preserves the condition of the integral equation. We first report an image reconstruction experiment of a sound-soft obstacle. In this case the object



(a) (b) (c)

Figure 4: Reconstructions of two circles with 13% error, (a) via SVDFI, (b) via MDP with correct error level, (c) via MDP with underestimated error level.



(a) (b)

Figure 5: SVDFI reconstructions of, (a) the aluminum triangle and (b) the "mystery" object.

to be reconstructed is a kite located in a grid of 100×100 points, the far-field matrix $\widetilde{F}_D$ is 60×60 (i.e., we use 60 incident and observed directions), and the relative noise level in $\widetilde{F}_D$ is 1% (which implied a relative noise level in $(\widetilde{F}_D^\star \widetilde{F}_D)^{1/4}$ of approximately 9%).

Reconstruction of the kite via our method is shown in Figures 3-(a). For comparison purposes, in 3-(b) we have included a reconstruction obtained through

the factorization method with Morozov discrepancy principle (MDP). Even though the reconstructions are comparable, notice that the Morozov discrepancy principle requires a-priori knowledge of the noise level in the data whether our method does not.

We now continue our numerical experiments by attempting to reconstruct two sound-soft obstacles. In this case the objects to be reconstructed are two circles located in a grid of 100×100, the far-field matrix $\widetilde{F}_D$ is 100×100, and the relative noise level in $\widetilde{F}_D$ is 2% (which implied a relative noise level in $(\widetilde{F}_D^{\star}\widetilde{F}_D)^{1/4}$ of 13%.). In this experiment we consider the MDP in two distinct circumstances: (i) when the exact error norm $\|(\widetilde{F}_D^{\star}\widetilde{F}_D)^{1/4} - (F_D^{\star}F_D)^{1/4}\|_2 = \epsilon$ is used as input data and (ii), when the error norm is underestimated and set to 0.03ϵ. The results are depicted in Figure 4. We note that while both our method and MDP (case (i)) produce reasonable reconstructions, MDP (case (ii)) yields reconstruction of poor quality due to the lack of a prior information about the noise level, and we see that our method may outperform MDP if the noise level is not correctly estimated.

We will now consider real data sets (The Ipswich Data) produced by using an echo-free chamber, a fixed transmitter and a receiver rotating around the scatterer. The incident and observation angles are 36 for both experiments. Initially we will attempt to reconstruct an aluminum triangle (IPS009) whose outer circle has radius equal to 6 cm and using SVDFI. Figure 5-(a) shows that SVDFI yields a good reconstruction. Consequently we attempt the reconstruction of the "mystery" object given by the data set (IPS007), with a priori information that the object is penetrable and lies inside a circle of radius 7.5 cm. The reconstruction results appear in figure 5-(b) and as clearly indicated by the SVDFI the object was a circular tube, with a smaller one in its interior. It is worthwhile to mention here that reconstructions via the Morozov's discrepancy principle are difficult to obtain for real data cases due to the fact that the level of error in the experimental far-field matrix is not available or difficult to estimate.

References

[1] T. Arens, A. Lechleiter and D. R. Luke, MUSIC for Extended Scatterers as an Instance of the Factorization Method, SIAM J. Appl. Math., 70 (2009) 1283-1304.

[2] M. Cheney, The linear sampling method and the MUSIC algorithm, Inverse Problems 17 (2001), 591-596.

[3] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer-Verlag, New York, 1992.

[4] D. Colton, A. Kirsch, A simple method for solving the inverse scattering problems in the resonance region, Inverse Problems 12 (1996) 383-393.

[5] A. Kirsch, Characterization of the shape of a scattering obstacle using the spectral data of the far-field operator, Inverse Problems, 14 (1998) 1489-1512.

[6] A. Kirsch, N. Grinberg, The Factorization Method for Inverse Problems, Oxford University Press, Oxford, 2008.

[7] R. McGahan, R. Kleinman, Third annual special session on image reconstruction using real data, part 1. IEEE Antennas Propag. Mag. 41(1999) 34-36.

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Applications of coupled $\mathcal{N}$-structures in BCC-algebras

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Abstract. The notions of a $\mathcal{N}$-subalgebra, a BCK-$\mathcal{N}$-ideal, a (strong) BCC-$\mathcal{N}$-ideal of BCC-algebras are introduced, and related properties are investigated. Characterizations of a coupled $\mathcal{N}$-subalgebra, a coupled BCK-$\mathcal{N}$-ideals, and a coupled (strong) BCC-$\mathcal{N}$-ideals of BCC-algebras are given. Relations among a coupled $\mathcal{N}$-subalgebra, a coupled BCK-$\mathcal{N}$-ideal and a coupled (strong) BCC-$\mathcal{N}$ of BCC-algebras are discussed.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7], [8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The class of all BCK-algebras is a qusivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori [11] introduced a notion of BCC-algebra, and W. A. Dudek redefined the notion of BCC-algebra by using a dual form of the ordinary definition in the sense of Y. Kormori. In [6], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. Dudek et al. ([3]) considered the fuzzification of ideals in BCC-algebras. Jun et.al ([9]), introduced the notion of coupled $\mathcal{N}$-structures and its applications in BCK/BCI-algebras were discussed.

In this paper, we introduce the notions of a coupled $\mathcal{N}$-subalgebra, a coupled BCK-$\mathcal{N}$-ideal, and a coupled (strong) BCC-$\mathcal{N}$-ideals of BCC-algebras are introduced, and related properties are investigated. Characterizations of a coupled $\mathcal{N}$-subalgebras, a coupled BCK-$\mathcal{N}$-ideals and a coupled (strong) BCC-$\mathcal{N}$-ideals of BCC-algebras are given. Relations among a coupled $\mathcal{N}$-subalgebra, a coupled BCK-$\mathcal{N}$-ideal and a coupled (strong) BCC-$\mathcal{N}$-ideal of BCC-algebras are discussed.

2. Preliminaries

For an abstract algebra $(X, *, 0)$ of type $(2, 0)$, consider the following axioms:

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(a1) $(\forall x, y, z \in X)\,(((x * y) * (z * y)) * (x * z) = 0)\,,$
(a2) $(\forall x \in X)\,(x * x = 0)\,,$
(a3) $(\forall x \in X)\,(0 * x = 0)\,,$
(a4) $(\forall x \in X)\,(x * 0 = x)\,,$
(a5) $(\forall x, y \in X)\,(x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)\,,$
(a6) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)\,,$
(a7) $(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,,$
(a8) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)\,,$

If $(X, *, 0)$ satisfies axioms (a1), (a2), (a4) and (a5), then we say $(X, *, 0)$ is a *weak BCC-algebra*. By a *BCC-algebra* we mean a weak BCC-algebra X satisfying the axiom (a3). If $(X, *, 0)$ satisfies axioms (a8), (a6), (a2) and (a5), then we say $(X, *, 0)$ is a *BCI-algebra*. By a *BCK-algebra* we mean a BCI-algebra X satisfying the axiom (a3). On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural order $\leq$ by putting

$$(\forall x, y \in X)\,(x \leq y \ \Leftrightarrow\ x * y = 0)\,. \tag{2.1}$$

It is not difficult to verify that this order is partial and 0 is its smallest element.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (see [1]). Note that a BCC-algebra X is a BCK-algebra if and only if it satisfies the axiom (a7).

Any BCC-algebra $(X, *, 0)$ satisfies the following conditions:

(b1) $(\forall x, y \in X)\,(x * y \leq x)\,,$
(b2) $(\forall x, y, z \in X)\,((x * y) * (z * y) \leq x * z)\,,$
(b3) $(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,.$

A non-empty subset S of a BCC-algebra X is called a *BCC-subalgebra*(briefly, *subalgebra*) of X if $x * y \in S$ whenever $x \in S$ and $y \in S$. Let I be a subset of X with $0 \in I$. We say that I is a

(c1) *BCK-ideal* of X (see [5]) if it satisfies:

$$(\forall x, y \in X)\,(y \in I,\ x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.2}$$

(c2) *BCC-ideal* of X (see [5]) if it satisfies:

$$(\forall x, y, z \in X)\,(y \in I,\ (x * y) * z \in I \ \Rightarrow\ x * z \in I)\,. \tag{2.3}$$

(c3) *strong BCC-ideal* of X (see [10]) if it satisfies:

$$(\forall x, y, z \in X)\,(y \in I, (x * y) * z \in I \Rightarrow x \in I)\,. \tag{2.4}$$

3. Coupled $\mathcal{N}$-structures applied to subalgebras and ideals in BCC-algebras

Definition 3.1.([9]) A *coupled $\mathcal{N}$-structure* $\mathcal{C}$ in a nonempty set X is an object of the form

$$\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}}\rangle : x \in X\}$$

where $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$ are $\mathcal{N}$-functions on X such that $-1 \leq f_{\mathcal{C}}(x) + g_{\mathcal{C}}(x) \leq 0$ for all $x \in X$.

A coupled $\mathcal{N}$-structure $\mathcal{C} = \{\langle x; f_\mathcal{C}, g_\mathcal{C}\rangle : x \in X\}$ in X can be identified to an ordered pair $(f_\mathcal{C}, g_\mathcal{C})$ in $\mathcal{F}(X, [-1, 0]) \times \mathcal{F}(X, [-1, 0])$. For the sake of simplicity, we shall use the notation $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ instead of $\mathcal{C} = \{\langle x; f_\mathcal{C}, g_\mathcal{C}\rangle : x \in X\}$.

For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in X and $t, s \in [-1, 0]$ with $t + s \geq -1$, the set

$$\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t, s)\} = \{x \in X \mid f_\mathcal{C}(x) \leq t,\ g_\mathcal{C}(x) \geq s\}$$

is called an $\mathcal{N}(t, s)$-level set of $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$. An $\mathcal{N}(t, t)$-level set of $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is called an $\mathcal{N}$-level set of $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$.

Definition 3.2.([9]) A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in a BCC-algebra X is called a *coupled $\mathcal{N}$-subalgebra* of X if it satisfies:

$$f_\mathcal{C}(x * y) \leq \bigvee \{f_\mathcal{C}(x), f_\mathcal{C}(y)\} \quad \text{and} \quad g_\mathcal{C}(x * y) \geq \bigwedge \{g_\mathcal{C}(x), g_\mathcal{C}(y)\} \tag{3.1}$$

for all $x, y \in X$.

Lemma 3.3.([9]) *Every coupled $\mathcal{N}$-subalgebra $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of a BCC-algebra X satisfies $f_\mathcal{C}(0) \leq f_\mathcal{C}(x)$ and $g_\mathcal{C}(0) \geq g_\mathcal{C}(x)$ for all $x \in X$.*

Proposition 3.4. *If every $\mathcal{N}$-subalgebra $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of a BCC-algebra X satisfies the inequalities $f_\mathcal{C}(x * y) \leq f_\mathcal{C}(y)$ and $g_\mathcal{C}(x * y) \geq g_\mathcal{C}(y)$ for any $x, y \in X$, then $f_\mathcal{C}$ and $g_\mathcal{C}$ are constant functions.*

Proof. Let $x \in X$. Using (a4) and assumption, we have $f_\mathcal{C}(x) = f_\mathcal{C}(x * 0) \leq f_\mathcal{C}(0)$ and $g_\mathcal{C}(x) = g_\mathcal{C}(x * 0) \geq g_\mathcal{C}(0)$. It follows from Lemma 3.3 that $f_\mathcal{C}(x) = f_\mathcal{C}(0)$ and $g_\mathcal{C}(x) = g_\mathcal{C}(0)$. Hence $f_\mathcal{C}$ and $g_\mathcal{C}$ are constant functions. $\square$

Definition 3.5.([9]) A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in a BCC-algebra X is called a *coupled BCK-$\mathcal{N}$-ideal* of X if it satisfies:
(c81) $f_\mathcal{C}(0) \leq f_\mathcal{C}(x)$ and $g_\mathcal{C}(0) \geq g_\mathcal{C}(x)$,
(c82) $f_\mathcal{C}(x) \leq \bigvee \{f_\mathcal{C}(x * y), f_\mathcal{C}(y)\}$ and $g_\mathcal{C}(x) \geq \bigwedge \{g_\mathcal{C}(x * y), g_\mathcal{C}(y)\}$,
for all $x, y \in X$.

Example 3.6. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra([2]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

(1) Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.8, -0.2\rangle, \quad \langle a; -0.6, -0.3\rangle, \langle b; -0.6, -0.3\rangle, \\ \langle c; -0.1, -0.5\rangle, \langle d; -0.1, -0.5\rangle\big\}.$$

Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-subalgebra, but not a coupled BCK-$\mathcal{N}$-ideal of X since

$$f_{\mathcal{C}}(c) = -0.1 \not\leq\ -0.6 = \bigvee \{f_{\mathcal{C}}(c * b) = f_{\mathcal{C}}(a), f_{\mathcal{C}}(b)\}$$

and/or

$$g_{\mathcal{C}}(c) = -0.5 \not\geq -0.3 = \bigwedge \{g_{\mathcal{C}}(c * b) = g_{\mathcal{C}}(a), g_{\mathcal{C}}(b)\}.$$

(2) Let $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{D} = \big\{\langle 0; -0.7, -0.1\rangle, \quad \langle a; -0.7, -0.1\rangle, \langle b; -0.5, -0.4\rangle, \\ \langle c; -0.5, -0.4\rangle, \langle d; -0.5, -0.4\rangle\big\}.$$

It is easy to show that $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ is both a coupled $\mathcal{N}$-subalgebra and a coupled BCK-$\mathcal{N}$-ideal of X.

Proposition 3.7.([9]) *Every coupled BCK-$\mathcal{N}$-ideal of a BCC-algebra X satisfies the following assertions:*

(i) $(\forall x, y, z \in X)(x * y \leq z \ \Rightarrow f_{\mathcal{C}}(x) \leq \bigvee \{f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\},\ g_{\mathcal{C}}(x) \geq \bigwedge \{g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\})$.
(ii) $(\forall x, y \in X)(x \leq y \Rightarrow f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(y), g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(y))$.

Definition 3.8. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BCC-algebra X is called a *coupled BCC-$\mathcal{N}$-ideal* of X if it satisfies (c81) and

(c83) $f_{\mathcal{C}}(x * z) \leq \bigvee \{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(x * z) \geq \bigwedge \{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\}$,

for all $x, y \in X$.

Example 3.9. (1) Consider a BCC-algebra $X = \{0, a, , b, c, d\}$ and a coupled $\mathcal{N}$-structure $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ as in Example 3.6(2). Then $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ is both a coupled $\mathcal{N}$-subalgebra of X and a coupled BCK-$\mathcal{N}$-ideal of X, but not a coupled BCC-$\mathcal{N}$-ideal of X since

$$f_{\mathcal{D}}(d * c) = f_{\mathcal{D}}(c) = -0.5 \not\leq\ -0.7 = \bigvee \{f_{\mathcal{D}}((d * a) * c) = f_{\mathcal{D}}(0), f_{\mathcal{D}}(a)\}$$

and/or

$$g_{\mathcal{D}}(d * c) = g_{\mathcal{D}}(c) = -0.4 \not\geq -0.1 = \bigwedge \{g_{\mathcal{D}}((d * a) * c) = g_{\mathcal{D}}(0), g_{\mathcal{D}}(a)\}.$$

(2) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra ([2]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	1
2	2	2	0	0	0	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \Big\{ \begin{array}{ll} \langle 0; -0.8, -0.2\rangle, & \langle 1; -0.6, -0.3\rangle, \langle 2; -0.6, -0.3\rangle, \\ \langle 3; -0.6, -0.3\rangle, & \langle 4; -0.6, -0.3\rangle, \langle 5; -0.2, -0.5\rangle \Big\}. \end{array}$$

It is easy to check that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is both a coupled $\mathcal{N}$-subalgebra of X and a coupled BCC-$\mathcal{N}$-ideal of X.

Theorem 3.10.([9]) *For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BCC-algebra X, the following are equivalent:*

(1) *$\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-ideal of X.*
(2) *The nonempty $\mathcal{N}(t,s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ is a BCK-ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Theorem 3.11. *For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BCC-algebra X, the following are equivalent:*

(1) *$\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCC-$\mathcal{N}$-ideal of X.*
(2) *The nonempty $\mathcal{N}(t,s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ is a BCC-ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Proof. Assume that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCC- $\mathcal{N}$-ideal of X. Let $t, s \in [-1, 0]$ be such that $t + s \geq -1$. Obviously, $0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Let $x, y, z \in X$ be such that $(x * y) * z, y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s))\}$. Then $f_{\mathcal{C}}((x*y)*z) \leq t$, $f_{\mathcal{C}}(y) \leq t$ and $g_{\mathcal{C}}((x*y)*z) \geq s$, $g_{\mathcal{C}}(y) \geq s$. It follows from (c83) that $f_{\mathcal{C}}(x*z) \leq \bigvee\{f_{\mathcal{C}}((x*y)*z), f_{\mathcal{C}}(y)\} \leq t$ and $g_{\mathcal{C}}(x*z) \geq \bigwedge\{g_{\mathcal{C}}((x*y)*z), g_{\mathcal{C}}(y)\} \geq s$, which imply that $x * z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Hence the nonempty $\mathcal{N}(t,s)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a BCC-ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Conversely, suppose that the nonempty $\mathcal{N}(t,s)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a BCC-ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Since $0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$, the condition (c81) is valid. Assume that there exist $a, b, c \in X$ such that $f_{\mathcal{C}}(a*c) > \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\}$ or $g_{\mathcal{C}}(a*c) < \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$. For the case $f_{\mathcal{C}}(a*c) > \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\}$ and $g_{\mathcal{C}}(a*c) \geq \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$, there exist $s_0, t_0 \in [-1, 0)$ such that $f_{\mathcal{C}}(a*c) > t_0 > \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\}$ and $s_0 = \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$. It follows that $(a*b)*c, b \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$. This is impossible. For the case $f_{\mathcal{C}}(a*c) \geq \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\}$ and $g_{\mathcal{C}}(a*c) < \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$, there exist $s_0, t_0 \in [-1, 0)$ such that $t_0 = f_{\mathcal{C}}(a*b)$ and

$g_{\mathcal{C}}(a*c) < s_0 < \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$. Then $(a*b)*c, b \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$. This is a contradiction. If $f_{\mathcal{C}}(a*c) > \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\}$ and $g_{\mathcal{C}}(a*c) < \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\}$, then $(a*b)*c, b \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_0, s_0)\}$, where $t_0 := \frac{1}{2}(f_{\mathcal{C}}(a*c) + \bigvee\{f_{\mathcal{C}}((a*b)*c), f_{\mathcal{C}}(b)\})$ and $s_0 := \frac{1}{2}(g_{\mathcal{C}}(a*c) + \bigwedge\{g_{\mathcal{C}}((a*b)*c), g_{\mathcal{C}}(b)\})$. This is a contradiction. Therefore $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCC-$\mathcal{N}$-ideal of X. □

Proposition 3.12. *Every coupled BCK-$\mathcal{N}$-ideal $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ of a BCC-algebra X is a coupled $\mathcal{N}$-subalgebra of X.*

Proof. Let a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-ideal of a BCC-algebra X and let $x, y \in X$. Then

$$f_{\mathcal{C}}(x*y) \leq \bigvee\{f_{\mathcal{C}}((x*y)*x), f_{\mathcal{C}}(x)\} = \bigvee\{f_{\mathcal{C}}(0), f_{\mathcal{C}}(x)\} \leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y)\}$$

and

$$g_{\mathcal{C}}(x*y) \geq \bigwedge\{g_{\mathcal{C}}((x*y)*x), g_{\mathcal{C}}(x)\} = \bigwedge\{g_{\mathcal{C}}(0), g_{\mathcal{C}}(x)\} \geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y)\}.$$

Hence $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-subalgebra of X. □

The converse of Proposition 3.12 may not be true in general (see Example 3.6(1)) as seen in the following example.

Proposition 3.13. *Every coupled BCC-$\mathcal{N}$-ideal $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ of a BCC-algebra X is a coupled BCK-$\mathcal{N}$-ideal of X.*

Proof. Put $z := 0$ in (c83). □

Corollary 3.14. *Every coupled BCC-$\mathcal{N}$-ideal of a BCC-algebra X is a coupled $\mathcal{N}$-subalgebra of X.*

Proof. It is easily verified from Proposition 3.12 and Proposition 3.13. □

Proposition 3.15. *Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled BCC- $\mathcal{N}$-ideal of a BCC-algebra X. Then the following hold:*

(i) *If $x \leq y$ for any $x, y \in X$, then $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(y)$, $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(y)$.*
(ii) *If $f_{\mathcal{C}}(x*y) = f_{\mathcal{C}}(0)$ for any $x, y \in X$, then $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(y)$.*
(iii) *If $g_{\mathcal{C}}(x*y) = g_{\mathcal{C}}(0)$ for any $x, y \in X$, then $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(y)$.*
(iv) $(\forall x, y \in X)(f_{\mathcal{C}}(x*y) \leq f_{\mathcal{C}}(x), g_{\mathcal{C}}(x*y) \geq g_{\mathcal{C}}(x))$.
(v) $(\forall x, y, z \in X)(f_{\mathcal{C}}(x*(y*z)) \leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\}, g_{\mathcal{C}}(x*(y*z)) \geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\})$.

Proof. (i) It follows from Proposition 3.7 and Proposition 3.13.
(ii) For any $x, y \in X$, we have

$$\begin{aligned} f_{\mathcal{C}}(x) = f_{\mathcal{C}}(x*0) &\leq \bigvee\{f_{\mathcal{C}}((x*y)*0), f_{\mathcal{C}}(y)\} = \bigvee\{f_{\mathcal{C}}(x*y), f_{\mathcal{C}}(y)\} \\ &= \bigvee\{f_{\mathcal{C}}(0), f_{\mathcal{C}}(y)\} = f_{\mathcal{C}}(y). \end{aligned}$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned} g_{\mathcal{C}}(x) = g_{\mathcal{C}}(x * 0) &\geq \bigwedge\{g_{\mathcal{C}}((x * y) * 0), g_{\mathcal{C}}(y)\} = \bigwedge\{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \\ &= \bigwedge\{g_{\mathcal{C}}(0), g_{\mathcal{C}}(y)\} = g_{\mathcal{C}}(y). \end{aligned}$$

(iv) By (b1), we have $x * y \leq x$ for any $x, y \in X$. Using (i), we obtain $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(x)$ for any $x, y \in X$.
(v) For any $x, y, z \in X$, using Corollary 3.14 we have

$$\begin{aligned} f_{\mathcal{C}}(x * (y * z)) &\leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y * z)\} \\ &\leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\} \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{C}}(x * (y * z)) &\geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y * z)\} \\ &\geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\}. \end{aligned}$$

□

Theorem 3.16. *In a BCK-algebra, every coupled BCK-$\mathcal{N}$-ideal is a coupled BCC-$\mathcal{N}$-ideal of X.*

Proof. Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled BCK-$\mathcal{N}$-ideal of a BCC-algebra X. Using (c82) and (a7), we have

$$\begin{aligned} f_{\mathcal{C}}(x * z) &\leq \bigvee\{f_{\mathcal{C}}((x * z) * y), f_{\mathcal{C}}(y)\} \\ &= \bigvee\{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y)\} \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{C}}(x * z) &\geq \bigwedge\{g_{\mathcal{C}}((x * z) * y), g_{\mathcal{C}}(y)\} \\ &= \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\} \end{aligned}$$

for all $x, y, z \in X$. Hence (c83) holds. This completes the proof. □

Definition 3.17. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BCC-algebra X is called a *coupled strong BCC-$\mathcal{N}$-ideal* of X if it satisfies (c81) and

(c84) $f_{\mathcal{C}}(x) \leq \bigvee\{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(x) \geq \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\}$,

for all $x, y \in X$.

Theorem 3.18. *For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BCC-algebra X, the following are equivalent:*

1. *$\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled strong BCC-$\mathcal{N}$-ideal of X.*
2. *The nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is a strong BCC-ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Proof. Straightforward. □

Theorem 3.19. *Every coupled strong BCC-$\mathcal{N}$-ideal of a BCC-algebra X is a coupled BCC-$\mathcal{N}$-ideal of X.*

Proof. Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled strong BCC-$\mathcal{N}$-ideal of a BCC-algebra X. Then the nonempty $\mathcal{N}(t,s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ is a strong BCC-ideal of X for all $t, s \in [-1, 0]$ with $t+s \geq -1$ by Theorem 3.18. Let $x, y, z \in X$ be such that $y, (x*y)*z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Then $x \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ by (2.4). It follows from (b1) that $((x*z)*x)*0 = 0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ which implies from (2.4) that $x*z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Hence $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$ is a BCC-ideal of X. By Theorem 3.11, $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCC-$\mathcal{N}$-ideal of X. □

The converse of Theorem 3.19 is not true in general as seen in the following example.

Example 3.20 Consider a BCC-algebra $X = \{0, 1, 2, 3, 4, 5\}$ and a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ as in Example 3.9 (2). Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCC-$\mathcal{N}$-ideal of X(see Example 3.9), but not a coupled strong BCC-$\mathcal{N}$-ideal of X since

$$f_{\mathcal{C}}(5) = -0.2 \not\leq -0.6 = \bigvee \{f_{\mathcal{C}}((5*2)*5) = f_{\mathcal{C}}(0), f_{\mathcal{C}}(2)\}$$

and/or

$$g_{\mathcal{C}}(5) = -0.5 \not\geq -0.3 = \bigwedge \{g_{\mathcal{C}}((5*2)*5) = g_{\mathcal{C}}(0), g_{\mathcal{C}}(2)\}.$$

Lemma 3.21.([4]) *If a and b are non-zero distinct atoms of a BCC-algebra X, then $a*b = a$.*

Theorem 3.22. *Let X be a BCC-algebra in which every non-zero element is an atom. Then every coupled BCC-$\mathcal{N}$-ideal of X is a coupled strong BCC-$\mathcal{N}$-ideal of X.*

Proof. Assume that every non-zero element is an atom in a BCC-algebra X. Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled BCC-$\mathcal{N}$-ideal of X. Let $x, y, z \in X$ be such that $y, (x*y)*z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Then $x*z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. It follows from Lemma 3.21 that $x = x*z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t,s)\}$. Hence $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled BCC-$\mathcal{N}$-ideal of X. □

For any element a of a BCC-algebra X, let

$$X_a := \{x \in X \mid f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a),\ g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)\}.$$

Obviously, X_a is a non-empty subset of X.

Theorem 3.23. *Let a be any element of a BCC-algebra X. If $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled (strong) $BCK(BCC)$-$\mathcal{N}$-ideal of X, then the set X_a is a (strong) $BCK(BCC)$-ideal of X.*

Proof. Since $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)$ for any $x \in X$, we have $0 \in X_a$. Let $x, y \in X$ be such that $x*y \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}(x*y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x*y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from (c82) that $f_{\mathcal{C}}(x) \leq \bigvee\{f_{\mathcal{C}}(x*y), f_{\mathcal{C}}(y)\} \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq \bigwedge\{g_{\mathcal{C}}(x*y), g_{\mathcal{C}}(y)\} \geq g_{\mathcal{C}}(a)$ so that $x \in X_a$. Therefore X_a is a BCK-ideal of X.

Let $x, y, z \in X$ be such that $(x*y)*z \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}((x*y)*z) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}((x*y)*z) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from (c83) that $f_{\mathcal{C}}(x*z) \leq \bigvee\{f_{\mathcal{C}}((x*

$y) * z), f_{\mathcal{C}}(y)\} \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x * z) \geq \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\} \geq g_{\mathcal{C}}(a)$ so that $x * z \in X_a$. Therefore X_a is a BCC-ideal of X.

Let $x, y, z \in X$ be such that $(x * y) * z \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}((x * y) * z) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}((x * y) * z) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from (c84) that $f_{\mathcal{C}}(x) \leq \bigvee\{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y)\} \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\} \geq g_{\mathcal{C}}(a)$ so that $x \in X_a$. Therefore X_a is a strong BCC-ideal of X. □

Proposition 3.24. *Let a be any element of a BCC-algebra X and let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X. Then*

(i) *If X_a is an ideal of X, then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies the following assertion:*

$$(\forall x, y, z \in X) \left(\begin{array}{l} f_{\mathcal{C}}(x) \geq \bigvee \{f_{\mathcal{C}}(y * z), f_{\mathcal{C}}(z)\} \Rightarrow f_{\mathcal{C}}(x) \geq f_{\mathcal{C}}(y) \\ g_{\mathcal{C}}(x) \leq \bigwedge \{g_{\mathcal{C}}(y * z), g_{\mathcal{C}}(z)\} \Rightarrow g_{\mathcal{C}}(x) \leq g_{\mathcal{C}}(y) \end{array} \right). \tag{3.2}$$

(ii) *If $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies (3.2) and*

$$(\forall x \in X) \left(f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x),\ g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)\right), \tag{3.3}$$

then X_a is an ideal of X.

Proof. (i) Assume that X_a is an ideal of X for all $a \in X$. Let $x, y, z \in X$ be such that $f_{\mathcal{C}}(x) \geq \bigvee \{f_{\mathcal{C}}(y * z), f_{\mathcal{C}}(z)\}$ and $g_{\mathcal{C}}(x) \leq \bigwedge \{g_{\mathcal{C}}(y * z), g_{\mathcal{C}}(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$ so that $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(x)$.

(ii) Suppose that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies two conditions (3.2) and (3.3). Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a)$, $g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a)$, $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. Hence $f_{\mathcal{C}}(a) \geq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(a) \leq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\}$, which imply from (3.2) that $f_{\mathcal{C}}(a) \geq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(a) \leq g_{\mathcal{C}}(x)$. Thus $x \in X_a$. Obviously, $0 \in X_a$. Therefore X_a is an ideal of X. □

Theorem 3.25. *Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in a BCC-algebra X. Then X_a is a BCK-ideal of X for any $a \in X$ if and only if*

(i) $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(a))$.
(ii) $(\forall x, y \in X)(f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a)$ *and* $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ *imply* $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a))$.
(iii) $(\forall x, y \in X)(g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a)$ *and* $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$ *imply* $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a))$.

Proof. Assume that X_a is a BCK-ideal of X. Then $0 \in X_a$ and so $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(a)$. Let $x, y, z \in X$ be such that $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$, and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. Then $x * y, y \in X_a$. Since X_a is a BCK-ideal of X, we have $x \in X_a$. Hence $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)$.

Conversely, consider X_a for any $a \in X$. Obviously, $0 \in X_a$ for any $a \in X$. Assume that $x * y, y \in X_a$. Then $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$, and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from hypothesis that $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)$. Hence $x \in X_a$. Thus X_a is a BCK-ideal of X.

Note that any coupled $\mathcal{N}$-subalgebra of a BCC-algebra may not be a coupled BCK-$\mathcal{N}$-ideal (See Example 3.6(1)). We now provide a condition for a coupled $\mathcal{N}$-subalgebra to be a coupled BCK-$\mathcal{N}$-ideal of X.

Theorem 3.26. *If every non-zero element of a BCC-algebra X is an atom, then every coupled $\mathcal{N}$-subalgebra of X is a coupled BCK-$\mathcal{N}$-ideal of X.*

Proof. Assume that every non-zero element is an atom in a BCC-algebra X. Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled BCC-$\mathcal{N}$-ideal of X. It follows from Lemma 3.3 that (c81) holds. Let $x, y \in X$ be such that $x * y, y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Since x is an atom, (b1) implies $x * y = 0$ or $x * y = x \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. If $x * y = 0$, then $x \leq y$ gives $x = 0$ or $x = y$. Hence $x \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Thus $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled BCK-$\mathcal{N}$-ideal of X. □

References

[1] W. A. Dudek, *On proper BCI-algebras*, Bull. Inst. MAth. Academia Sinica **20**(1992) 137-150.

[2] W. A. Dudek and Y. B. Jun, *Fuzzy BCC-ideals in BCC-algebras,* Math. Montisnigri **10**(1999) 21-30.

[3] W. A. Dudek, Y. B. Jun and Z. Stojaković, *On fuzzy BCC-ideals in BCC-algebras,* Fuzzy Sets and Systems **123**(2001) 251-258.

[4] W. A. Dudek and X. H. Zhang, *On atoms in BCC-algeberas*, Discuss. Math. Gen. Algebra Appl. **15**(1995) 82-85.

[5] W. A. Dudek and X. H. Zhang, *On ideals and congruences in BCC-algebras,* Czec. Math. J. **48**(1998) 21-29.

[6] W. A. Dudek and X. H. Zhang, *On ideals and fuzzy BCC-ideals in BCC-algebras,* Fuzzy Sets and Systems **123**(2001) 251-258.

[7] K. Iséki, On BCI-algebras, Math. Seminar Notes 8(1980) 125–130.

[8] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Jpn. **23**(1978) 1-26.

[9] Y. B. Jun, S. S. Ahn and D. R. Prince Williams, Coupled $\mathcal{N}$-structures and its applications in BCK/BCI-algebras, Iranian J. Sci. Tech. **37**(2013), 133-140.

[10] B. Karamdin and S. A. Bhatti, *Ideals and branches of BCC-algebras*, East. Asian Math. J. **23**(2007) 247-255.

[11] Y. Komori, *The class of BCC-algebras is not a variety,* Math. Jpn. **29**(1984) 391-394.

[12] A. Wroński, *A BCK-algebras do not form a variety,* Math. Jpn. **28**(1983) 211-213.

Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method

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Abstract

This paper investigates the numerical solution of nonlinear second-order periodic boundary value problems by using reproducing kernel Hilbert space method. The solution was calculated in the form of a convergent series in the space W_2^3 with easily computable components. In the proposed method, the n-term approximation is obtained and is proved to converge to the analytical solution. Meanwhile, the error of the approximate solution is monotone decreasing in the sense of the norm of W_2^3. The proposed technique is applied to several examples to illustrate the accuracy, efficiency, and applicability of the method. The results reveal that the method is very effective, straightforward, and simple.

Keywords: periodic boundary value problems; Reproducing kernel Hilbert space method

AMS Subject Classification: 34K28; 47B32; 34B15

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1 Introduction

Second-order boundary value problems (BVPs) for ordinary differential equations arise very frequently in many branches of applied mathematics and physics such as atomic calculations, gas dynamics, nuclear physics, atomic structures, deformation of beams and plate deflection theory, chemical reactions, and so on [1–4]. In recent years, the nonlinear second-order periodic BVPs which are a combination of second-order ordinary differential equations and periodic boundary conditions have been widely studied by many authors [5–8], due to a wide range of applications in applied mathematics, physics, and engineering, particularly in the homogenization of composite materials with a periodic microstructure [9, 10]. In most cases, nonlinear second-order periodic BVPs do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve by this technique, these problems must be attacked by various approximate and numerical methods.

This paper discusses and investigates the analytical approximate solution using reproducing kernel Hilbert space (RKHS) method for nonlinear second-order BVP with periodic boundary conditions which is as follows:

$$u''(x) = F(x, u(x), u'(x)), \; 0 \le x \le 1, \tag{1}$$

subject to the periodic boundary conditions

$$\begin{aligned} u(0) &= u(1), \\ u'(0) &= u'(1), \end{aligned} \tag{2}$$

where $u \in W_2^3[0,1]$ is an unknown function to be determined, $F(x,y,z)$ is continuous term in $W_2^1[0,1]$ as $y = y(x), z = z(x) \in W_2^3[0,1]$, $0 \le x \le 1$, $-\infty < y, z < \infty$, and is depending on the problem discussed, and $W_2^1[0,1], W_2^3[0,1]$ are two reproducing kernel spaces.

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Recently, many authors have discussed the numerical solvability of periodic BVPs. To mention a few, the existence and multiplicity of positive solutions have been discussed to first-order periodic BVPs as described in [11]. In [12] the authors have discussed the existence of nontrivial periodic solutions for second-order periodic BVPs. In [13] also, the author has provided the existence and multiplicity of positive solutions to further investigation to second-order periodic BVPs. Furthermore, the existence of solutions is carried out in [14] for third-order periodic BVPs. The existence of positive solution has been investigated to solve fourth-order periodic BVPs as presented in [15]. However, we assume that Eq. (1) subject to the periodic boundary conditions (2) has a unique solution on $[0,1]$. But on the other aspects as well, the numerical solvability of differential and integro-differential equations of different types and orders can be found in [16–20] and references therein. Also, for numerical solvability of different categories of second-order BVPs one can consult the references [21–24].

Investigation about second-order periodic BVPs numerically is scarce. In this work, we utilize an a methodical way to solve these type of differential equations. The new method is accurate, need less effort to achieve the results, and is developed especially for nonlinear case. Meanwhile, the proposed method has an advantage that it is possible to pick any point in the interval $[0,1]$ and as well the approximate solutions and all its derivatives up to order two will be applicable.

Reproducing kernel theory has important application in numerical analysis, differential equations, integral equations, probability and statistics, and so fourth [25–27]. In the last years, extensive work has been done using RKHS method, which provides numerical approximations for linear and nonlinear equations. This method has been implemented in several operator, differential, integral, and integro-differential equations, such as nonlinear operator equations [28], nonlinear system of second-order BVPs [29], linear initial-boundary-value problems, [30], nonlinear second-order singular BVPs [31, 32], nonlinear partial differential equations [33], nonlinear Fredholm-Volterra integral equation [34], nonlinear fourth-order integro-differential equations [35, 36], nonlinear Fredholm-Volterra integro-differential equations [37], and others.

The rest of the paper is organized as follows: in the next section, several reproducing kernel spaces are described. In section 3 a linear operator, a complete normal orthogonal system, and some essential results are introduced. Also, a method for the existence of solutions for Eqs. (1) and (2) based on reproducing kernel space is described. In section 4, we give an iterative method to solve Eqs. (1) and (2) numerically in the space $W_2^3[0,1]$. A numerical examples are presented in section 5. Section 6 ends this paper with a brief conclusion.

2 Several reproducing kernel spaces

In this section, two reproducing kernels needed are constructed in order to solve Eqs. (1) and (2) using RKHS method. Before the construction, we utilize the reproducing kernel concept. Throughout this paper $\mathbb{C}$ the set of complex numbers, the superscript (i) in $u^{(i)}(x)$ denotes the i-th derivative of $u(x)$, $L^2[a,b] = \{u \mid \int_a^b u^2(x)\,dx < \infty\}$, and $l^2 = \{A \mid \sum\limits_{i=1}^{\infty} (A_i)^2 < \infty\}$.

An abstract set is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for the number of elements.

Definition .1 [31] Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if

1. for each $x \in E$, $K(\cdot,x) \in H$.
2. for each $x \in E$ and $\varphi \in H$, $\langle \varphi(\cdot), K(\cdot,x)\rangle = \varphi(x)$.

The condition (2) is called "the reproducing property": the value of the function φ at the point x is reproducing by the inner product of $\varphi(\cdot)$ with $K(\cdot,x)$. A Hilbert space which possesses a reproducing kernel is called a RKHS [31].

Next, we first utilize the reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the periodic boundary conditions (2) and then construct the space $W_2^1[0,1]$.

Definition .2 [32] The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real-valued functions on $[0,1]$, $u, u', u'', u''' \in L^2[0,1]$, and $u(0) = u(1)$, $u'(0) = u'(1)\}$. On the other hand, the inner product and norm in $W_2^3[0,1]$ are defined, respectively, by

$$\langle u(x), v(x)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)\, v^{(i)}(0) + \int_0^1 u'''(x)v'''(x)dx, \tag{3}$$

and $||u||_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

The Hilbert space $W_2^3[0,1]$ is called a reproducing kernel if for each fixed $x \in [0,1]$, there exist $R(x,y) \in W_2^3[0,1]$ (simply $R_x(y)$) such that $\langle u(y), R_x(y)\rangle_{W_2^3} = u(x)$ for any $u(y) \in W_2^3[0,1]$ and $y \in [0,1]$. Next theorem utilize the reproducing kernel function $R_x(y)$ on the space $W_2^3[0,1]$.

Theorem .1 The Hilbert space $W_2^3[0,1]$ is a complete reproducing kernel and its reproducing kernel function $R_x(y)$ can be written as

$$R_x(y) = \begin{cases} a_1(x) + a_2(x)y + a_3(x)y^2 + a_4(x)y^3 + a_5(x)y^4 + a_6(x)y^5, & y \le x, \\ b_1(x) + b_2(x)y + b_3(x)y^2 + b_4(x)y^3 + b_5(x)y^4 + b_6(x)y^5, & y > x, \end{cases} \tag{4}$$

where $a_i(x)$ and $b_i(x)$, $i = 1, 2, ..., 6$, are unknown coefficients of $R_x(y)$.

Proof. The proof of the completeness and reproducing property of $W_2^3[0,1]$ is similar to the proof in [30]. Now, let us find out the expression form of the reproducing kernel function $R_x(y)$ in the space $W_2^3[0,1]$. Through several integration by parts, we obtain $\int_0^1 u'''(y)\partial_y^3 R_x(y)dy = \sum_{i=0}^{2}(-1)^i u^{(i)}(y)\,\partial_y^{5-i}R_x(y)\,|_{y=0}^{y=1} - \int_0^1 u(y)\,\partial_y^6 R_x(y)\,dy$. Thus, from Eq. (3) we can write

$$\begin{aligned} \langle u(y), R_x(y)\rangle_{W_2^3} &= \sum_{i=0}^{2} u^{(i)}(0)\,[\partial_y^i R_x(0) + (-1)^{i+1}\partial_y^{5-i}R_x(0)] + \sum_{i=0}^{2}(-1)^i u^{(i)}(1)\,\partial_y^{5-i}R_x(1) \\ &\quad - \int_0^1 u(y)\,\partial_y^6 R_x(y)\,dy. \end{aligned}$$

Since $R_x(y) \in W_2^3[0,1]$, it follows that $R_x(0) = R_x(1)$ and $\partial_y^1 R_x(0) = \partial_y^1 R_x(1)$. Again, since $u(x) \in W_2^3[0,1]$, it yield that $u^{(i)}(a) = u^{(i)}(b)$, $i = 0, 1$. Hence,

$$\begin{aligned} \langle u(y), R_x(y)\rangle_{W_2^3} = {}& \sum_{i=0}^{2} u^{(i)}(0)\,[\partial_y^i R_x(0) + (-1)^{i+1}\partial_y^{5-i}R_x(0)] + \sum_{i=0}^{2}(-1)^i u^{(i)}(1)\,\partial_y^{5-i}R_x(1) \\ & - \int_0^1 u(y)\,\partial_y^6 R_x(y)\,dy + c_1(u(0) - u(1)) + c_2(u'(0) - u'(1)). \end{aligned} \tag{5}$$

On the other hand, if $\partial_y^3 R_x(1) = 0$, $R_x(0) - \partial_y^5 R_x(0) + c_1 = 0$, $\partial_y^2 R_x(0) - \partial_y^3 R_x(0) = 0$, $\partial_y^5 R_x(1) - c_1 = 0$, $\partial_y^1 R_x(0) + \partial_y^4 R_x(0) + c_2 = 0$, and $\partial_y^4 R_x(1) + c_2 = 0$, then Eq. (5) implies that $\langle u(y), R_x(y)\rangle_{W_2^3} = \int_0^1 u(y)\,(-\partial_y^6 R_x(y))dy$. Now, for any $x \in [0,1]$, if $R_x(y)$ satisfies

$$\partial_y^6 R_x(y) = -\delta(x - y),\ \delta \text{ dirac-delta function}, \tag{6}$$

then $\langle u(y), R_x(y)\rangle_{W_2^3} = u(x)$. Obviously, $R_x(y)$ is the reproducing kernel function of the space $W_2^3[0,1]$.

Next, we give the expression of the reproducing kernel function $R_x(y)$. The auxiliary equation of Eq. (6) is given by $\lambda^6 = 0$, and their auxiliary values are $\lambda = 0$ with multiplicity 6. So, let the expression of the reproducing kernel function $R_x(y)$ be as defined in Eq. (4).

But on the other aspect as well, for Eq. (6) let $R_x(y)$ satisfy the equation $\partial_y^m R_x(x+0) = \partial_y^m R_x(x-0)$, $m = 0, 1, 2, 3, 4$. Integrating $\partial_y^6 R_x(y) = -\delta(x-y)$ from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $\partial_y^5 R_x(y)$ at $y = x$ given by $\partial_y^5 R_x(x+0) - \partial_y^5 R_x(x-0) = -1$. Through the last descriptions the unknown coefficients $a_i(x)$ and $b_i(x)$, $i = 1, 2, ..., 6$ of Eq. (4) can be obtained. This completes the proof. ■

Remark .1 By using Mathematica 7.0 software package, the coefficients $a_i(x)$ and $b_i(x)$, $i = 1, 2, ..., 6$ of the reproducing kernel function $R_x(y)$ in the space $W_2^3[0,1]$ are obtained and are given as follows:

$$
\begin{aligned}
a_1(x) &= 1, \\
a_2(x) &= \frac{1}{3867}x(27 - 60x - 20x^2 + 85x^3 - 32x^4), \\
a_3(x) &= -\frac{1}{15468}x(240 - 963x + 968x^2 - 247x^3 + 2x^4), \\
a_4(x) &= -\frac{1}{46404}x(240 - 963x + 968x^2 - 247x^3 + 2x^4), \\
a_5(x) &= \frac{1}{9208}x(-1827 + 1482x + 494x^2 - 166x^3 + 17x^4), \\
a_6(x) &= \frac{1}{464040}(3867 - 3840x - 60x^2 - 20x^3 + 85x^4 - 32x^5),
\end{aligned}
$$

$$
\begin{aligned}
b_1(x) &= 1 + \frac{1}{120}x^5, \\
b_2(x) &= \frac{1}{30936}x(216 - 480x - 160x^2 - 609x^3 - 256x^4), \\
b_3(x) &= -\frac{1}{15468}x(240 - 963x - 321x^2 - 247x^3 + 2x^4), \\
b_4(x) &= -\frac{1}{46404}x(240 + 2904x + 968x^2 - 247x^3 + 2x^4), \\
b_5(x) &= \frac{1}{92808}x(2040 + 1482x + 494x^2 - 166x^3 + 17x^4), \\
b_6(x) &= -\frac{1}{464040}x(3840 + 60x + 20x^2 - 85x^3 + 32x^4).
\end{aligned}
$$

The reproducing kernel function $R_x(y)$ posses some important properties such as: $R_x(y)$ is symmetric, unique, and nonnegative for any fixed $x \in [0,1]$.

Definition .3 [33] The inner product space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real-valued function on $[0,1]$ and $u' \in L^2[0,1]\}$. On the other hand, the inner product and norm in $W_2^1[0,1]$ are defined, respectively, by

$$\langle u(x), v(x) \rangle_{W_2^1} = u(0)\,v(0) + \int_0^1 u'(x)v'(x)dx,$$

and $||u||_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}}$, where $u, v \in W_2^1[0,1]$.

Remark .2 In [33], it has been proved that the Hilbert space $W_2^1[0,1]$ is a complete reproducing kernel and its reproducing kernel function is given by $G_x(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$

3 Structure representation of solution

In this section, the representation of the analytical solution of Eqs. (1) and (2) and the implementation method are given in the reproducing kernel space $W_2^3[0,1]$. After that, we construct an orthogonal function system of the space $W_2^3[0,1]$ based on the Gram-Schmidt orthogonalization process.

To do this, we define a differential operator L as

$$L : W_2^3[0,1] \to W_2^1[0,1],$$

such that

$$Lu(x) = u''(x).$$

As a result, Eqs. (1) and (2) can be converted into the equivalent form as follows:

$$\begin{aligned} &Lu(x) = F(x, u(x), u'(x)), \ 0 \leq x \leq 1, \\ &u(0) - u(1) = 0, \\ &u'(0) - u'(1) = 0, \end{aligned} \tag{7}$$

where $u(x) \in W_2^3[0,1]$ and $F(x,y,z) \in W_2^1[0,1]$ for $y = y(x), z = z(x) \in W_2^3[0,1]$, $-\infty < y, z < \infty$, and $0 \leq x \leq 1$. It is easy to show that L is a bounded linear operator from the space $W_2^3[0,1]$ into space $W_2^1[0,1]$.

Initially, we construct an orthogonal function system of $W_2^3[0,1]$. To do so, put $\varphi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L_i^* \varphi(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and L^* is the adjoint operator of L. In terms of the properties of reproducing kernel $R_x(y)$, one obtains $\langle u(x), \psi_i(x) \rangle_{W_2^3} = \langle u(x), L^* \varphi_i(x) \rangle_{W_2^3} = \langle Lu(x), \varphi_i(x) \rangle_{W_2^1} = Lu(x_i)$, $i = 1, 2, ...$

For the orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^3[0,1]$, it can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \tag{8}$$

where β_{ik} are orthogonalization coefficients and are given by the following subroutine:

$$\begin{aligned} \beta_{ij} &= \frac{1}{\|\psi_1\|}, \text{ for } i = j = 1, \\ \beta_{ij} &= \frac{1}{d_{ik}}, \text{ for } i = j \neq 1, \\ \beta_{ij} &= -\frac{1}{d_{ik}} \sum_{k=j}^{i-1} c_{ik} \beta_{kj}, \text{ for } i > j, \end{aligned}$$

such that $d_{ik} = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$, $c_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{W_2^3}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^3[0,1]$.

It is easy to see that, $\psi_i(x) = L^* \varphi_i(x) = \langle L^* \varphi_i(x), K_x(y) \rangle_{W_2^3} = \langle \varphi_i(x), L_y K_x(y) \rangle_{W_2^1} = L_y K_x(y)|_{y=x_i} \in W_2^3[0,1]$. Thus, $\psi_i(x)$ can be written in the form $\psi_i(x) = L_y K_x(y)|_{y=x_i}$, where L_y indicates that the operator L applies to the function of y.

Theorem .2 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of the space $W_2^3[0,1]$.

Proof. For each fixed $u(x) \in W_2^3[0,1]$, let $\langle u(x), \psi_i(x) \rangle_{W_2^3} = 0$, $i = 1, 2, ...$ In other word, $\langle u(x), \psi_i(x) \rangle_{W_2^3} = \langle u(x), L^* \varphi_i(x) \rangle_{W_2^3} = \langle Lu(x), \varphi_i(x) \rangle_{W_2^1} = Lu(x_i) = 0$. Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1}. So, the proof of the theorem is complete. ■

Lemma .1 If $u(x) \in W_2^3[0,1]$, then there exists a positive constant M such that $\left\|u^{(i)}(x)\right\|_C \leq M \|u(x)\|_{W_2^3}$, $i = 0, 1, 2$, where $\|u(x)\|_C = \max_{0 \leq x \leq 1} |u(x)|$.

Proof. For any $x, y \in [0,1]$, we have $u^{(i)}(x) = \langle u(y), \partial_x^i R_x(y) \rangle_{W_2^3}$, $i = 0, 1, 2$. By the expression of $R_x(y)$, it follows that $\|\partial_x^i R_x(y)\|_{W_2^3} \leq M_i$, $i = 0, 1, 2$. Thus, $|u^{(i)}(x)| = \left|\langle u(x), \partial_x^i R_x(x) \rangle_{W_2^3}\right| \leq \|\partial_x^i R_x(x)\|_{W_2^3} \|u(x)\|_{W_2^3} \leq M_i \|u(x)\|_{W_2^3}$, $i = 0, 1, 2$. Hence, $\left\|u^{(i)}(x)\right\|_C \leq \max_{i=0,1,2} \{M_i\} \|u(x)\|_{W_2^3}$, $i = 0, 1, 2$. ■

The internal structure of the following theorem is as follows: firstly, we will give the representation of the exact solution of Eqs. (1) and (2) in the space $W_2^3[0,1]$. After that, the convergence of approximate solution $u_n(x)$ to the analytic solution $u(x)$ will be proved.

Theorem .3 For each u in the space $W_2^3[0,1]$, the series $\sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^3[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the following are hold:

(i) the exact solution of Eq. (7) could be represented by

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k, u(x_k), u'(x_k))\,\bar{\psi}_i(x). \tag{9}$$

(ii) the approximate solution of Eq. (7)

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(x_k, u(x_k), u'(x_k))\,\bar{\psi}_i(x), \tag{10}$$

and $u_n^{(i)}(x)$, $i = 0, 1, 2$ are converging uniformly to the exact solution $u(x)$ and all its derivative as $n \to \infty$, respectively.

Proof. For the first part, let $u(x)$ be solution of Eq. (7) in the space $W_2^3[0,1]$. Since $u(x) \in W_2^3[0,1]$, $\sum_{i=1}^{\infty}\left\langle u(x), \bar{\psi}_i(x)\right\rangle \bar{\psi}_i(x)$ is the Fourier series expansion about normal orthogonal system $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$, and $W_2^3[0,1]$ is the Hilbert space, then the series $\sum_{i=1}^{\infty}\left\langle u(x), \bar{\psi}_i(x)\right\rangle \bar{\psi}_i(x)$ is convergent in the sense of $\|\cdot\|_{W_2^3}$. On the other hand, using Eq. (8), it easy to see that

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x), \bar{\psi}_i(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x), \psi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x), L^*\varphi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x), \varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F(x, u(x), u'(x)), \varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k, u(x_k), u'(x_k))\,\bar{\psi}_i(x).
\end{aligned}$$

Therefore, the form of Eq. (9) is the exact solution of Eq. (7).

For the second part, it easy to see that by Lemma .1, for any $x \in [0,1]$

$$\begin{aligned}
|u_n(x) - u(x)| &= \left|\left\langle u_n(x) - u(x), R_x(x)\right\rangle_{W_2^3}\right| \\
&\leq \|R_x(x)\|_{W_2^3}\|u_n(x) - u(x)\|_{W_2^3} \\
&\leq M_0\|u_n(x) - u(x)\|_{W_2^3}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|u_n^{(i)}(x) - u^{(i)}(x)\right| &= \left|\left\langle u_n(x) - u(x), R_x^{(i)}(x)\right\rangle_{W_2^3}\right| \\
&\leq \left\|\partial_x^i R_x(x)\right\|_{W_2^3}\|u_n(x) - u(x)\|_{W_2^3} \\
&\leq M_i\|u_n(x) - u(x)\|_{W_2^3},\ i = 1, 2.
\end{aligned}$$

where M_i, $i = 0, 1, 2$ are positive constants. Hence, if $\|u_n(x) - u(x)\|_{W_2^3} \to 0$ as $n \to \infty$, the approximate solution $u_n(x)$ and $u_n^{(i)}(x)$, $i = 0, 1, 2$ are converge uniformly to the exact solution $u(x)$ and all its derivative, respectively. So, the proof of the theorem is complete. ■

Remark .3 We mention here that, the approximate solution $u_n(x)$ in Eq. (10) can be obtained directly by taking finitely many terms in the series representation for $u(x)$ of Eq. (9).

4 Procedure of constructing iterative method

In this section, an iterative method of obtaining the solution of Eq. (5) is represented in the reproducing kernel space $W_2^3[0,1]$ for linear and nonlinear case. Initially, we will mention the following remark about the exact and approximate solutions of Eqs. (1) and (2).

Remark .4 In order to apply the RKHS technique for solve Eqs. (1) and (2), we have the following two cases based on the structure of the function F.

Case 1: if Eq. (1) is linear, that is $F(x,u(x),u'(x)) = p(x)u'(x)+q(x)u(x)+r(x)$, then the exact and approximate solutions can be obtained directly from Eqs. (9) and (10), respectively.

Case 2: if Eq. (1) is nonlinear, that is $F(x,u(x),u'(x))$ is not a linear combination of $u'(x)$ and $u(x)$, then in this case the exact and approximate solutions can be obtained by using the following iterative algorithm:

Algorithm 1 According to Eq. (9), the representation of the solution of Eq. (1) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x), \tag{11}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right)$. In fact, B_i, $i=1,2,...$ in Eq. (11) are unknown, we will approximate B_i using known A_i. For a numerical computations, we define initial function $u_0(x_1)=0$, put $u_0(x_1)=u(x_1)$, and define the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} A_i \bar{\psi}_i(x), \tag{12}$$

where the coefficients A_i of $\bar{\psi}_i(x)$, $i=1,2,...,n$ are given as

$$\begin{aligned}
&A_1 = \beta_{11} F(x_1, u_0(x_1), u_0'(x_1)), \\
&u_1(x) = A_1 \bar{\psi}_1(x), \\
&A_2 = \textstyle\sum_{k=1}^{2} \beta_{2k} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right), \\
&u_2(x) = \textstyle\sum_{i=1}^{2} A_i \bar{\psi}_i(x), \\
&\vdots \\
&u_{n-1}(x) = \textstyle\sum_{i=1}^{n-1} A_i \bar{\psi}_i(x), \\
&A_n = \textstyle\sum_{k=1}^{n} \beta_{nk} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right).
\end{aligned} \tag{13}$$

Here, we note that: in the iterative process of Eq. (12), we can guarantee that the approximation $u_n(x)$ satisfies the periodic boundary conditions (2). Now, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$ and

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{n-1}(x_k), u'_{n-1}(x_k)\right) \bar{\psi}_i(x). \tag{14}$$

Now, we will proof that $u_n(x)$ in the iterative formula (12) is converge to the exact solution $u(x)$ of Eq. (1), in fact this result is a fundamental in the RKHS theory and its applications. The next two lemmas are collected in order to prove the recent theorem.

Lemma .2 If $\|u_n(x) - u(x)\|_{W_2^3} \to 0$, $x_n \to y$ as $n \to \infty$, and $F(x,v,w)$ is continuous in $[0,1]$ with respect to x,v,w for $x \in [0,1]$ and $v,w \in (-\infty,\infty)$, then $F(x_n, u_{n-1}(x_n), u'_{n-1}(x_n)) \to F(y,u(y),u'(y))$ as $n \to \infty$.

Proof. Since $\|u_n(x) - u(x)\|_{W_2^3} \to 0$ as $n \to \infty$, by Theorem .3 and Lemma .1, we know that $u_{n-1}(x_n)$ and $u'_{n-1}(x_n)$ are convergent uniformly to $u(x)$ and $u'(x)$, respectively, as $x_n \to y$ and $n \to \infty$. Hence, the continuity of F gives the result. ■

Lemma .3 $Lu_n(x_j) = Lu(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j)\right)$ as $j \leq n$.

Proof. The proof of $Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j)\right)$ will be obtained by induction as follows: if $j \leq n$, then $Lu_n(x_j) = \sum\limits_{i=1}^{n} A_i L\bar{\psi}_i(x_j) = \sum\limits_{i=1}^{n} A_i \left\langle L\bar{\psi}_i(x), \varphi_j(x)\right\rangle_{W_2^1} = \sum\limits_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), L_j^*\varphi(x)\right\rangle_{W_2^3} = \sum\limits_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^3}$. Using the orthogonality of $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$, yields that

$$
\begin{aligned}
\sum_{l=1}^{j} \beta_{jl} Lu_n(x_l) &= \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \sum_{l=1}^{j} \beta_{jl}\psi_l(x)\right\rangle_{W_2^3} \\
&= \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \bar{\psi}_j(x)\right\rangle_{W_2^3} = A_j \\
&= \sum_{l=1}^{j} \beta_{jl} F\left(x_l, u_{l-1}(x_l), u'_{l-1}(x_l)\right).
\end{aligned}
$$

Now, if $j = 1$, then $Lu_n(x_1) = F(x_1, u_0(x_1), u'_0(x_1))$. Again, if $j = 2$, then $\beta_{21}Lu_n(x_1) + \beta_{22}Lu_n(x_2) = \beta_{21}F(x_1, u_0(x_1), u'_0(x_1)) + \beta_{22}F(x_2, u_1(x_2), u'_1(x_2))$. Thus, $Lu_n(x_2) = F(x_2, u_1(x_2), u'_1(x_2))$. It is easy to see that $Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j)\right)$ by using mathematical induction.

On the other hand, from Theorem .3, $u_n(x)$ converge uniformly to $u(x)$. It follows that, on taking limits in Eq. (12), $u(x) = \sum\limits_{i=1}^{\infty} A_i\bar{\psi}_i(x)$. Therefore, $u_n(x) = P_n u(x)$, where P_n is an orthogonal projector from the space $W_2^3[0,1]$ to Span$\{\psi_1, \psi_2, ..., \psi_n\}$. Thus, $Lu_n(x_j) = \left\langle Lu_n(x), \varphi_j(x)\right\rangle_{W_2^1} = \left\langle u_n(x), L_j^*\varphi(x)\right\rangle_{W_2^3} = \left\langle P_n u(x), \psi_j(x)\right\rangle_{W_2^3} = \left\langle u(x), P_n\psi_j(x)\right\rangle_{W_2^3} = \left\langle u(x), \psi_j(x)\right\rangle_{W_2^3} = \left\langle Lu(x), \varphi_j(x)\right\rangle_{W_2^1} = Lu(x_j)$. ■

Theorem .4 If $||u_n||_{W_2^3}$ is bounded and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term approximate solution $u_n(x)$ in the iterative formula (12) converges to the exact solution $u(x)$ of Eq. (7) in the space $W_2^3[0,1]$ and $u(x) = \sum\limits_{i=1}^{\infty} A_i\bar{\psi}_i(x)$, where A_i is given by Eq. (13).

Proof. The proof consists of the following three steps: firstly, we will prove that the sequence $\{u_n\}_{n=1}^{\infty}$ in Eq. (12) is monotone increasing in the sense of $\|\cdot\|_{W_2^3}$. By Theorem .2, $\left\{\bar{\psi}_i\right\}_{i=1}^{\infty}$ is the complete orthonormal system in the space $W_2^3[0,1]$. Hence, we have $\|u_n\|^2_{W_2^3} = \langle u_n(x), u_n(x)\rangle_{W_2^3} = \left\langle \sum\limits_{i=1}^{n} A_i\bar{\psi}_i(x), \sum\limits_{i=1}^{n} A_i\bar{\psi}_i(x)\right\rangle_{W_2^3} = \sum\limits_{i=1}^{n} (A_i)^2$. Therefore, $\|u_n\|_{W_2^3}$ is monotone increasing.

Secondly, we will prove the convergence of $u_n(x)$. From Eq. (12), we have $u_{n+1}(x) = u_n(x) + A_{n+1}\bar{\psi}_{n+1}(x)$. From the orthogonality of $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$, it follows that $||u_{n+1}||^2_{W_2^3} = ||u_n||^2_{W_2^3} + (A_{n+1})^2 = ||u_{n-1}||^2_{W_2^3} + (A_n)^2 + (A_{n+1})^2 = ... = ||u_0||^2_{W_2^3} + \sum\limits_{i=1}^{n+1}(A_i)^2$. Since, the sequence $\|u_n\|^2_{W_2^3}$ is monotone increasing. Due to the condition that $||u_n||_{W_2^3}$ is bounded, $||u_n||_{W_2^3}$ is convergent as $n \to \infty$. Then, there exists a constant c such that $\sum\limits_{i=1}^{\infty}(A_i)^2 = c$. It implies that $A_i = \sum\limits_{k=1}^{i} \beta_{ik} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right) \in l^2$, $i = 1, 2, ...$ On the other hand, since $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp ... \perp (u_{n+1} - u_n)$ it follows for $m > n$ that

$$
\begin{aligned}
||u_m(x) - u_n(x)||^2_{W_2^3} &= ||u_m(x) - u_{m-1}(x) + u_{m-1}(x) - ... + u_{n+1}(x) - u_n(x)||^2_{W_2^3} \\
&= ||u_m(x) - u_{m-1}(x)||^2_{W_2^3} + ... + ||u_{n+1}(x) - u_n(x)||^2_{W_2^3}.
\end{aligned}
$$

Furthermore, $||u_m(x) - u_{m-1}(x)||^2_{W_2^3} = (A_m)^2$. Consequently, as $n, m \to \infty$, we have $||u_m(x) - u_n(x)||^2_{W_2^3} = \sum\limits_{i=n+1}^{m} (A_i)^2 \to 0$. Considering the completeness of $W_2^3[0,1]$, there exists a $u(x) \in W_2^3[0,1]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in the sense of $||\cdot||_{W_2^3}$.

Thirdly, we will prove that $u(x)$ is the solutions of Eq. (7). Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, for any $x \in [0,1]$, there exists subsequence $\left\{x_{n_j}\right\}_{j=1}^{\infty}$, such that $x_{n_j} \to x$ as $j \to \infty$. From Lemma .3, It is clear that $Lu\left(x_{n_j}\right) =$

$F\left(x_{n_j}, u_{n_j-1}(x_k), u'_{n_j-1}(x_k)\right)$. Hence, let $j \to \infty$, by lemma .2 and the continuity of F, we have $Lu(x) = F(x, u(x), u'(x))$. That is, $u(x)$ satisfies Eq. (1). Also, since $\bar{\psi}_i(x) \in W_2^3[0,1]$, clearly, $u(x)$ satisfies the periodic boundary conditions (2). In other words, $u(x)$ is the solution of Eqs. (1) and (2), where $u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$ and A_i are given by Eq. (13). The proof is complete. ■

It obvious that, if we let $u(x)$ denote the exact solution of Eq. (7), $u_n(x)$ denote the approximate solution obtained by the RKHS method as given by Eq. (12), and $r_n(x)$ is the difference between $u_n(x)$ and $u(x)$, where $x \in [0,1]$, then $||r_n(x)||^2_{W_2^5} = ||u(x) - u_n(x)||^2_{W_2^3} = \left\| \sum_{i=n+1}^{\infty} A_i \bar{\psi}_i(x) \right\|^2_{W_2^3} = \sum_{i=n+1}^{\infty} (A_i)^2$ and $||r_{n-1}(x)||^2_{W_2^3} = \sum_{i=n}^{\infty} (A_i)^2$ or $||r_n(x)||_{W_2^3} \le ||r_{n-1}(x)||_{W_2^3}$. Consequently, this show that the difference $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$.

5 Numerical outcomes

In this section, we propose few numerical simulations implemented by Mathematica 7.0 software package for solving some specific examples of Eqs. (1) and (2). However, we apply the algorithm described in the previous sections to some linear and nonlinear test examples in order to demonstrate the efficiency, accuracy, and applicability of the proposed method. Results obtained by the method are compared with the analytical solution of each example by computing the exact and relative errors and are found to be in good agreement with each other.

Example 1 Consider the following linear nonhomogeneous equation:

$$u''(x) + u(x) = f(x), \; 0 \le x \le 1,$$

subject to the periodic boundary conditions

$$u(0) = u(1),$$
$$u'(0) = u'(1),$$

where $f(x) = (4x^4 - 8x^3 + 4x^2 + 3)(1-2x)^2 e^{x^2(x-1)^2}$. The exact solution is $u(x) = e^{x^2(x-1)^2}$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ with the reproducing kernel function $R_x(y)$ on $[0,1]$, the approximate solution $u_n(x)$ is calculated by Eq. (10). The numerical results at some selected grid points for $n = 26$ are given in Table 1.

Table 1. Numerical results for Example 1.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	1.0182274892397234	1.0182295775058678	2.08827×10^{-6}	2.05088×10^{-6}
0.32	1.0484886643504874	1.0484907460220763	2.08167×10^{-6}	1.98540×10^{-6}
0.48	1.0642817515996124	1.0642838267814154	2.07518×10^{-6}	1.94984×10^{-6}
0.64	1.0545183896526067	1.0545204690600505	2.07941×10^{-6}	1.97190×10^{-6}
0.80	1.0259304941903820	1.0259325816419569	2.08745×10^{-6}	2.03469×10^{-6}
0.96	1.0014756476981566	1.0014777346809174	2.08698×10^{-6}	2.08391×10^{-6}

Example 2 Consider the following nonlinear nonhomogeneous equation:

$$u''(x) + 2u(x) + \frac{1}{1 + (u(x))^2} = f(x), \; 0 \le x \le 1,$$

subject to the periodic boundary conditions

$$u(0) = u(1),$$
$$u'(0) = u'(1),$$

where $f(x) = \dfrac{1}{x^4(x-1)^4+1} + 8x^3 - 8x + 2$. The exact solution is $u(x) = x^4 - 2x^3 + x^2$.

Using RKHS method, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ with the reproducing kernel function $R_x(y)$ on $[0, 1]$, the approximate solution $u_n^N(x)$ is calculated by Eq. (14). The numerical results at some selected grid points for $N = 51$ and $n = 3$ are given in Table 2.

Table 2. Numerical results for Example 2.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	0.01806336	0.01806360646957179	2.46470×10^{-7}	1.36447×10^{-5}
0.32	0.04734976	0.04734996786169693	2.07862×10^{-7}	4.38992×10^{-6}
0.48	0.06230016	0.06230018632258992	2.63226×10^{-8}	4.22512×10^{-7}
0.64	0.05308416	0.05308398923694464	1.70763×10^{-7}	3.21684×10^{-6}
0.80	0.02560000	0.02559974238385324	2.57616×10^{-7}	1.00631×10^{-5}
0.96	0.00147456	0.00147446292690514	9.70731×10^{-8}	6.58319×10^{-5}

Example 3 Consider the following nonlinear nonhomogeneous equation:

$$u''(x) + u'(x) - (2x-1)^2 u(x) + \cosh^{-1}(u(x)) = f(x), \; 0 \le x \le 1,$$

subject to the periodic boundary conditions

$$\begin{aligned} u(0) &= u(1), \\ u'(0) &= u'(1), \end{aligned}$$

where $f(x) = (1+2x)\sinh(x(x-1)) + x - x^2$. The exact solution is $u(x) = \cosh\left(x^2 - x\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ with the reproducing kernel function $R_x(y)$ on $[0, 1]$, the approximate solution $u_n^N(x)$ is calculated by Eq. (14). The numerical results at some selected grid points for $N = 51$ and $n = 3$ are given in Table 3.

Table 3. Numerical results for Example 3.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	1.0090452833957488	1.0090454991505815	2.15755×10^{-7}	2.13821×10^{-7}
0.32	1.0237684442237780	1.0237671617164725	1.28251×10^{-6}	1.25273×10^{-6}
0.48	1.0313121374632044	1.0313104084921125	1.72897×10^{-6}	1.67648×10^{-6}
0.64	1.0266597016257117	1.0266584708541906	1.23077×10^{-6}	1.19881×10^{-6}
0.80	1.0128273299790107	1.0128274421174757	1.12138×10^{-7}	1.10718×10^{-7}
0.96	1.0007373706014195	1.0007395991224600	2.22852×10^{-6}	2.22688×10^{-6}

Example 4 Consider the following nonlinear nonhomogeneous equation:

$$u''(x) + (u'(x))^2 + e^{-u(x)} = f(x), \; 0 \le x \le 1,$$

subject to the periodic boundary conditions

$$\begin{aligned} u(0) &= u(1), \\ u'(0) &= u'(1), \end{aligned}$$

where $f(x) = \dfrac{3(2x-1)^2}{x^2(1-x)^2+1}$. The exact solution is $u(x) = \ln\left(x^4 - 2x^3 + x^2 + 1\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ with the reproducing kernel function $R_x(y)$ on $[0, 1]$, the approximate solution $u_n^N(x)$ is calculated by Eq. (14). The numerical results at some selected grid points for $N = 51$ and $n = 3$ are given in Table 4.

Table 4. Numerical results for Example 4.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	0.017902155877180255	0.017902119683171697	3.61940×10^{-8}	2.02177×10^{-6}
0.32	0.046262935319852880	0.046262940200774260	4.88092×10^{-9}	1.05504×10^{-7}
0.48	0.060436519420405940	0.060436573149274840	5.37289×10^{-7}	8.89013×10^{-7}
0.64	0.051723153984674340	0.051723175796916536	2.18122×10^{-8}	4.21711×10^{-7}
0.80	0.025277807184268607	0.025277772899866330	3.42844×10^{-8}	1.35630×10^{-6}
0.96	0.001473473903947873	0.001473467202387802	6.70156×10^{-9}	4.54814×10^{-6}

As we mentioned earlier, it is possible to pick any point in $[0,1]$ and as well the approximate solutions and all its derivative up to order two will be applicable. Next, new numerical results for Example 4 which include the absolute error at some selected gird nodes in $[0,1]$ for $u^{(i)}(x)$, $i = 0,1,2$, where $x_i = \frac{i-1}{N-1}$, $i = 1,2,...,N$, $N = 51$, and $n = 3$ are given in Table 5.

Table 5. Absolute error of $u^{(i)}(x)$, $i = 0,1,2$ for Example 4.

i	$x = 0.16$	$x = 0.48$	$x = 0.64$	$x = 0.96$
0	3.61940×10^{-8}	5.37289×10^{-7}	2.18122×10^{-8}	6.70156×10^{-9}
1	2.32325×10^{-7}	1.95001×10^{-6}	5.04347×10^{-7}	4.52410×10^{-8}
2	9.12947×10^{-6}	9.88580×10^{-6}	5.81507×10^{-6}	3.72029×10^{-7}

6 Conclusions

The main concern of this work has been to propose an efficient algorithm for the solutions of second-order periodic BVPs. The main goal has been achieved by introducing the RKHS method to solve this class of differential equations. We can conclude that the RKHS method is powerful and efficient technique in finding approximate solution for linear and nonlinear second-order periodic BVPs. In the proposed algorithm, the solution $u(x)$ and the approximate solution $u_n(x)$ are represented in the form of series in W_2^3. Moreover, the approximate solution and all its derivatives converge uniformly to the exact solution and all its derivatives up to order two, respectively. There is an important point to make here, the results obtained by the RKHS method are very effective and convenient in linear and nonlinear cases with less computational, iteration steps, work, and time. This confirms our belief that the efficiency of our technique gives it much wider applicability in the future for general classes of linear and nonlinear periodic problems.

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References

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover, New York, 1981.

[2] W. Soedel, Vibrations of Shells and Plates, Dekker, New York, 1993.

[3] T.Y. Na, Computational Methods in Engineering Boundary Value Problems, Academic Press, New York, 1979.

[4] E. Dulacska, Soil Settlement Effects on Buildings (Developments in Geotechnical Engineering), Elsevier Science Ltd., Amsterdam, 1992.

[5] F.M. Atici, G.S. Guseinov, On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions, Journal of Computational and Applied Mathematics 132 (2001) 341-356.

[6] F. Li, Z. Liang, Existence of positive periodic solutions to nonlinear second-order differential equations, Applied Mathematics Letters 18 (2005) 1256-1264.

[7] R.P. Agarwal, M.E. Filippakis, D. O'Regan, N.S. Papageorgiou, Degree theoretic methods in the study of nonlinear periodic problems with nonsmooth potentials, Differential and Integral Equations 19 (2006) 279-296.

[8] V. Seda, J.J. Nieto, M. Gera, Periodic boundary value problem for nonlinear higher ordinary differential equations, Applied Mathematics and Computation 48 (1992) 71-82.

[9] A. Bensoussan, J.L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[10] E.S. Palencia, Non-Homogeneous Media and Vibration Theory (Lecture Notes in Physics), Springer-Verlag, Berlin, 1980.

[11] S. Peng, Positive solutions for first order periodic boundary value problem, Applied Mathematics and Computation 158 (2004) 345-351.

[12] B. Liu, L. Liu, Y. Wuc, Existence of nontrivial periodic solutions for a nonlinear second order periodic boundary value problem, Nonlinear Analysis 72 (2010) 3337-3345.

[13] Q. Yao, Positive solutions of nonlinear second-order periodic boundary value problems, Applied Mathematics Letters 20 (2007) 583-590.

[14] H. Yu, M. Pei, Solvability of a nonlinear third-order periodic boundary value problem, Applied Mathematics Letters 23 (2010) 892-896.

[15] Y. Lia, Positive solutions of fourth-order periodic boundary value problems, Nonlinear Analysis 54 (2003) 1069-1078.

[16] O. Abu Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability, Journal of Advanced Research in Applied Mathematics 5 (2013) 31-52.

[17] O. Abu Arqub, A. El-Ajou, S. Momani, N. Shawagfeh, Analytical solutions of fuzzy initial value problems by HAM, Applied Mathematics and Information Sciences, in press.

[18] O. Abu Arqub, A. El-Ajou, Solution of the fractional epidemic model by homotopy analysis method, Journal of King Saud University (Science) 25 (2013) 73-81.

[19] A. El-Ajou, O .Abu Arqup, Solving fractional two-point boundary value problems using continuous analytic method, Ain Shams Engineering Journal, in press.

[20] A. El-Ajou, O. Abu Arqub, S. Momani, Homotopy analysis method for second-order boundary value problems of integro-differential equations, Discrete Dynamics in Nature and Society, vol. 2012, Article ID 365792, 18 pages, 2012. doi:10.1155/2012/365792.

[21] O. Abu Arqub, Z. Abo-Hammour, S. Momani, Application of continuous genetic algorithm for nonlinear system of second-order boundary value problems, Applied Mathematics and Information Sciences. In press.

[22] O. Abu Arqub, Z. Abo-Hammour, S. Momani, N. Shawagfeh, Solving singular two-point boundary value problems using continuous genetic algorithm, Abstract and Applied Analysis, vol. 2012, Article ID 205391, 25 page, 2012, doi.10.1155/2012/205391.

[23] G. Adomian, R. Rach, N. Shawagfeh, On the analytic solution of Lane-Emden equation, Foundations of Physics Letters 8 (1995) 161-181.

[24] N. Shawagfeh, Nonperturbative approximate solution for Lane-Emden equation, Journal of Mathematical Physics 34 (1993) 4364-4369.

[25] A. Berlinet, C.T. Agnan, Reproducing Kernel Hilbert Space in Probability and Statistics, Kluwer Academic Publishers, 2004.

[26] M. Cui, Y. Lin, Nonlinear Numercial Analysis in the Reproducing Kernel Space, Nova Science Publisher, New York, 2008.

[27] A. Daniel, Reproducing Kernel Spaces and Applications, Springer, 2003.

[28] C. Li, M. Cui, The exact solution for solving a class nonlinear operator equations in the reproducing kernel space, Applied Mathematics and Computation 143 (2003) 393-399.

[29] F. Geng, M. Cui, Solving a nonlinear system of second order boundary value problems, Journal of Mathematical Analysis and Applications 327 (2007) 1167-1181.

[30] L.H. Yang, Y. Lin, Reproducing kernel methods for solving linear initial-boundary-value problems, Electronic Journal of Differential Equations 2008 (2008) 1-11.

[31] F. Geng, Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method, Applied Mathematics and Computation 215 (2009) 2095-2102.

[32] W. Wang, M. Cui, B. Han, A new method for solving a class of singular two-point boundary value problems, Applied Mathematics and Computation 206 (2008) 721-727.

[33] Y. Lin, M. Cui, L. Yang, Representation of the exact solution for a kind of nonlinear partial differential equations, Applied Mathematics Letters 19 (2006) 808-813.

[34] M. Cui, H. Du, Representation of exact solution for the nonlinear Volterra-Fredholm integral equations, Applied Mathematics and Computation 182 (2006) 1795-1802.

[35] M. Al-Smadi, O. Abu Arqub, N. Shawagfeh, Approximate solution of BVPs for 4th-order IDEs by using RKHS method, Applied Mathematical Sciences 6 (2012) 2453-2464.

[36] M. Al-Smadi, O. Abu Arqub, S. Momani, A computational method for two-point boundary value problems offourth-order mixed integro-differential equations, Mathematical Problems in Engineering, In press

[37] O. Abu Arqub, M. Al-Smadi, S. Momani, Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations, Abstract and Applied Analysis, vol. 2012, Article ID 839836, 16 pages, 2012. doi:10.1155/2012/839836.

GENERALIZED CHEBYSHEV INEQUALITIES WITH APPLICATIONS

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Abstract

In this paper, we generalize the Pecaric work on Montgomery's identity via an arbitrary weight function, which no longer needs to be a probability density function, and apply it to derive some generalized Chebyshev type inequalities for any absolutely continuous function.

Key words: Chebyshev inequality, Absolutely continuous functions, Peano kernel, Montgomery identity, Weight function, Optimal constants
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Introduction

Let $L_p[a,b]$ $(1 \leq p \leq \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

and $L_\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)| \, .$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$ and the positive function $w : [a,b] \to \mathbb{R}^+$ such that $wf, wg, wfg \in L_1[a,b]$, the weighted Chebyshev functional [10] is defined by

$$T(w,f,g) = \int_a^b w(t)f(t)g(t)dt - \left(\int_a^b w(t)f(t)dt \right) \left(\int_a^b w(t)g(t)dt \right). \tag{1.1}$$

If $w(t)$ is uniformly distributed on $[a,b]$ then (1.1) is reduced to the usual Chebyshev functional

$$T(f,g) = \frac{1}{b-a} \int_a^b f(t)g(t)dt - \left(\frac{1}{b-a} \int_a^b f(t)dt \right) \left(\frac{1}{b-a} \int_a^b g(t)dt \right). \tag{1.2}$$

To date, extensive research has been done on the bounds of Chebyshev functional, see e.g. [1, 3]. The first work dates back to 1882, when Chebyshev [2] proved that if $f', g' \in L_\infty[a,b]$ then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \, . \tag{1.3}$$

Later on, in 1934 Grüss [4] showed that

$$|T(f,g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{1.4}$$

where m_1, m_2, M_1 and M_2 and are real numbers satisfying the conditions

$$m_1 \leq f(t) \leq M_1 \text{ and } m_2 \leq g(t) \leq M_2 \text{ for all } t \in [a,b]. \tag{1.5}$$

The optimal constant $\frac{1}{4}$ is the best possible number in (1.4) in the sense that it cannot be replaced by a smaller quantity.

A mixture type of inequalities (1.3) and (1.4) was introduced in [10] as

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M_1 - m_1) \|g'\|_\infty, \tag{1.6}$$

in which f is a Lebesgue integrable function satisfying (1.5) and g is absolutely continuous so that $g' \in L_\infty[a,b]$. The optimal constant $\frac{1}{8}$ is also the best possible number in (1.6).

Probably the most recent work about appropriate bounds of the usual Chebyshev functional is due to Niezgoda [8]: Let $f, \alpha, \beta \in L_p[a,b]$ and $g \in L_q[a,b]$ $\left(\frac{1}{p}+\frac{1}{q}=1,\ 1 \leq p \leq \infty\right)$ be functions such that $\alpha(t)+\beta(t)$ is a constant function and $\alpha(t) \leq f(t) \leq \beta(t)$ for all $t \in [a,b]$. Then we have

$$|T(f,g)| \leq \frac{1}{2(b-a)} \|\beta - \alpha\|_p \left\| g - \frac{1}{b-a}\int_a^b g(t)dt \right\|_q. \tag{1.7}$$

For $p = 2 = q$, (1.7) leads to the well-known inequality [7]

$$|T(f,g)| \leq \frac{1}{2}(M_1 - m_1)\sqrt{T(g,g)} \tag{1.8}$$

such that $m_1 \leq f(t) \leq M_1$ for all $t \in [a,b]$.

See [1, 3, 5-6] for further works on Chebyshev functional.

In this paper, we first generalize the Pecaric work on Montgomery's identity via an arbitrary weight function, which no longer needs to be a probability density function and then apply it to derive some generalized Chebyshev type inequalities for any absolutely continuous functions.

1 Main Results

Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$, then the Montgomery type identity [7] reads as:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b K(x,t)\,f'(t)\,dt, \tag{2.1}$$

where $K(x,t)$ is the Peano kernel defined by:

$$K(x,t)=\begin{cases}\frac{t-a}{b-a}, a\leq t\leq x\\ \frac{t-b}{b-a}, x\leq t\leq b.\end{cases}$$

The weighted version of identity (2.1) given by Pečarić in [10] is in the form:

$$f(x)=\int_a^b r(t)\,f(t)\,dt+\int_a^b K_w(x,t)\,f'(t)\,dt, \tag{2.2}$$

where $r(t)$ is a probability density function and the weighted Peano kernel is defined by:

$$K_w(x,t)=\begin{cases}\int_a^t r(s)\,ds\ , a\leq t\leq x\\ \int_a^t r(s)\,ds-1,\ x\leq t\leq b.\end{cases}$$

We now introduce a further generalization of (2.2) by considering the positive function $w:[a,b]\to[0,+\infty)$ which is not necessarily a probability density function, integrable and

$$\int_a^b w(s)\,ds<\infty.$$

The domain of w may be finite or infinite. If $m(a,b)=\int_a^b w(s)\,ds$ as total area of w such that $m(a,x)=0$ for any $x<a$, then a generalized type of weighted Peano kernel can be defined as follows:

$$K_{w,\theta}(x,t)=\begin{cases}\int_{(1-\theta)a+\theta b}^t w(s)\,ds,\ a\leq t\leq x\\ \int_{(1-\theta)a+\theta b}^t w(s)\,ds-\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds,\ x\leq t\leq b,\end{cases} \tag{2.3}$$

where $\theta\in[0,1)$; $\theta\neq\frac{1}{2}$, and $x\in[(1-\theta)a+\theta b,\theta a+(1-\theta)b]$.

To simplify the details of presentation, let us define

$$
\begin{aligned}
&\tilde{T}(w,f,g,\theta) \qquad (2.4)\\
&= \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \times \\
&\quad \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \\
&\quad + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b)}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\
&\quad \left(1 - \frac{m(a,b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,dx}\right) \int_a^b w(x)\,g(x)\,dx \\
&\quad + \frac{\left(\left(\int_a^{(1-\theta)a+\theta b} w(s)\,dx\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) g(b)\right)}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\
&\quad \left(1 - \frac{m(a,b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\right) \int_a^b w(x)\,f(x)\,dx \\
&\quad + \frac{1}{m(a,b)} \int_a^b w(x)\,f(x)\,g(x)\,dx \\
&\quad - \frac{2}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\
&\quad \left(\int_a^b w(x)\,f(x)\,dx\right)\left(\int_a^b w(x)\,g(x)\,dx\right) \\
&\quad + \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2} \times \\
&\quad \left(\int_a^b w(x)\,f(x)\,dx\right)\left(\int_a^b w(x)\,g(x)\,dx\right),
\end{aligned}
$$

$$
\begin{aligned}
S_{w,f,g,\theta}(x) = {}& f(x)g(x) \qquad (2.5)\\
&+ \frac{1}{2}\left(\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} f(x)\right. \\
&\left. + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} g(x)\right) \\
&- \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)}\left(f(x)\int_a^b w(x)g(x)dx + g(x)\int_a^b w(x)f(x)dx\right),
\end{aligned}
$$

and

$$\begin{aligned}\tilde{T}_{w,f,g,\theta} = & \frac{1}{m(a,b)} \int_a^b w(x) f(x) g(x)\, dx \qquad (2.6)\\ & + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\, ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^b w(s)\, ds\right) f(b)}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\, ds\right) m(a,b)} \int_a^b w(x) g(x)\, dx \\ & - \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\, ds\right) m(a,b)} \left(\int_a^b w(x) f(x)\, dx\right)\left(\int_a^b w(x) g(x)\, dx\right).\end{aligned}$$

It is easy to note that

$$\frac{1}{m(a,b)} \int_a^b w(x) S_{w,f,g,\theta}(x) dx = T_{w,f,g,\theta}, \qquad (2.7)$$

where

$$\begin{aligned}T_{w,f,g,\theta} = & \frac{1}{m(a,b)} \int_a^b w(x) f(x) g(x)\, dx \qquad (2.8)\\ & + \frac{1}{2m(a,b)} \left(\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\, ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^b w(s)\, ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\, ds} \times \right. \\ & \int_a^b w(x) g(x)\, dx \\ & \left. + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\, ds\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^b w(s)\, ds\right) g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\, ds} \int_a^b w(x) f(x)\, dx \right) \\ & - \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\, ds\right) m(a,b)} \left(\int_a^b w(x) f(x)\, dx\right)\left(\int_a^b w(x) g(x)\, dx\right).\end{aligned}$$

It is worth to mention that for $\theta = 0$, (2.4), (2.6) and (2.8) reduce to the Chebyshev functional (1.2).

We have given the generalized weighted Montgomery's identity in the following result:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, then*

$$\begin{aligned} f(x) = &-\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \\ &+\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \int_a^b w(t)\, f(t)\, dt \\ &+\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \int_a^b K_{w,\theta}(x,t)\, f'(t)\, dt, \end{aligned}$$

for all $x \in [a,b]$, *where* $K_{w,\theta}(x,t)$ *is defined in (2.3).*

PROOF. Consider the kernel defined in (2.3), we have

$$\begin{aligned} \int_a^b K_{w,\theta}(x,t)\, f'(t)\, dt = &\int_a^x \left(\int_{(1-\theta)a+\theta b}^{t} w(s)\,ds\right) f'(t)\, dt \\ &+\int_x^b \left(\int_{(1-\theta)a+\theta b}^{t} w(s)\,ds - \int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) f'(t)\, dt \\ = &\int_a^b \left(\int_{(1-\theta)a+\theta b}^{t} w(s)\,ds\right) f'(t)\, dt - \left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) \int_x^b f'(t)\, dt. \end{aligned}$$

Integrating by parts and simplifying we obtain

$$\begin{aligned} \int_a^b K_{w,\theta}(x,t)\, f'(t)\, dt = &\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) \\ &+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b) + \left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) f(x) \\ &-\int_a^b w(t)\, f(t)\, dt. \end{aligned}$$

Hence proved.

2 Applications of generalized Montgomery identity

We now give a generalization of the Chebyshev inequality in the following result:

Theorem 2 *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions on* $[a,b]$. *Also let the function* w *satisfies the conditions given in Theorem 1. Suppose*

$$f', g' \in L_\infty [a,b],$$

then

$$\left|\tilde{T}(w,f,g,\theta)\right| \leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2 m(a,b)} \|f'\|_\infty \|g'\|_\infty \int_a^b w(x)\,\eta^2_{\theta,a,b}(x)dx, \tag{3.1}$$

for all $x \in [a,b]$, *where*

$$\begin{aligned}\eta_{\theta,a,b}(x) = & -a\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) + x\left(\int_{(1-\theta)a+\theta b}^{x} w(s)\,ds\right) \\ & -x\left(\int_x^{\theta a+(1-\theta)b} w(s)\,ds\right) + b\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) \\ & +\left(\int_a^{(1-\theta)a+\theta b} sw(s)\,ds\right) - \left(\int_{(1-\theta)a+\theta b}^{x} sw(s)\,ds\right) \\ & +\left(\int_x^{\theta a+(1-\theta)b} sw(s)\,ds\right) - \left(\int_{\theta a+(1-\theta)b}^{b} sw(s)\,ds\right).\end{aligned} \tag{3.2}$$

PROOF. Since the functions f and g are absolutely continuous, we have

$$\begin{aligned} & f(x) + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \\ & -\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b w(t)\,f(t)\,dt \\ = & \frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b K_{w,\theta}(x,t)\,f'(t)\,dt,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned} & g(x) + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \\ & -\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b w(t)\,g(t)\,dt \\ = & \frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b K_{w,\theta}(x,t)\,g'(t)\,dt.\end{aligned}$$

Using (3.3) and (3.4) we have

$$
\begin{aligned}
&\frac{1}{\left[\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right]^2}\left(\int_a^b K_{w,\theta}(x,t)\,f'(t)\,dt\right)\left(\int_a^b K_{w,\theta}(x,t)\,g'(t)\,dt\right)\\
&=\Bigg(f(x)+\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)f(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\\
&\quad-\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b w(t)\,f(t)\,dt\Bigg)\times\\
&\quad\Bigg(g(x)+\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)g(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\\
&\quad-\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\int_a^b w(t)\,g(t)\,dt\Bigg)\\
&=f(x)\,g(x)+\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)g(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}f(x)\\
&\quad-\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}f(x)\int_a^b w(t)\,g(t)\,dt\\
&\quad+\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)f(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}g(x)\\
&\quad+\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)f(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\times\\
&\quad\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)g(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}\\
&\quad-\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)f(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)f(b)}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2}\int_a^b w(t)\,g(t)\,dt\\
&\quad-\frac{1}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds}g(x)\int_a^b w(t)\,f(t)\,dt\\
&\quad-\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right)g(a)+\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right)g(b)}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2}\int_a^b w(t)\,f(t)\,dt\\
&\quad+\frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2}\left(\int_a^b w(t)\,f(t)\,dt\right)\left(\int_a^b w(t)\,g(t)\,dt\right).
\end{aligned}
$$

Now first multiplying both sides by $\frac{w(x)}{m(a,b)}$ and then integrating both sides with respect to x over the interval $[a,b]$ and simplifying by using (2.4), we get

$$\tilde{T}(w,f,g,\theta) = \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2 m(a,b)} \times \int_a^b w(x)\left(\int_a^b K_{w,\theta}(x,t)\, f'(t)\,dt\right)\left(\int_a^b K_{w,\theta}(x,t)\, g'(t)\,dt\right)dx,$$

which implies that

$$\left|\tilde{T}(w,f,g,\theta)\right| \leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)^2 m(a,b)} \|f'\|_\infty \|g'\|_\infty \int_a^b w(x)\left(\int_a^b |K_{w,\theta}(x,t)|\,dt\right)^2 dx. \tag{3.5}$$

It can be easily seen that

$$\begin{aligned}
\int_a^b |K_{w,\theta}(x,t)|\,dt &= \int_a^x |K_{w,\theta}(x,t)|\,dt + \int_x^b |K_{w,\theta}(x,t)|\,dt \\
&= \int_a^{(1-\theta)a+\theta b}\left(\int_t^{(1-\theta)a+\theta b} w(s)\,ds\right)dt \\
&\quad + \int_{(1-\theta)a+\theta b}^{x}\left(\int_{(1-\theta)a+\theta b}^{t} w(s)\,ds\right)dt \\
&\quad + \int_x^{\theta a+(1-\theta)b}\left(\int_t^{\theta a+(1-\theta)b} w(s)\,ds\right)dt \\
&\quad + \int_{\theta a+(1-\theta)b}^{b}\left(\int_{\theta a+(1-\theta)b}^{t} w(s)\,ds\right)dt \\
&= -a\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) + x\left(\int_{(1-\theta)a+\theta b}^{x} w(s)\,ds\right) \\
&\quad - x\left(\int_x^{\theta a+(1-\theta)b} w(s)\,ds\right) + b\left(\int_{\theta a+(1-\theta)b}^{b} w(s)\,ds\right) \\
&\quad + \left(\int_a^{(1-\theta)a+\theta b} s w(s)\,ds\right) - \left(\int_{(1-\theta)a+\theta b}^{x} s w(s)\,ds\right) \\
&\quad + \left(\int_x^{\theta a+(1-\theta)b} s w(s)\,ds\right) - \left(\int_{\theta a+(1-\theta)b}^{b} s w(s)\,ds\right),
\end{aligned}$$

so that (3.5) turns in (3.1). Hence proved.

Remark 3 *If in (2.10) and (3.1), $\theta = 0$ and w is the probability density function, then we recapture the results obtained in [10].*

We now give another generalization of the Chebyshev inequality in the following result:

Theorem 4 *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$.*

Also let the function w satisfies the conditions given in Theorem 1. Suppose

$$f', g' \in L_\infty [a, b],$$

then

$$|S_{w,f,g,\theta}(x)| \leq \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)} \left(|f(x)| \, \|g'\|_\infty + |g(x)| \, \|f'\|_\infty\right) \eta_{\theta,a,b}(x), \tag{3.6}$$

and

$$\begin{aligned} |T_{w,f,g,\theta}| \leq & \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \left(\|f\|_\infty \|g'\|_\infty + \|g\|_\infty \|f'\|_\infty\right) \times \\ & \int_a^b w(x)\, \eta_{\theta,a,b}(x) dx \\ \leq & \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \left(\|f\|_\infty \|g'\|_\infty + \|g\|_\infty \|f'\|_\infty\right) \times \\ & \begin{cases} \|w\|_\infty \left\|\eta_{\theta,a,b}\right\|_1 \ \textit{if } w \in L_\infty[a,b] \textit{ and } \eta_{\theta,a,b} \in L_1[a,b], \\ \|w\|_p \left\|\eta_{\theta,a,b}\right\|_q \ \textit{if } w \in L_p[a,b] \textit{ and } \eta_{\theta,a,b} \in L_q[a,b], \textit{ where } p>1, \ \frac{1}{p}+\frac{1}{q}=1, \\ \|w\|_1 \left\|\eta_{\theta,a,b}\right\|_\infty \ \textit{if } w \in L_1[a,b] \textit{ and } \eta_{\theta,a,b} \in L_\infty[a,b], \end{cases} \end{aligned} \tag{3.7}$$

for all $x \in [a, b]$, where $\eta_{\theta,a,b}(x) = \int_a^b |K_{w,\theta}(x,t)|\, dt$ is given by (3.2).

PROOF. Since the functions f and g are absolutely continuous, multiplying both sides of (3.3) and (3.4) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting, we have

$$\begin{aligned} & f(x)g(x) + \\ & \frac{1}{2}\left(\frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) g(a) + \left(\int_{\theta a+(1-\theta)b}^b w(s)\,ds\right) g(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} f(x) \right. \\ & \left. + \frac{\left(\int_a^{(1-\theta)a+\theta b} w(s)\,ds\right) f(a) + \left(\int_{\theta a+(1-\theta)b}^b w(s)\,ds\right) f(b)}{\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} g(x) \right) \\ & - \frac{1}{2\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \left(f(x) \int_a^b w(t)g(t)dt + g(x) \int_a^b w(t)f(t)dt \right) \\ = & \frac{1}{2\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \left(f(x) \int_a^b K_{w,\theta}(x,t)\, g'(t)\, dt + g(x) \int_a^b K_{w,\theta}(x,t)\, f'(t)\, dt \right), \end{aligned}$$

implies

$$S_{w,f,g,\theta}(x) = \frac{1}{2\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds} \times \left(f(x)\int_a^b K_{w,\theta}(x,t)\,g'(t)\,dt + g(x)\int_a^b K_{w,\theta}(x,t)\,f'(t)\,dt \right), \tag{3.8}$$

gives us

$$\begin{aligned} |S_{w,f,g,\theta}(x)| &\le \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)} \times \\ &\quad \left(|f(x)|\int_a^b |K_{w,\theta}(x,t)|\,|g'(t)|\,dt + |g(x)|\int_a^b |K_{w,\theta}(x,t)|\,|f'(t)|\,dt \right) \\ &\le \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right)} \left(|f(x)|\,\|g'\|_\infty + |g(x)|\,\|f'\|_\infty\right)\int_a^b |K_{w,\theta}(x,t)|\,dt, \end{aligned}$$

for all $x \in [a,b]$.

This completes the proof of (3.6).

Now multiplying both sides of (3.8) by $\frac{w(x)}{m(a,b)}$ and then integrating with respect to x from a to b and rewriting, we get

$$\begin{aligned} T_{w,f,g,\theta} &= \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\ &\quad \int_a^b w(x)\left(\int_a^b K_{w,\theta}(x,t)\left(f(x)g'(t) + g(x)f'(t)\right)dt\right)dx, \end{aligned}$$

implies

$$\begin{aligned} |T_{w,f,g,\theta}| &\le \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\ &\quad \int_a^b w(x)\left(\int_a^b |K_{w,\theta}(x,t)|\left(|f(x)|\,|g'(t)| + |g(x)|\,|f'(t)|\right)dt\right)dx \\ &\le \frac{1}{2\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \times \\ &\quad \left(\|f\|_\infty\,\|g'\|_\infty + \|g\|_\infty\,\|f'\|_\infty\right)\int_a^b w(x)\left(\int_a^b |K_{w,\theta}(x,t)|\,dt\right)dx. \end{aligned}$$

Now (3.7) can be easily derived form the last inequality.

Another variant of Chebyshev inequality is given in the form of the following result:

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$. Also let the function w satisfies the conditions given in Theorem 1. Suppose*

$$f', g \in L_\infty[a,b],$$

then

$$\begin{aligned} \left|\tilde{T}_{w,f,g,\theta}\right| &\leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \|f'\|_\infty \|g\|_\infty \int_a^b w(x)\eta_{\theta,a,b}(x)dx \qquad (3.9) \\ &\leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \|f'\|_\infty \|g\|_\infty \times \\ &\begin{cases} \|w\|_\infty \left\|\eta_{\theta,a,b}\right\|_1 \ \text{if } w \in L_\infty[a,b] \text{ and } \eta_{\theta,a,b} \in L_1[a,b], \\ \|w\|_p \left\|\eta_{\theta,a,b}\right\|_q \ \text{if } w \in L_p[a,b] \text{ and } \eta_{\theta,a,b} \in L_q[a,b], \text{ where } p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \|w\|_1 \left\|\eta_{\theta,a,b}\right\|_\infty \ \text{if } w \in L_1[a,b] \text{ and } \eta_{\theta,a,b} \in L_\infty[a,b], \end{cases} \end{aligned}$$

where $\eta_{\theta,a,b}$ is as defined in (3.2).

PROOF. Since the function f is absolutely continuous, multiplying both sides of (3.3) by $\frac{w(x)g(x)}{m(a,b)}$ and then integrating with respect to x from a to b and rewriting, we obtain

$$\tilde{T}_{w,f,g,\theta} = \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \int_a^b w(x)g(x)\left(\int_a^b K_{w,\theta}(x,t)\, f'(t)dt\right)dx,$$

which implies, by taking the modulus on both sides,

$$\begin{aligned} &\left|\tilde{T}_{w,f,g,\theta}\right| \\ &\leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \int_a^b w(x)\,|g(x)| \left(\int_a^b |K_{w,\theta}(x,t)|\,|f'(t)|\,dt\right)dx \\ &\leq \frac{1}{\left(\int_{(1-\theta)a+\theta b}^{\theta a+(1-\theta)b} w(s)\,ds\right) m(a,b)} \|f'\|_\infty \|g\|_\infty \int_a^b w(x)\left(\int_a^b |K_{w,\theta}(x,t)|\,dt\right)dx. \end{aligned}$$

Hence proved.

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References

(1) P. Cerone, S.S. Dragomir, New bounds for the Chebyshev functional, Appl. Math. Lett. 18 (2005) 603-611.
(2) P. L.Chebyshev, Sur les expressions approximative des integrals par les auters prises entre les mêmes limites, Proc. Math. Soc. Charkov, 2 (1882) 93-98.
(3) S. S. Dragomir, Bounds for some perturbed Chebyshev functionals, J. Inequal. Pure Appl. Math. 9 (3) (2008) Art. 64.
(4) G. Grüss, Uber das Maximum des absoluten Betrages von , Math. Z., 39 (1935) 215-226.
(5) M. Masjed-Jamei, A Main inequality for several special functions, Comput. Math. Appl., 60 (2010) 1280-1289.
(6) M. Masjed-Jamei, Feng Qi, H.M Srivastava, Generalizations of some classical inequalities via a special functional property, Integral Transforms Spec. Funct, 21 (2010) 327-336.
(7) D. S. Mitrinovic, J.E. Pecaric and A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 1993.
(8) M. Niezgoda, A new inequality of Ostrowski-Gruss type and applications to some numerical quadrature rules, Comput. Math. Appl, 58 (2009) 589-596.
(9) A.M. Ostrowski, On an integral inequality, Aequat. Math. 4 (1970) 358-373.
(10) B. G. PACHPATTE, On Chebyshev-Grüss type inequalities via Pečarić's extention of the Montgomery identity, *J. Inequal. Pure and Appl. Math.*, 7(1) (2006), Art 108.

Comment on "Generalized ternary bi-derivations on ternary Banach algebras: a fixed point approach" [Esgahi Gordji et al., J. Comput. Anal. Appl. 15 (2013) 45-54]

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Abstract. Eshaghi Gordji et al. [10] proved the Hyers-Ulam stability of generalized ternary bi-derivations on ternary Banach algebras.

It is easy to show that the resulting generalized ternary bi-derivations of [10] are meaningless, since the definition of generalized ternary bi-derivation, given in [10], is meaningless.

In this paper, we correct the definition of generalized ternary bi-derivation, and correct the statements of the results and prove the corrected results.

1. Introduction and preliminaries

A C^*-ternary algebra is a complex Banach space $\mathcal{A}$, equipped with a ternary product $(x, y, z) \rightarrowtail [xyz]$ of $\mathcal{A}^3$ into $\mathcal{A}$, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[xy[zvw]] = [x[wzy]v] = [[xyz]wv]$, and satisfies $\|[xyz]\| \leq \|x\|\|y\|\|z\|$ and $\|[xxx]\| = \|x\|^3$ (see [3]).

Definition 1.1. ([2]) Let $\mathcal{A}$ be a C^*-ternary algebra. A $\mathbb{C}$-bilinear mapping $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *ternary bi-derivation* if it satisfies

$$\delta([abc], d) = [\delta(a, d)bc] + [a\delta(b, d)c] + [ab\delta(c, d)],$$
$$\delta(a, [bcd]) = [\delta(a, b)cd] + [b\delta(a, c)d] + [bc\delta(a, d)]$$

for all $a, b, c, d \in \mathcal{A}$.

Note that the d-variable in the left side of the first equality and the a-variable in the left side of the second equality are $\mathbb{C}$-linear. But the d-variable in the right side of the first equality and the a-variable in the right side of the second equality are not $\mathbb{C}$-linear. So we correct the definition of ternary bi-derivation as follows.

Definition 1.2. Let $\mathcal{A}$ be a C^*-ternary algebra. A $\mathbb{C}$-bilinear mapping $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *ternary bi-derivation* if it satisfies

$$\delta([abc], d) = [\delta(a, d)bc] + [a\delta(b, d^*)c] + [ab\delta(c, d)],$$
$$\delta(a, [bcd]) = [\delta(a, b)cd] + [b\delta(a^*, c)d] + [bc\delta(a, d)]$$

for all $a, b, c, d \in \mathcal{A}$.

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Definition 1.3. ([10]) Let $\mathcal{A}$ be a C^*-ternary algebra. A $\mathbb{C}$-bilinear mapping $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *generalized ternary bi-derivation* if there exists a bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$D([abc], d) = [D(a, d)bc] + [a\delta(b, d)c] + [ab\delta(c, d)],$$
$$D(a, [bcd]) = [D(a, b)cd] + [b\delta(a, c)d] + [bc\delta(a, d)]$$

for all $a, b, c, d \in \mathcal{A}$.

Note that the d-variable in the left side of the first equality and the a-variable in the left side of the second equality are $\mathbb{C}$-linear. But the d-variable in the right side of the first equality and the a-variable in the right side of the second equality are not $\mathbb{C}$-linear. So we correct the definition of generalized ternary bi-derivation as follows.

Definition 1.4. Let $\mathcal{A}$ be a C^*-ternary algebra. A $\mathbb{C}$-bilinear mapping $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *generalized ternary bi-derivation* if there exists a bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$D([abc], d) = [D(a, d)bc] + [a\delta(b, d^*)c] + [ab\delta(c, d)],$$
$$D(a, [bcd]) = [D(a, b)cd] + [b\delta(a^*, c)d] + [bc\delta(a, d)]$$

for all $a, b, c, d \in \mathcal{A}$.

The stability problem of functional equations originated from a question of Ulam [14] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. Găvruta [11] obtained the generalized result of Rassias's theorem which allows the Cauchy difference to be controlled by a general unbounded function. The stability problems of various functional equations have been extensively investigated by a number of authors (see [5, 6, 7, 8, 9]).

We recall a fundamental result in fixed point theory.

Theorem 1.5. ([4]) *Suppose that a complete generalized metric space $(\mathcal{X}, d)$ and a strictly contractive mapping $J : \mathcal{X} \to \mathcal{X}$ with Lipschits constant $0 < L < 1$ are given. Then, for a given element $x \in \mathcal{X}$, exactly one of the following assertions is true:*
either
(1) *$d(J^n x, J^{n+1}x) = \infty$ for all $n \geq 0$ or*
(2) *there exists n_0 such that $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$.*
Actually, if (2) holds, then the sequence $J^n x$ is convergent to a fixed point x^* of J and
(3) *x^* is the unique fixed point of J in $\Lambda := \{y \in \mathcal{X}, d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, x^*) \leq \frac{d(y,Jy)}{1-L}$ for all $y \in \Lambda$.*

In this paper, we correct the statements of the results and prove the corrected results.

2. Main results

From now on, we assume that $\mathcal{A}$ is a C^*-ternary algebra.

For a given mapping $f : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, we define the difference operator $E_{\lambda,\mu}f : \mathcal{A}^4 \to \mathcal{A}$ by

$$E_{\lambda,\mu}f(a, b, c, d) = f(\lambda a - \lambda b, \mu c) + f(\lambda a, \mu c - \mu d) - \lambda\mu(2f(a, c) - f(b, c) - f(a, d))$$

for all $\lambda, \mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $a, b, c, d \in \mathcal{A}$.

We prove the Hyers-Ulam stability of generalized ternary bi-derivations.

Theorem 2.1. *Let $f, g : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be uniformly continuous mappings such that $g(0, 0) = f(0, 0) = 0$. Let $\varphi : \mathcal{A}^4 \to [0, \infty)$ be a function such that*

$$max\{\|E_{\lambda,\mu}f(a, b, c, d)\|, \|E_{\lambda,\mu}g(a, b, c, d)\|\} \leq \varphi(a, b, c, d), \tag{2.1}$$

$$\begin{aligned} &max\{\|f([abc], d) - [f(a, d)bc] - [af(b, d^*)c] - [abf(c, d)]\|, \\ &\|f(a, [bcd]) - [f(a, b)cd] - [bf(a^*, c)d] - [bcf(a, d)]\|\} \leq \varphi(a, b, c, d), \end{aligned} \tag{2.2}$$

$$\begin{aligned}&max\{\|g([abc],d)-[g(a,d)bc]-[af(b,d^*)c]-[abf(c,d)]\|,\\ &\|g(a,[bcd])-[g(a,b)cd]-[bf(a^*,c)d]-[bcf(a,d)]\|\}\le\varphi(a,b,c,d),\end{aligned} \tag{2.3}$$

$$\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^na,2^nb,2^nc,2^nd)=0$$

for all $\lambda,\mu\in\mathbb{T}^1$ *and all* $a,b,c,d\in\mathcal{A}$*. If there exists an* $L<1$ *such that* $\Psi(a,b)\le 4L\Psi(\frac{a}{2},\frac{b}{2})$ *for all* $a,b\in\mathcal{A}$*, where*

$$\begin{aligned}\Psi(a,b):=\varphi(0,a,2b,0)+\varphi(a,-a,2b,b)+\varphi(0,0,2b,0)+3(\varphi(a,0,b,-b)\\ +\varphi(a,0,0,b)+\varphi(a,0,0,0)+\varphi(0,0,b,0)),\end{aligned}$$

then there exist a unique ternary bi-derivation $\delta:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ *and a unique generalized ternary bi-derivation* $D:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ *(related to* δ*) such that*

$$max\{\|g(a,c)-D(a,c)\|,\|f(a,c)-\delta(a,c)\|\}\le\frac{L}{1-L}\Psi\left(\frac{a}{2},\frac{c}{2}\right) \tag{2.4}$$

for all $a,c\in\mathcal{A}$*.*

Proof. By the same reasoning as in the proof of [10, Theorem 2.1], there exist a unique $\mathbb{C}$-bilinear mapping $\delta:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and a unique $\mathbb{C}$-bilinear mapping $D:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ satisfying (2.4). The $\mathbb{C}$-bilinear mapping $\delta:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and the $\mathbb{C}$-bilinear mapping $D:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ are given by

$$\delta(a,d):=\lim_{n\to\infty}\frac{1}{4^n}f(2^na,2^nd),$$

$$D(a,d):=\lim_{n\to\infty}\frac{1}{4^n}g(2^na,2^nd)$$

for all $a,d\in\mathcal{A}$, respectively.

It is easy to show that

$$\delta(a,d)=\lim_{n\to\infty}\frac{1}{16^n}f\left(8^na,2^nd\right)=\lim_{n\to\infty}\frac{1}{16^n}f\left(2^na,8^nd\right),$$

$$D(a,d)=\lim_{n\to\infty}\frac{1}{16^n}g\left(8^na,2^nd\right)=\lim_{n\to\infty}\frac{1}{16^n}g\left(2^na,8^nd\right)$$

for all $a,d\in\mathcal{A}$, since δ,D are bi-additive and f,g are uniformly continuous.

It follows from (2.2) that

$$\begin{aligned}&\|\delta([abc],d)-[\delta(a,d)bc]-[a\delta(b,d^*)c]-[ab\delta(c,d)]\|\\ &=\lim_{n\to\infty}\frac{1}{16^n}(\|f([(2^na)(2^nb)(2^nc)],2^nd)-[f(2^na,2^nd)(2^nb)(2^nc)]\\ &-[(2^na)f(2^nb,2^nd)(2^nc)]-[(2^na)(2^nb)f(2^nc,2^nd^*)]\|)\\ &\le\lim_{n\to\infty}\frac{1}{16^n}\varphi(2^na,2^nb,2^nc,2^nd)\le\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^na,2^nb,2^nc,2^nd)=0\end{aligned}$$

for all $a,b,c,d\in\mathcal{A}$. This means that

$$\delta([abc],d)=[\delta(a,d)bc]+[a\delta(b,d^*)c]+[ab\delta(c,d)]$$

for all $a,b,c,d\in\mathcal{A}$. Similarly, we can show that

$$\delta(a,[bcd])=[\delta(a,b)cd]+[b\delta(a^*,c)d]+[bc\delta(a,d)]$$

for all $a,b,c,d\in\mathcal{A}$. Hence δ is a bi-derivation.

On the other hand, by (2.3), we have

$$\begin{aligned}&\|D([abc],d)-[D(a,d)bc]-[a\delta(b,d^*)c]-[ab\delta(c,d)]\|\\ &=\lim_{n\to\infty}\frac{1}{16^n}(\|g([(2^na)(2^nb)(2^nc)],2^nd)-[g(2^na,2^nd)(2^nb)(2^nc)]\\ &-[(2^na)f(2^nb,2^nd^*)(2^nc)]-[(2^na)(2^nb)f(2^nc,2^nd)]\|)\\ &\le\lim_{n\to\infty}\frac{1}{16^n}\varphi(2^na,2^nb,2^nc,2^nd)\le\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^na,2^nb,2^nc,2^nd)=0\end{aligned}$$

for all $a, b, c, d \in \mathcal{A}$. It follows that

$$D([abc], d) = [D(a, d)bc] + [a\delta(b, d^*)c] + [ab\delta(c, d)]$$

for all $a, b, c, d \in \mathcal{A}$. Similarly, we can show that

$$D(a, [bcd]) = [D(a, b)cd] + [b\delta(a^*, c)d] + [bc\delta(a, d)]$$

for all $a, b, c, d \in \mathcal{A}$. This means that D is a generalized bi-derivation related to δ. □

Corollary 2.2. *Let $p \in (0, 2)$ and $q \in (0, \infty)$ be real numbers. Suppose that $f, g : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are uniformly continuous mappings satisfying $g(0, 0) = f(0, 0) = 0$ and*

$$\begin{aligned}
\max\{&\|E_{\lambda,\mu} f(a, b, c, d)\|, \|E_{\lambda,\mu} g(a, b, c, d)\|, \\
&\|f([abc], d) - [f(a, d)bc] - [af(b, d^*)c] - [abf(c, d)]\|, \\
&\|f(a, [bcd]) - [f(a, b)cd] - [bf(a^*, c)d] - [bcf(a, d)]\|, \\
&\|g([abc], d) - [g(a, d)bc] - [af(b, d^*)c] - [abf(c, d)]\|, \\
&\|g(a, [bcd]) - [g(a, b)cd] - [bf(a^*, c)d] - [bcf(a, d)]\} \\
&\le q(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $a, b, c, d \in \mathcal{A}$. Then there exist a unique ternary bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unique generalized ternary bi-derivation $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (related to δ) such that

$$max\{\|g(a, c) - D(a, c)\|, \|f(a, c) - \delta(a, c)\|\} \le \frac{5q}{4 - 2^p}(\|a\|^p + \|c\|^p)$$

for all $a, c \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.1 by putting

$$\varphi(a, b, c, d) := q(\|a\|^p + \|b\|^p + \|c\|^p + \|c\|^p)$$

for all $a, b, c, d \in \mathcal{A}$ and $L = 2^{p-2}$. □

Theorem 2.3. *Let $f, g : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be uniformly continuous mappings satisfying $g(0, 0) = f(0, 0) = 0$. Let $\varphi : \mathcal{A}^4 \to [0, \infty)$ be a function satisfying (2.1), (2.2) and (2.3). Let*

$$\lim_{n\to\infty} 16^n \varphi(2^{-n}a, 2^{-n}b, 2^{-n}c, 2^{-n}d) = 0$$

for all $a, b, c, d \in \mathcal{A}$. If there exists an $L < 1$ such that $\Psi(a, b) \le \frac{L}{4}\Psi(2a, 2b)$ for all $a, b \in \mathcal{A}$, where $\Psi(a, b)$ is defined in Theorem 2.1, then there exist a unique ternary bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unique generalized ternary bi-derivation $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (related to δ) such that

$$max\{\|g(a, c) - D(a, c)\|, \|f(a, c) - \delta(a, c)\|\} \le \frac{L}{4 - 4L}\Psi(a, c)$$

for all $a, c \in \mathcal{A}$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $p \in (4, \infty)$ and $q \in (0, \infty)$ be real numbers. Suppose that $f, g : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are uniformly continuous mappings satisfying $g(0, 0) = f(0, 0) = 0$ and*

$$\begin{aligned}
\max\{&\|E_{\lambda,\mu} f(a, b, c, d)\|, \|E_{\lambda,\mu} g(a, b, c, d)\|, \\
&\|f([abc], d) - [f(a, d)bc] - [af(b, d^*)c] - [abf(c, d)]\|, \\
&\|f(a, [bcd]) - [f(a, b)cd] - [bf(a^*, c)d] - [bcf(a, d)]\|, \\
&\|g([abc], d) - [g(a, d)bc] - [af(b, d^*)c] - [abf(c, d)]\|, \\
&\|g(a, [bcd]) - [g(a, b)cd] - [bf(a^*, c)d] - [bcf(a, d)]\} \\
&\le q(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $a, b, c, d \in \mathcal{A}$. *Then there exist a unique ternary bi-derivation* $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *and a unique generalized ternary bi-derivation* $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *(related to* δ*) such that*

$$max\{\|g(a,c) - D(a,c)\|, \|f(a,c) - \delta(a,c)\|\} \leq \frac{5q}{2^p - 4}(\|a\|^p + \|c\|^p)$$

for all $a, c \in \mathcal{A}$.

Proof. It follows from Theorem 2.3 by taking

$$\varphi(a,b,c,d) := q(\|a\|^p + \|b\|^p + \|c\|^p + \|c\|^p)$$

for all $a, b, c, d \in \mathcal{A}$ and $L = 2^{2-p}$. □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.

[2] J. Bae and W. Park, *Approximate bi-homomorphisms and bi-derivations in* C^**-ternary algebras,* Bull. Korean Math. Soc. **47** (2010) 195–209.

[3] A. Cayley, *On the* 34 *concomitants of the ternary cubic*, Amer. J. Math. **4** (1881) 1–15.

[4] J.B. Diaz, B. Margolis, *A fixed point theorem of the alternative for the contractions on generaliuzed complete metric space,* Bull. Amer. Math. Soc. **74** (1968) 305–309.

[5] A. Ebadian, N. Ghobadipour and H. Baghban, *Stability of bi-*θ*-derivations on* JB^**-triples,* Int. J. Geom. Methods Mod. Phys. **9** (2012), No. 7, Art. ID 1250051, 12 pages.

[6] A. Ebadian, I. Nikoufar and M. Eshaghi Gordji, *Nearly* $(\theta_1, \theta_2, \theta_3, \phi)$*-derivations on* C^**-modules,* Int. J. Geom. Methods Mod. Phys. **9** (2012), No. 3, Art. ID 1250019, 12 pages.

[7] M. Eshaghi Gordji, A. Fazeli and C. Park, *3-Lie multipliers on Banach 3-Lie algebras,* Int. J. Geom. Methods Mod. Phys. **9** (2012), No. 7, Art. ID 1250052, 15 pages.

[8] M. Eshaghi Gordji, M.B. Ghaemi and B. Alizadeh, *A fixed point method for perturbation of higher ring derivationsin non-Archimedean Banach algebras,* Int. J. Geom. Methods Mod. Phys. **8** (2011) 1611–1625.

[9] M. Eshaghi Gordji and N. Ghobadipour, *Stability of* (α, β, γ)*-derivations on Lie* C^**-algebras,* Int. J. Geom. Methods Mod. Phys. **7** (2010) 1097–1102.

[10] M. Eshaghi Gordji, G.H. Kim, J.R. Lee and C. Park, *Generalized ternary bi-derivations on ternary Banach algebras: a fixed point approach,* J. Comput. Anal. Appl. **15** (2013) 45–54.

[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994) 431–436.

[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941) 222–224.

[13] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.

[14] S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science ed., Wiley, New York, 1940.

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EXAMPLES OF UMBRAL CALCULUS

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ABSTRACT. In this paper, we introduce some interesting Sheffer sequences of polynomials. From the properties of those sequences of polynomials, we derive some identities involving multiple power and alternating(power) sums.

1. INTRODUCTION

For $\alpha \in \mathbb{R}$, the Frobenious-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x \mid \lambda)\frac{t^n}{n!} \tag{1}$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. In the special case, $x=0$, $H_n^{(\alpha)}(0 \mid \lambda) = H_n^{(\alpha)}(\lambda)$ are called the $n-$th Frobenius-Euler numbers,(see [1-15]).

As is well knowen, the Bernoulli polynomials of order α are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{2}$$

(see[3,4,5,6]). In the special case, $x=0$, $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$ are called the $n-$th Bernoulli numbers of order α.

From (1) and (2), we have

$$H_n^{(\alpha)}(x \mid \lambda) = \sum_{l=0}^{n}\binom{n}{l} H_l^{(\alpha)}(\lambda)x^{n-l}, \quad B_n^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l} B_l^{(\alpha)}(\lambda)x^{n-l}. \tag{3}$$

Let

$$\mathcal{F} = \{f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C}\}. \tag{4}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the single variable x over $\mathbb{C}$ and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$.

The action of the linear functional on a polynomial $p(x)$ is denoted by $< L \mid p(x) >$. We remind that the vector space structure on $\mathbb{P}^*$ are defined by $< L+M \mid p(x) >=< L \mid p(x) > + < M \mid p(x) >$, $< cL \mid p(x) >= c < L \mid p(x) >$, where c is a complex constant (see[11,12,13,15]). For $f(t) \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$< f(t) \mid x^n >= a_n, \tag{5}$$

for all $n \geq 0$, (see[11,12,15]).

From (4) and (5), we note that

$$< t^k \mid x^n >= n!\delta_{n,k}, \tag{6}$$

for all $n, k \geq 0$, where $\delta_{n,k}$ is the Kronecker's symbol (see[13,15]).

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For $f_L(t) = \sum_{k=0}^{\infty} \frac{<L|x^k>}{k!} t^k$, we have $< f_L(t) \mid x^n >=< L \mid x^n >$. So the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra(see[12,15]). The order $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish(see[11,15]). If $O(f(t)) = 1$, then $f(t)$ is called a delta series. If $O(f(t)) = 0$, then $f(t)$ is called an invertible series. Let $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $< g(t)f(t)^k \mid S_n(x) >= n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see[13,15]).

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{< f(t) \mid x^k >}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{< t^k \mid p(x) >}{k!} x^k, \quad \text{(see[15])}. \tag{7}$$

By (7), we easily get

$$p^k(0) =< t^k \mid p(x) > ; \ < 1 \mid p^{(k)} >= p^{(k)}(0). \tag{8}$$

From (8), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad k \geq 0. \tag{9}$$

For $S_n(x) \sim (g(t), f(t))$, the generating function of Sheffer sequence $S_n(x)$ is given by

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{10}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, (see[12,15]).

Let $S_n(x) \sim (1, g(t))$ and $t_n(x) \sim (1, f(t))$. Then we have

$$S_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} t_n(x), \quad n \geq 0, \tag{11}$$

(see[11,15]). This equation (11) is important in deriving our main results in this paper.

The purpose of this paper is to introduce some interesting Sheffer sequences of polynomials and to investigate some properties of those sequences of polynomials.

2. Sheffer sequences and Applications

Let us consider the following Sheffer sequences

$$S_n(x) \sim \left(1, \frac{t}{1 - \lambda e^t}\right), \ t_n(x) \sim \left(1, \frac{t}{1 - \lambda^m e^{mt}}\right), \tag{12}$$

where $\lambda \in \mathbb{C}$ with $\lambda^m \neq 1$. From $x^n \sim (1,t)$, (11) and (12), we have

$$\begin{aligned}
S_n(x) &= x\left(\frac{t}{\left(\frac{t}{1-\lambda e^t}\right)}\right)^n x^{-1}x^n = x\left(1-\lambda e^t\right)^n x^{n-1} \\
&= x(1-\lambda-\lambda(e^t-1))^n x^{n-1} \\
&= x\sum_{l=0}^{n}\binom{n}{l}(1-\lambda)^{n-l}(-\lambda)^l(e^t-1)^l x^{n-1} \\
&= x\sum_{l=0}^{n-1}\binom{n}{l}(1-\lambda)^{n-l}(-\lambda)^l\sum_{m=0}^{n-l-1}\frac{l!}{(l+m)!}S_2(l+m,l)t^{l+m}x^{n-1} \\
&= x\sum_{l=0}^{n-1}\sum_{m=0}^{n-l-1}\binom{n}{l}\binom{n-1}{l+m}l!(1-\lambda)^{n-l}(-\lambda)^l S_2(l+m,l)x^{n-l-m-1} \\
&= \sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n-1}{r}\binom{n}{l}l!(1-\lambda)^{n-l}(-\lambda)^l S_2(n-1-r,l)x^{r+1}, \qquad (13)
\end{aligned}$$

where $S_2(n,k)$ is the Stirling number of the second kind.

For $n \geq 1$, by (11) and (12), we get

$$\begin{aligned}
t_n(x) &= x\left(\frac{t}{\left(\frac{t}{1-\lambda^m e^{mt}}\right)}\right)^n x^{n-1} = x\left(1-\lambda^m e^{mt}\right)^n x^{n-1} \\
&= x(1-\lambda^m-\lambda^m(e^{mt}-1))^n x^{n-1} \\
&= x\sum_{l=0}^{n}\binom{n}{l}(1-\lambda^m)^{n-l}(-\lambda^m)^l(e^{mt}-1)^l x^{n-1} \\
&= x\sum_{l=0}^{n-1}\sum_{k=0}^{n-l-1}\binom{n}{l}\binom{n-1}{k+l}(1-\lambda^m)^{n-l}(-\lambda^m)^l l! m^{k+l}S_2(l+k,l)x^{n-k-l-1} \\
&= x\sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n}{l}\binom{n-1}{n-1-r}(1-\lambda^m)^{n-l}(-\lambda^m)^l l! m^{n-1-r}S_2(n-1-r,l)x^r \\
&= \sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n}{l}\binom{n-1}{r}(1-\lambda^m)^{n-l}(-\lambda^m)^l l! m^{n-1-r}S_2(n-1-r,l)x^{r+1}. \qquad (14)
\end{aligned}$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 1. *For $n \geq 1$, let*

$$S_n(x) \sim \left(1, \frac{t}{1-\lambda e^t}\right), \ t_n(x) \sim \left(1, \frac{t}{1-\lambda^m e^{mt}}\right)$$

where $\lambda \in \mathbb{C}$ with $\lambda^m \neq 1$. Then we have

$$S_n(x) = \sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n-1}{r}\binom{n}{l}l!(1-\lambda)^{n-l}(-\lambda)^l S_2(n-1-r,l)x^{r+1},$$

and

$$t_n(x) = \sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n}{l}\binom{n-1}{r}(1-\lambda^m)^{n-l}(-\lambda^m)^l l! m^{n-1-r}S_2(n-1-r,l)x^{r+1}.$$

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By (11) and (12), we get

$$\begin{aligned}
t_n(x) &= x\left(\frac{\frac{t}{1-\lambda e^t}}{\frac{t}{1-\lambda^m e^{mt}}}\right)^n x^{-1}S_n(x) = x\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^n x^{-1}S_n(x)\\
&= x\left(\frac{1}{\lambda e^t}\sum_{k=1}^{m}\lambda^k e^{kt}\right)^n x^{-1}S_n(x)\\
&= \lambda^{-n}xe^{-nt}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}(\lambda e^t)^{v_1+2v_2+\cdots+mv_m}x^{-1}S_n(x)\\
&= \lambda^{-n}x\left(\sum_{l=0}^{\infty}\frac{(-n)^l}{l!}t^l\right)\left(\sum_{k=0}^{\infty}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m}\right.\\
&\qquad\left.\times(v_1+2v_2+\cdots+mv_m)^k\right)\frac{t^k}{k!}x^{-1}S_n(x)\\
&= \lambda^{-n}x\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m}\right.\\
&\qquad\left.\times(v_1+2v_2+\cdots+mv_m)^k\right\}\frac{t^s}{s!}x^{-1}S_n(x)\\
&= \lambda^{-n}x\sum_{s=0}^{n-1}\left\{\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m}\right.\\
&\qquad\left.\times(v_1+2v_2+\cdots+mv_m)^k\right\}\frac{t^s}{s!}x^{-1}S_n(x). \qquad (15)
\end{aligned}$$

Let

$$D_k^{(n)}(m\mid\lambda) = \sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m}(v_1+2v_2+\cdots+mv_m)^k. \qquad (16)$$

By (13), (15) and (16), we get

$$\begin{aligned}
t_n(x) &= \lambda^{-n}x\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}D_k^{(n)}(x\mid\lambda)\frac{t^s}{s!}\sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\binom{n-1}{r}\binom{n}{l}l!(1-\lambda)^{n-l}\\
&\quad\times(-\lambda)^l S_2(n-1-r,l)x^r\\
&= \lambda^{-n}x\sum_{s=0}^{n-1}\sum_{k=0}^{s}\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\binom{n-1}{r}\binom{s}{k}\binom{n}{l}\binom{r}{s}(-n)^{s-k}D_k^{(n)}(x\mid\lambda)l!(1-\lambda)^{n-l}\\
&\quad\times(-\lambda)^l S_2(n-1-r,l)x^{r-s}. \qquad (17)
\end{aligned}$$

Therefore, by Theorem1 and (17), we obtain the following theorem.

Theorem 2. *For $n \geq 1$, $\lambda \in \mathbb{C}$ with $\lambda^m \neq 1$, we have*

$$\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\binom{n}{l}\binom{n-1}{r}(1-\lambda^m)^{n-l}(-\lambda^m)^l l! m^{n-1-r} S_2(n-1-r,l)x^{r+1}$$
$$= \lambda^{-n}\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{n-1}{r}\binom{s}{k}\binom{n}{l}\binom{r}{s}(-n)^{s-k}D_k^{(n)}(x\mid\lambda)l!(1-\lambda)^{n-l}$$
$$\times(-\lambda)^l S_2(n-1-r,l)x^{r+1-s}.$$

Let us consider the following Sheffer sequences

$$S_n(x) \sim (1, t(1-\lambda e^t)),\ \ t_n(x) \sim (1, t(1-\lambda^m e^{mt})), \tag{18}$$

where $\lambda^m \neq 1$.

For $n \geq 1$, by (18) and $x^n \sim (1,t)$, we have

$$\begin{aligned} S_n(x) &= x\left(\frac{t}{t(1-\lambda e^t)}\right)^n x^{-1}x^n = x(1-\lambda e^t)^{-n}x^{n-1} \\ &= x(1-\lambda-\lambda(e^t-1))^{-n}x^{n-1} \\ &= \frac{x^n}{(1-\lambda)^n} + \frac{x}{(1-\lambda)^n}\sum_{l=1}^{n-1}\binom{-n}{l}(-1)^l\lambda^l\left(\frac{e^t-1}{1-\lambda}\right)^l x^{n-1} \\ &= \frac{x}{(1-\lambda)^n}\sum_{l=0}^{n-1}\binom{n+l-1}{l}\left(\frac{\lambda}{1-\lambda}\right)^l\sum_{r=0}^{l}\binom{l}{r}(-1)^{l-r}e^{rt}x^{n-1} \\ &= \frac{x}{(1-\lambda)^n}\sum_{l=0}^{n-1}\sum_{r=0}^{l}\binom{n+l-1}{l}\binom{l}{r}\left(\frac{\lambda}{1-\lambda}\right)^l(-1)^{l-r}(x+r)^{n-1}. \end{aligned} \tag{19}$$

From the generating function of the Stirling number of the second kind, we can derive

$$\begin{aligned} S_n(x) &= x(1-\lambda e^t)^{-n}x^{n-1} = \frac{x}{(1-\lambda)^n}\sum_{l=0}^{n-1}\binom{n+l-1}{l}\left(\frac{\lambda}{1-\lambda}\right)^l(e^t-1)^l x^{n-1} \\ &= \frac{x}{(1-\lambda)^n}\sum_{l=0}^{n-1}\sum_{m=0}^{n-1-l}\binom{n+l-1}{l}\left(\frac{\lambda}{1-\lambda}\right)^l l!S_2(m+l,l)\binom{n-1}{m+l}x^{n-m-l-1} \\ &= \frac{x}{(1-\lambda)^n}\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\binom{n+l-1}{l}\left(\frac{\lambda}{1-\lambda}\right)^l l!S_2(n-1-r,l)\binom{n-1}{r}x^r \\ &= \frac{1}{(1-\lambda)^n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n-1}{r}\binom{n+l-1}{l}l!\left(\frac{\lambda}{1-\lambda}\right)^l S_2(n-1-r,l)x^{r+1}. \end{aligned} \tag{20}$$

For $n \geq 1$, by (18) and $x^n \sim (1,t)$, we get

$$\begin{aligned} t_n(x) &= x\left(\frac{t}{t(1-\lambda^m e^{mt})}\right)^n x^{n-1} = x(1-\lambda^m e^{mt})^{-n}x^{n-1} \\ &= x(1-\lambda^m)^{-n}\sum_{l=0}^{n-1}\binom{-n}{l}(-1)^l\left(\frac{\lambda^m}{1-\lambda^m}\right)^l\sum_{r=0}^{l}\binom{l}{r}(-1)^{l-r}e^{rmt}x^{n-1} \\ &= x(1-\lambda^m)^{-n}\sum_{l=0}^{n-1}\sum_{r=0}^{l}\binom{n+l-1}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l\binom{l}{r}(-1)^{l-r}(x+rm)^{n-1}. \end{aligned} \tag{21}$$

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By the same method as (20), we get

$$\begin{aligned}
t_n(x) &= x(1-\lambda^m e^{mt})^{-n}x^{n-1} \\
&= x(1-\lambda^m)^{-n}\sum_{l=0}^{n-1}\binom{-n}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l(-1)^l(e^{mt}-1)^l x^{n-1} \\
&= \frac{x}{(1-\lambda^m)^n}\sum_{l=0}^{n-1}\binom{-n}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l(-1)^l\sum_{r=0}^{n-1-l}\frac{l!S_2(r+l,l)}{(r+l)!} \\
&\quad\times(n-1)_{r+l}m^{r+l}x^{n-r-l-1} \\
&= \frac{x}{(1-\lambda^m)^n}\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\binom{-n}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l(-1)^l\frac{l!S_2(n-r-1,l)}{(n-r-1)!} \\
&\quad\times(n-1)_{n-r-1}m^{n-r-1}x^r \\
&= \frac{1}{(1-\lambda^m)^n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}(-1)^l\binom{-n}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l\binom{n-1}{r}l!S_2(n-r-1,l) \\
&\quad\times m^{n-r-1}x^{r+1}.
\end{aligned} \tag{22}$$

Therefore, by (20) and (22), we obtain the following theorem.

Theorem 3. *For $n \geq 1$, let*

$$S_n(x) \sim (1, t(1-\lambda e^t)),\ \ t_n(x) \sim (1, t(1-\lambda^m e^{mt})),$$

where $\lambda \in \mathbb{C}$ with $\lambda^m \neq 1$. Then we have

$$S_n(x) = \frac{1}{(1-\lambda)^n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n-1}{r}\binom{n+l-1}{l}l!\left(\frac{\lambda}{1-\lambda}\right)^l S_2(n-1-r,l)x^{r+1},$$

and

$$\begin{aligned}
t_n(x) = \frac{1}{(1-\lambda^m)^n}&\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n+l-1}{l}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l\binom{n-1}{r}l!S_2(n-r-1,l) \\
&\times m^{n-r-1}x^{r+1}.
\end{aligned}$$

By (11) and (18), we get

$$\begin{aligned}
S_n(x) &= x\left(\frac{t(1-\lambda^m e^{mt})}{t(1-\lambda e^t)}\right)^n x^{-1}t_n(x) = x\left(\frac{1}{\lambda e^t}\sum_{l=1}^{m}\lambda^l e^{lt}\right)^n x^{-1}t_n(x) \\
&= \frac{x}{\lambda^n}(e^{-nt})\sum_{\substack{0\leq v_1,\cdots,v_m\leq n \\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}(\lambda e^t)^{v_1+2v_2+\cdots+mv_m}x^{-1}t_n(x) \\
&= \frac{x}{\lambda^n}(e^{-nt})\Bigg(\sum_{k=0}^{\infty}\sum_{\substack{0\leq v_1,\cdots,v_m\leq n \\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m} \\
&\quad\times(v_1+2v_2+\cdots+mv_m)^k\Bigg)\frac{t^k}{k!}x^{-1}t_n(x).
\end{aligned} \tag{23}$$

From (16), (21) and (23), we have

$$\begin{aligned}
S_n(x) &= \lambda^{-n}x\sum_{s=0}^{\infty}\left(\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}D_k^{(n)}(m\mid\lambda)\right)\frac{t^s}{s!}x^{-1}t_n(x)\\
&= \lambda^{-n}x\sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}D_k^{(n)}(m\mid\lambda)(1-\lambda^m)^{-n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n-1}{r}\\
&\qquad\times\binom{n+l-1}{l}l!\left(\frac{\lambda^m}{1-\lambda^m}\right)^l m^{n-1-r}S_2(n-1-r,l)\frac{t^s}{s!}x^r\\
&= (\lambda-\lambda^{m+1})^{-n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\sum_{s=0}^{r}\sum_{k=0}^{s}\binom{r}{s}\binom{s}{k}\binom{n-1}{r}\binom{n+l-1}{l}\\
&\qquad\times(-n)^{s-k}m^{n-1-r}\left(\frac{\lambda^m}{1-\lambda^m}\right)^l l!D_k^{(n)}(m\mid\lambda)S_2(n-1-r,l)x^{r-s+1}.
\end{aligned}\tag{24}$$

Therefore, by Theorem 3 and (24), we obtain the following theorem.

Theorem 4. *For $n\geq 1$, $\lambda\in\mathbb{C}$ with $\lambda^m\neq 1$, we have*

$$\begin{aligned}
&\frac{1}{(1-\lambda)^n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n+l-1}{l}\left(\frac{\lambda}{1-\lambda}\right)^l\binom{n-1}{r}l!S_2(n-r-1,l)x^{r+1}\\
&= (\lambda-\lambda^{m+1})^{-n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\sum_{s=0}^{r}\sum_{k=0}^{s}\binom{r}{s}\binom{s}{k}\binom{n-1}{r}\binom{n+l-1}{l}(-n)^{s-k}m^{n-1-r}\\
&\quad\times\left(\frac{\lambda^m}{1-\lambda^m}\right)^l l!D_k^{(n)}(m\mid\lambda)S_2(n-1-r,l)x^{r-s+1}.
\end{aligned}$$

Let

$$S_n(x)\sim\left(1,\frac{t}{1+\lambda e^t}\right),\ t_n(x)\sim\left(1,\frac{t}{1+(-1)^{m+1}\lambda^m e^{mt}}\right),\tag{25}$$

where $(-\lambda)^m\neq 1$.

From $x\sim(1,t)$ and (25), we have

$$S_n(x)=x(1+\lambda e^t)^n x^{n-1}=x\sum_{l=0}^{n}\binom{n}{l}\lambda^l(x+l)^{n-1}.\tag{26}$$

By(1), we get

$$\begin{aligned}
S_n(x) &= x(1+\lambda e^t)^n x^{n-1}=x(1+\lambda)^n\left(\frac{\lambda^{-1}+e^t}{\lambda^{-1}+1}\right)^n x^{n-1}\\
&= (1+\lambda)^n\sum_{l=0}^{n-1}\binom{n-1}{l}H_l^{(-n)}(-\lambda^{-1})x^{n-l},\quad\text{where}\quad\lambda\neq 0.
\end{aligned}\tag{27}$$

For $n\geq 1$, by $x^n\sim(1,t)$ and (25), we get

$$t_n(x)=x(1+(-1)^{m+1}\lambda^m e^{mt})^n x^{n-1}=x\sum_{l=0}^{n}\binom{n}{l}(-1)^l(-\lambda)^{ml}(x+lm)^{n-1}.\tag{28}$$

From the generating function of the Stirling number of the second kind and (28), we have

$$
\begin{aligned}
t_n(x) &= x(1+(-1)^{m+1}\lambda^m e^{mt})^n x^{n-1}\\
&= (1-(-\lambda)^m)^n x \sum_{l=0}^{n}\binom{n}{l}(-1)^l(-\lambda)^{ml}\left(\frac{e^{mt}-1}{1-(-\lambda)^m}\right)^l x^{n-1}\\
&= (1-(-\lambda)^m)^n \sum_{l=0}^{n-1}\sum_{r=0}^{n-l-1}\frac{(-\lambda)^{m+l}\binom{n}{l}}{(1-(-\lambda)^m)^l}(-1)^l\binom{n-1}{r} l! S_2(n-r-1,l) m^{n-r-1} x^{r+1}.
\end{aligned}
\tag{29}
$$

By (11) and (25), we get

$$
\begin{aligned}
t_n(x) &= x\left(\frac{1+(-1)^{m+1}\lambda^m e^{mt}}{1+\lambda e^t}\right)^n x^{-1}S_n(x)\\
&= x\left(\left(\frac{1}{-\lambda e^t}\right)\sum_{l=1}^{m}(-\lambda e^t)^l\right)^n x^{-1}S_n(x)\\
&= x(-\lambda)^{-n}e^{-nt}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}(-\lambda e^t)^{v_1+2v_2+\cdot+mv_m}x^{-1}S_n(x)\\
&= x(-\lambda)^{-n}\left(\sum_{l=0}^{\infty}\frac{(-n)^l}{l!}t^l\right)\left(\sum_{k=0}^{\infty}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\right.\\
&\qquad \left.\times(-\lambda)^{v_1+2v_2+\cdot+mv_m}(v_1+2v_2+\cdot+mv_m)^k\frac{t^k}{k!}x^{-1}S_n(x)\right)\\
&= x(-\lambda)^{-n}\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{s}(-n)^{s-k}\binom{s}{k}\sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}\right.\\
&\qquad \left.\times(-\lambda)^{v_1+2v_2+\cdot+mv_m}(v_1+2v_2+\cdot+mv_m)^k\right\}\frac{t^s}{s!}x^{-1}S_n(x).
\end{aligned}
\tag{30}
$$

Let us define $T_k^{(n)}(m \mid \lambda)$ as follows:

$$
T_k^{(n)}(m \mid \lambda) = \sum_{\substack{0\le v_1,\cdots,v_m\le n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\cdots,v_m}(-\lambda)^{v_1+2v_2+\cdot+mv_m} \times (v_1+2v_2+\cdot+mv_m)^k. \tag{31}
$$

From (26), (30) and (31), we have

$$
\begin{aligned}
t_n(x) &= x(-\lambda)^{-n}\sum_{s=0}^{\infty}\sum_{k=0}^{s}(-n)^{s-k}\binom{s}{k}T_k^{(n)}(m\mid\lambda)\frac{t^s}{s!}\sum_{l=0}^{n}\binom{n}{l}\lambda^l(x+l)^{n-1}\\
&= x(-\lambda)^{-n}\sum_{l=0}^{n}\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{n}{l}\binom{s}{k}\binom{n-1}{s}(-n)^{s-k}\lambda^l T_k^{(n)}(m\mid\lambda)(x+l)^{n-1-s}.
\end{aligned}
\tag{32}
$$

Therefore, by (28) and (32), we obtain the following theorem.

Theorem 5. *For $n \geq 1$, $\lambda \in \mathbb{C}$ with $(-\lambda)^m \neq 1$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^l(-\lambda)^{ml}(x+lm)^{n-1}$$
$$= (-\lambda)^{-n}\sum_{l=0}^{n}\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{n}{l}\binom{s}{k}\binom{n-1}{s}(-n)^{s-k}\lambda^l T_k^{(n)}(m\mid\lambda)(x+l)^{n-1-s}.$$

Remark. Let

$$S_n(x) \sim (1, t(1+\lambda e^t)),\ t_n(x) \sim (1, t(1+(-1)^{m+1}\lambda^m e^{mt})),\quad (-\lambda)^m \neq 1. \tag{33}$$

For $n \geq 1$, by $x^n \sim (1,t)$ and (33), we get

$$\begin{aligned} S_n(x) &= \frac{x}{(1+\lambda)^n}\sum_{l=0}^{n-1}\sum_{r=0}^{l}\binom{n+l-1}{l}\binom{l}{r}\left(\frac{\lambda}{1+\lambda}\right)^l(-1)^r(x+r)^{n-1} \\ &= \frac{x}{(1+\lambda)^n}\sum_{r=0}^{n-1}\sum_{l=0}^{n-1-r}\binom{n+l-1}{l}\binom{n-1}{r}\left(\frac{-\lambda}{1+\lambda}\right)^l l! S_2(n-r-1,l)x^r, \end{aligned} \tag{34}$$

and

$$\begin{aligned} t_n(x) = (1-(-\lambda)^m)^{-n}\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\binom{n+l-1}{l}\binom{n-1}{r}\left(\frac{(-\lambda)^m}{1-(-\lambda)^m}\right)^l l! \\ \times S_2(n-1-r,l)m^{n-1-r}x^{r+1}. \end{aligned} \tag{35}$$

From (11), (33), (34) and (35), we have

$$\begin{aligned} S_n(x) = (\lambda+(-\lambda)^{m+1})^{-n}\sum_{l=0}^{n-1}\sum_{r=0}^{l}\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{n-1}{s}\binom{s}{k}\binom{n+l-1}{l}\binom{l}{r}(-n)^{s-k} \\ \times (-1)^{l-r}\left(\frac{(-\lambda)^m}{1-(-\lambda)^m}\right)^l T_k^{(n)}(m\mid\lambda)x(x+rm)^{n-1-s}. \end{aligned} \tag{36}$$

Let

$$S_n(x) \sim (1, e^{bt}-1),\ t_n(x) \sim (1, \frac{t^2}{e^{bt}-1}),\ b \neq 0. \tag{37}$$

From $x^n \sim (1,t)$ and (37), we have

$$\begin{aligned} S_n(x) &= x\left(\frac{t}{e^{bt}-1}\right)^n x^{n-1} = \frac{x}{b^n}(b^{n-1}B_{n-1}^{(n)}\left(\frac{x}{b}\right)) \\ &= \frac{x}{b^n}\sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(n)}b^l x^{n-1-l}, \end{aligned} \tag{38}$$

and

$$t_n(x) = x\left(\frac{e^{bt}-1}{t}\right)^n x^{n-1} = b^n x\sum_{l=0}^{n-1}b^l\frac{\binom{n-1}{l}}{\binom{n+l}{l}}S_2(n+l,l)x^{n-1-l}. \tag{39}$$

Therefore, by (37), (38) and (39), we obtain the following theorem.

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Theorem 6. *For $n \geq 1$, $b \neq 0$, let*

$$S_n(x) = \left(\frac{x}{b}\right)_n \sim (1, e^{bt}-1), \;\; t_n(x) \sim (1, \frac{t^2}{e^{bt}-1}).$$

Then we have

$$S_n(x) = \left(\frac{x}{b}\right)_n = \frac{x}{b^n}\sum_{l=0}^{n-1} b^l \binom{n-1}{l} B_l^{(n)} x^{n-1-l},$$

and

$$t_n(x) = b^n x \sum_{l=0}^{n-1} b^l \frac{\binom{n-1}{l}}{\binom{n+l}{l}} S_2(n+l,l) x^{n-1-l}.$$

From Theorem 6, we note that

$$\left(\frac{x}{m}\right)_n = m^{-n} x \sum_{l=0}^{n-1} m^l \binom{n-1}{l} B_l^{(n)} x^{n-1-l} \sim (1, e^{mt}-1). \tag{40}$$

Let $b = 1$. Then, by Theorem 6, we get

$$(x)_n = x \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} x^{n-1-l} \sim (1, e^t - 1). \tag{41}$$

From (11), (40) and (41), we have

$$\begin{aligned}
(x)_n &= x\sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(n)}x^{n-1-l} = x\left(\frac{e^{mt}-1}{e^t-1}\right)^n x^{-1}\left(\frac{x}{m}\right)_n \\
&= x\left(\frac{1}{e^t}\sum_{l=1}^{m}e^{lt}\right)^n x^{-1}\left(\frac{x}{m}\right)_n = xe^{-nt}\left(e^t+e^{2t}+\cdots+e^{mt}\right)^n x^{-1}\left(\frac{x}{m}\right)_n \\
&= xe^{-nt}\sum_{\substack{0\leq v_1,\cdots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,v_2,\cdots,v_m}e^{(v_1+2v_2+\cdots+mv_m)t}x^{-1}\left(\frac{x}{m}\right)_n \\
&= x\left(\sum_{l=0}^{\infty}\frac{(-n)^l}{l!}t^l\right)\left(\sum_{k=0}^{\infty}\left(\sum_{\substack{0\leq v_1,\cdots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,v_2,\cdots,v_m}\right.\right. \\
&\qquad\left.\left.\times(v_1+2v_2+\cdots+mv_m)^k\frac{t^k}{k!}\right)x^{-1}\left(\frac{x}{m}\right)_n\right) \\
&= x\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}\sum_{\substack{0\leq v_1,\cdots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,v_2,\cdots,v_m}\right. \\
&\qquad\left.\times(v_1+2v_2+\cdots+mv_m)^k\right\}\frac{t^s}{s!}x^{-1}\left(\frac{x}{m}\right)_n.
\end{aligned} \tag{42}$$

Let us define $S_k^{(n)}(m)$ as follows:

$$S_k^{(n)}(m) = \sum_{\substack{0\leq v_1,\cdots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,v_2,\cdots,v_m}(v_1+2v_2+\cdots+mv_m)^k. \tag{43}$$

Then, by (40), (42) and (43), we get

$$\begin{aligned}(x)_n &= x\sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}S_k^{(n)}(m)\frac{t^s}{s!}\left\{m^{-n}\sum_{l=0}^{n-1}m^l\binom{n-1}{l}B_l^{(n)}x^{n-1-l}\right\}\\ &= x\sum_{l=0}^{n-1}\sum_{s=0}^{n-1-l}\sum_{k=0}^{s}m^{-n+l}\binom{s}{k}\binom{n-1}{l}(-n)^{s-k}S_k^{(n)}(m)B_l^{(n)}\frac{t^s}{s!}x^{n-1-l}\\ &= x\sum_{l=0}^{n-1}\sum_{s=0}^{n-1-l}\sum_{k=0}^{s}m^{-n+l}\binom{s}{k}\binom{n-1}{l}\binom{n-1-l}{s}(-n)^{s-k}S_k^{(n)}(m)B_l^{(n)}x^{n-1-l-s}\\ &= x\sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\sum_{k=0}^{n-1-l-r}m^{-n+l}\binom{n-1-l-r}{k}\binom{n-1}{l}\binom{n-1-l}{r}\\ &\qquad\times(-n)^{n-1-l-r-k}S_k^{(n)}(m)B_l^{(n)}x^r. \end{aligned}\tag{44}$$

Therefore, by (44), we obtain the following theorem.

Theorem 7. *For $n \geq 1$, we have*

$$\begin{aligned}(x)_n = \sum_{l=0}^{n-1}\sum_{r=0}^{n-1-l}\sum_{k=0}^{n-1-l-r}m^{-n+l}\binom{n-1-l-r}{k}\binom{n-1}{l}\binom{n-1-l}{r}\\ \times(-n)^{n-1-l-r-k}S_k^{(n)}(m)B_l^{(n)}x^{r+1}.\end{aligned}$$

References

[1] S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math . **22** (2012) no.3, 399-406.

[2] S. Araci, M. Acikgoz, H. Jolany and J. J. Seo, *A unied generating function of the q-Genocchi polynomials with their interpolation functions* , Proc. Jangjeon Math. Soc., **15** (2012) no.2, 227-233.

[3] M. Can, M. Cenkci, V. Kurt and Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Eulerl-functions*, Adv. Stud. Contemp. Math. **18** (2009), no. 2, 135-160.

[4] L.Carlitz, *The product of two Eulerian polynomials*, Math. Mag. **368** (1963) 37-41.

[5] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. **22** (2012) 433-438.

[6] J. Y. Kang and C. S. Ryoo, *n multiple interpolation functions of the q-Genocchi numbers and polynomials with weight and weak weight* , Adv. Stud. Contemp. Math., **22** (2012) no. 3, 407-420.

[7] D. S. Kim and T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials* , Int. J. Math. Math. Sci. 2012, Art. ID 463659, 12 pp.

[8] T. Kim, *q-Bernoulli numbers and polynomials associated with Gaussian binomial coecients* , Russ. J. Math. Phys. **15** (2008), no. 1, 51-57.

[9] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear dierential equations*, J. Number Theory. **132** (2012), no. 1, 2854-2865.

[10] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on Z_p* , Rocky Mountain J. Math. **41** (2011), no. 1, 239-247.

[11] T. Kim *A note on q-Bernstein polynomials* , Russ. J. Math. Phys. **18** (2011), no. 1, 73-82.

[12] T. Kim *Symmetry p-adic invariant integral on Z_p for Bernoulli and Euler polynomials* , J. Difference Equ. Appl. **14** (2008), no. 12, 1267-1277.

[13] D. S. Kim, T. Kim, S. H. Lee and S. H. Rim *ome identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus* , Adv. Difference Equ. (2013), 2013:15.

[14] H. Ozden, I. N. Cangul and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers* , Adv. Stud. Contemp. Math. **18** (2009), no. 1, 41-48.

[15] S. Roman, *The umbral calculus,* Dover Publ. Inc. New York, (2005).

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Dynamics and Behavior of a Second Order Rational Difference equation

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ABSTRACT

In this paper we investigate the global convergence result, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}, \quad n = 0, 1, ...,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

Keywords: stability, periodic solutions, boundedness, difference equations.
Mathematics Subject Classification: 39A10

1 Introduction

Difference equations have been used to describe evolution phenomena since most measurements of time-evolving variables are discrete. More significantly, difference equations are used in the study of discretization methods for differential equations. The theory of difference equations has some results that have been acquired approximately as natural discrete analogues of corresponding results of differential equations [35].

The study of rational difference equations of order greater than one is quite ambitious and worthwhile since some paradigms for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results of rational difference equations. However, there have not been any useful general methods to study the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one deserves further consideration. Many research have been done to study the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example, Agarwal et al. [2]

looked at the global stability, periodicity character and found the solution form of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

The form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}},$$

was obtained by Aloqeili [4]. The dynamics, the global stability, periodicity character and the solution of special case of the recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}},$$

was investigated by Elabbasy et al in [8].
Elabbasy et al. [9] studied the behavior of the difference equation, especially global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Karatas et al. [31] researched the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

In [36] Simsek et al. acquired the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

The dynamics of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k},$$

was studied by Yalçınkaya et al. in [44].
Zayed et al. [46], [47] looked at the behavior of the following rational recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}, \quad x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

Other related results on rational difference equations and systems can be found in refs. [1-45]. This paper aims to study the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}, \tag{1}$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

2 Some Basic Properties and Definitions

In this section, we state some basic definitions and theorems that we need in this paper. Suppose that I is an interval of real numbers and let

$$F : I \times I \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 2.1. (Equilibrium Point)
A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = F(\overline{x}, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0,$ is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2.2. (Periodicity)
A sequence $\{x_n\}_{n=-1}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -1$.

Definition 2.3. (Stability)
(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0,$ there exists $\delta > 0$ such that for all $x_{-1}, x_0 \in I$ with

$$|x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -1.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0,$ such that for all $x_{-1},\, x_0 \in I$ with

$$|x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-1}, x_0 \in I,$ we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation associated with Eq.(2) about the equilibrium point$\overline{x}$ is the linear difference equations

$$y_{n+1} = py_n + qy_{n-1}.$$

where

$$p = \frac{\partial F}{\partial x_n}(\overline{x}, \overline{x}), \quad q = \frac{\partial F}{\partial x_{n-1}}(\overline{x}, \overline{x}).$$

Theorem A [34]: (Linearized Stability)
(a) If both roots of the quadratic equation

$$\lambda^2 - p\lambda - q = 0. \tag{3}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Eq.(2) is locally asymptotically stable.
(b) If at least one of the roots of Eq.(3) has absolute value greater than one, then the equilibrium $\overline{x}$ of Eq.(2) is unstable.
(c) A necessary and sufficient condition for both roots of Eq.(3) to lie in the open unit disk $|\lambda| < 1$, is

$$|p| < 1 - q < 2. \tag{4}$$

In this case the locally asymptotically stable equilibrium $\overline{x}$ is also called a sink.
Now, consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}). \tag{5}$$

The following two theorems will be useful for the proof of our results in this paper.
Theorem B [34]: Suppose that $[\alpha, \beta]$ is an interval of real numbers and assume that

$$g : [\alpha, \beta]^2 \to [\alpha, \beta],$$

is a continuous function satisfying the following properties:
(a) $g(x, y)$ is non-decreasing in each of its arguments;
(b) The equation

$$g(x, x) = x,$$

has a unique positive solution. Then Eq.(5) has a unique equilibrium point $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(5) converges to $\overline{x}$.
Theorem C [34]: Suppose that $[\alpha, \beta]$ is an interval of real numbers and let

$$g : [\alpha, \beta]^2 \to [\alpha, \beta],$$

be a continuous function that satisfies the following properties :
(a) $g(x, y)$ is non-decreasing in x in $[\alpha, \beta]$ for each $y \in [\alpha, \beta]$, and is non-increasing in $y \in [\alpha, \beta]$ for each x in $[\alpha, \beta]$;
(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$M = g(M, m) \quad and \quad m = g(m, M),$$

then

$$m = M.$$

Then Eq.(5) has a unique equilibrium point $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(5) converges to $\overline{x}$.

3 Local Stability of the Equilibrium Point of Eq.(1)

In this section, we study the local stability character of the equilibrium point of Eq.(1). Eq.(1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b + c\overline{x}}{d + e\overline{x}},$$

or

$$e(1-a)\overline{x}^2 + (d - da - c)\overline{x} - b = 0.$$

Then the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \frac{(c-d+da)+\sqrt{(c-d+da)^2+4be(1-a)}}{2e(1-a)}.$$

Theorem 3.1. *The equilibrium $\overline{x}$ of Eq. (1) is locally asymptotically stable if and only if*

$$(d + e\overline{x})^2 > \frac{|cd - be|}{(1-a)}, \qquad a < 1.$$

Proof: Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuous function defined by

$$f(u,v) = au + \frac{b + cv}{d + ev}. \tag{6}$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = a, \ \frac{\partial f(u,v)}{\partial v} = \frac{(cd - be)}{(d + ev)^2}.$$

So, we can write

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = a = p, \ \frac{\partial f(\overline{x},\overline{x})}{\partial v} = \frac{(cd - be)}{(d + e\overline{x})^2} = q.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} - py_{n-1} - qy_n = 0. \tag{7}$$

It follows by Theorem A that, Eq.(1) is asymptotically stable if and only if

$$|p| < 1 - q < 2.$$

Thus,

$$|a| + \left|\frac{(cd - be)}{(d + e\overline{x})^2}\right| < 1,$$

and so

$$\begin{aligned} \left|\frac{(cd - be)}{(d + e\overline{x})^2}\right| &< (1-a), \qquad a < 1, \\ |cd - be| &< (d + e\overline{x})^2(1-a), \qquad a < 1, \end{aligned}$$

or

$$\frac{|cd - be|}{(1-a)} < (d + e\overline{x})^2, \qquad a < 1.$$

The proof is complete.

4 Existence of Bounded and Unbounded Solutions of Eq.(1)

Here we look at the boundedness nature of solutions of Eq.(1).

Theorem 4.1. *Every solution of Eq.(1) is bounded if* $a < 1$.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}} = ax_n + \frac{b}{d + ex_{n-1}} + \frac{cx_{n-1}}{d + ex_{n-1}}.$$

Then

$$x_{n+1} \leq ax_n + \frac{b}{d} + \frac{cx_{n-1}}{ex_{n-1}} = ax_n + \frac{b}{d} + \frac{c}{e} \quad \text{for all} \quad n \geq 1.$$

By using a comparison, the right hand side can be written as follows

$$y_{n+1} = ay_n + \frac{b}{d} + \frac{c}{e}.$$

So, we can write

$$y_n = a^n y_0 + constant,$$

and this equation is locally asymptotically stable because $a < 1$, and converges to the equilibrium point $\overline{y} = \dfrac{be + cd}{de(1-a)}$.

Therefore

$$\limsup_{n\to\infty} x_n \leq \frac{be + cd}{de(1-a)}.$$

Hence, the solution is bounded.

Theorem 4.2. *Every solution of Eq.(1) is unbounded if* $a > 1$.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we see that

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}} > ax_n \quad \text{for all} \quad n \geq 1.$$

The right hand side can be written as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-1}^{\infty}$ is unbounded from above.

5 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 5.1. *Eq.(1) has positive prime period two solutions if and only if*

$$(i) \;\; [c - ad - d]^2 (1 + a) + 4 [be + (c - ad - d)ad] > 0. \tag{8}$$

Proof: Firstly, suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(1). We will show that Condition (i) holds.
From Eq.(1), we get

$$p = aq + \frac{b + cp}{d + ep},$$

and

$$q = ap + \frac{b + cq}{d + eq}.$$

Therefore,

$$dp + ep^2 = adq + aepq + b + cp, \tag{9}$$

and

$$dq + eq^2 = adp + aepq + b + cq. \tag{10}$$

Subtracting (10) from (9) gives

$$d(p - q) + e(p^2 - q^2) = -ad(p - q) + c(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{c - ad - d}{e}. \tag{11}$$

Again, adding (9) and (10) yields

$$d(p + q) + e(p^2 + q^2) = ad(p + q) + 2aepq + 2b + c(p + q),$$

$$e(p^2 + q^2) = (ad - d + c)(p + q) + 2aepq + 2b. \tag{12}$$

By using (11), (12) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all} \quad p, q \in R,$$

we obtain

$$\begin{aligned} e((p + q)^2 - 2pq) &= (ad - d + c)(p + q) + 2aepq + 2b \\ 2(1 + a)epq &= -2ad(p + q) - 2b. \end{aligned}$$

Then,

$$pq = \frac{-(c - ad - d)ad - be}{(1 + a)e^2}. \tag{13}$$

Now it is obvious from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$t^2 - \left(\frac{c - ad - d}{e}\right)t - \left(\frac{be + (c - ad - d)ad}{(1 + a)e^2}\right) = 0,$$

$$et^2 - (c - ad - d)t - \left(\frac{be + (c - ad - d)ad}{(1 + a)e}\right) = 0, \tag{14}$$

and so

$$[c - ad - d]^2 + \tfrac{4[be + (c - ad - d)ad]}{(1 + a)} > 0,$$

or

$$[c-ad-d]^2(1+a)+4[be+(c-ad-d)ad]>0.$$

Therefore inequality (i) holds.
Conversely, suppose that inequality (i) is true. We will prove that Eq.(1) has a prime period two solution.
Suppose that

$$p=\frac{c-ad-d+\zeta}{2e},$$

and

$$q=\frac{c-ad-d-\zeta}{2e},$$

where $\zeta=\sqrt{[c-ad-d]^2+\frac{4[be+(c-ad-d)ad]}{(1+a)}}$.
We see from the inequality (i) that

$$[c-ad-d]^2(1+a)+4[be+(c-ad-d)ad]>0,$$

which equivalents to

$$[c-ad-d]^2+\frac{4[be+(c-ad-d)ad]}{(1+a)}>0.$$

Therefore p and q are distinct real numbers.
Set

$$x_{-1}=p \text{ and } x_0=q.$$

We would like to show that

$$x_1=x_{-1}=p \quad \text{and} \quad x_2=x_0=q.$$

It follows from Eq.(1) that

$$x_1=aq+\frac{b+cp}{d+ep}=a\left(\frac{c-ad-d-\zeta}{2e}\right)+\frac{b+c\left(\frac{c-ad-d+\zeta}{2e}\right)}{d+e\left(\frac{c-ad-d+\zeta}{2e}\right)}.$$

Dividing the denominator and numerator by $2(d+ae)$ we get

$$x_1=a\left(\frac{c-ad-d-\zeta}{2e}\right)+\frac{2eb+c(c-ad-d+\zeta)}{2ed+e(c-ad-d+\zeta)}.$$

Multiplying the denominator and numerator of the right side by $2ed+e(c-ad-d-\zeta)$ and by computation we obtain

$$x_1=p.$$

Similarly as before, it is easy to show that

$$x_2=q.$$

Then by induction we get

$$x_{2n}=q \quad \text{and} \quad x_{2n+1}=p \quad \text{for all} \quad n\geq -1.$$

Thus Eq.(1) has the prime period two solution

$$\ldots,p,q,p,q,\ldots,$$

where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

6 Global Attractivity of the Equilibrium Point of Eq.(1)

In this section, the global asymptotic stability of Eq.(1) is studied.

Lemma 6.1. *For any values of the quotient $\frac{b}{d}$ and $\frac{c}{e}$, the function $f(u,v)$ defined by Eq.(6) has the monotonicity behavior in its two arguments.*

Proof: The proof follows by some computations and it will be omitted.

Theorem 6.2. *The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds*

$$(1)\ cd \geq be \ \text{ and } \ c > d(1-a),\ a < 1. \tag{15}$$

$$(2)\ cd \leq be \ \text{ and } \ a < 1. \tag{16}$$

Proof: Suppose that α and β are real numbers and assume that $g : [\alpha,\beta]^2 \longrightarrow [\alpha,\beta]$ is a function defined by

$$g(u,v) = au + \frac{b+cv}{d+ev}.$$

Then

$$\frac{\partial g(u,v)}{\partial u} = a, \quad \frac{\partial g(u,v)}{\partial v} = \frac{(cd-be)}{(d+ev)^2}.$$

Now, two cases must be considered :

Case (1): Suppose that (15) is true, then we can easily see that the function $g(u,v)$ increasing in u,v.

Let x be a solution of the equation $x = g(x,x)$. Then from Eq.(1), we can write

$$x = ax + \frac{b+cx}{d+ex},$$

or

$$x(1-a) = \frac{b+cx}{d+ex},$$

then the equation

$$e(1-a)x^2 + \{d(1-a)-c\}x - b = 0,$$

has a unique positive solution when $c > d(1-a),\ a < 1$ which is

$$x = \frac{(c-d(1-a)) + \sqrt{(c-d(1-a))^2 + 4be(1-a)}}{2e(1-a)},$$

By using Theorem B, it follows that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

Case (2): Suppose that (16) is true, let α and β be real numbers and assume that $g : [\alpha,\beta]^2 \longrightarrow [\alpha,\beta]$ be a function defined by $g(u,v) = au + \dfrac{b+cv}{d+ev}$, then we can easily see that the function $g(u,v)$ increasing in u and decreasing in v.

Let (m,M) be a solution of the system $M = g(M,m)$ and $m = g(m,M)$. Then from Eq.(1), we see that

$$M = aM + \frac{b+cm}{d+em}, \quad m = am + \frac{b+cM}{d+eM},$$

or

$$M(1-a) = \frac{b+cm}{d+em}, \quad m(1-a) = \frac{b+cM}{d+eM},$$

then

$$d(1-a)M + e(1-a)Mm = b + cm,\ d(1-a)m + e(1-a)mM = b + cM.$$

Subtracting we obtain

$$(M-m)\{d(1-a)(M+m)+c\} = 0,$$

under the condition $a < 1$, we see that

$$M = m.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(1). This completes the proof of the theorem.

7 Numerical examples

To confirm the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume that $x_{-1} = 7,\ x_0 = 11,\ a = 0.1,\ b = 2,\ c = 5,\ d = 3,\ e = 7$**.** See Fig. 1.

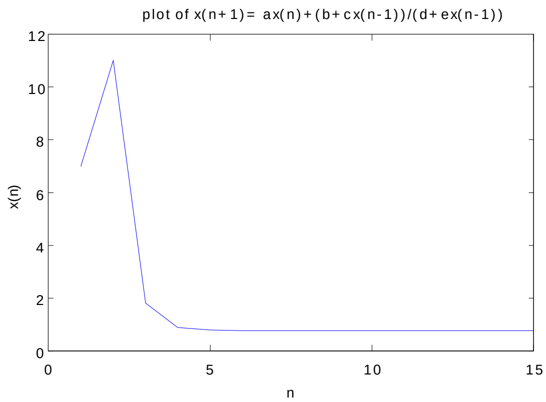

Figure 1.

Example 2. See Fig. 2, since $x_{-1} = 13,\ x_0 = 5,\ a = 0.8,\ b = 7,\ c = 2,\ d = 0.4,\ e = 2.$

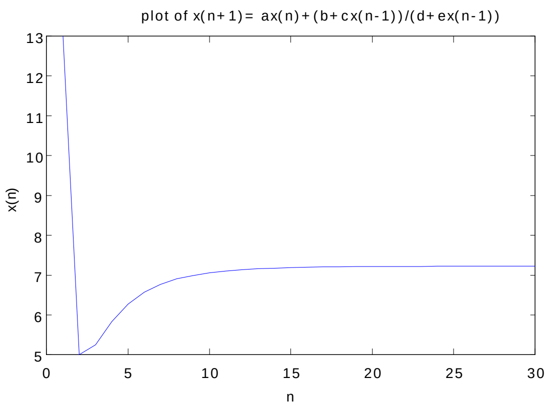

Figure 2.

Example 3. We consider $x_{-1} = 2,\ x_0 = 5,\ a = 1.2,\ b = 8,\ c = 5,\ d = 4,\ e = 1$. See Fig. 3.

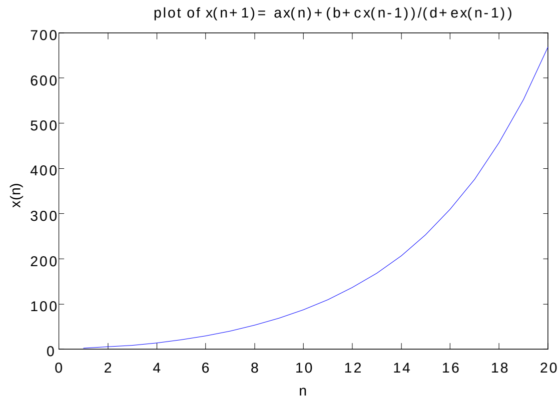

Figure 3.

Example 4. Fig. 4. shows the solutions when $a = 2,\ b = 1,\ c = 11,\ d = 3,\ e = 4,\ x_{-1} = p,$ $x_0 = q.$ $\left(\text{Since } p, q = \frac{c-ad-d\pm\sqrt{[c-ad-d]^2+\frac{4[be+(c-ad-d)ad]}{(1+a)}}}{2e}.\right)$

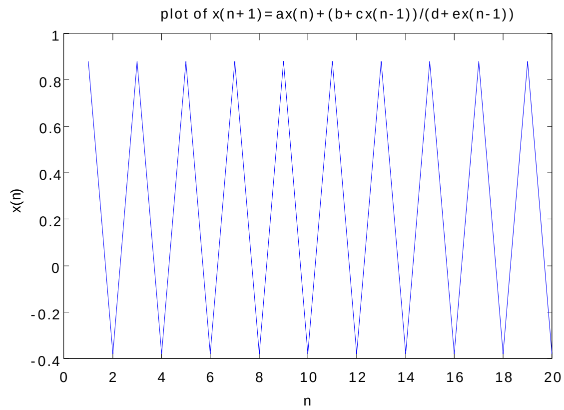

Figure 4.

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References

[1] R. P. Agarwal, Difference Equations and Inequalities, 1st edition, Marcel Dekker, New York, 1992, 2nd edition, 2000.

[2] R. P. Agarwal and E. M. Elsayed, Periodicity and stability of solutions of higher order rational difference equation, Adv. Stud. Contem. Math., 17 (2) (2008), 181-201.

[3] R. P. Agarwal and E. M. Elsayed, On the solution of fourth-order rational recursive sequence, Adv. Stud. Contem. Math., 20 (4), (2010), 525–545.

[4] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comp., 176(2), (2006), 768-774.

[5] N. Battaloglu, C. Cinar and I. Yalçınkaya, The dynamics of the difference equation, ARS Combinatoria, 97 (2010), 281-288.

[6] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_n x_{n-1}}$, Appl. Math. Comp., 150 (2004), 21-24.

[7] C. Cinar, On the difference equation $x_{n+1} = \frac{x_{n-1}}{-1+x_n x_{n-1}}$, Appl. Math. Comp., 158 (2004), 813-816.

[8] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ., Volume 2006 (2006), Article ID 82579,1–10.

[9] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5(2) (2007), 101-113.

[10] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Some properties and expressions of solutions for a class of nonlinear difference equation, Utilitas Mathematica, 87 (2012), 93-110.

[11] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Adv. Differ. Equ., 2011, 2011: 28.

[12] E. M. Elabbasy and E. M. Elsayed, Global attractivity and periodic nature of a difference equation, World Appl. Sci. J., 12 (1) (2011), 39–47.

[13] H. El-Metwally and E. M. Elsayed, Solution and behavior of a third rational difference equation, Utilitas Mathematica, 88 (2012), 27–42.

[14] H. El-Metwally and E. M. Elsayed, Form of solutions and periodicity for systems of difference equations, J. Comp. Anal. Appl., 15 (5) (2013), 852-857.

[15] E. M. Elsayed, Qualitative behavior of difference equation of order two, Math. Comp. Mod., 50 (2009), 1130–1141.

[16] E. M. Elsayed, Solution of a recursive sequence of order ten, General Mathematics, 19 (1) (2011), 145–162.

[17] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Comp. Anal. Appl., 15 (1) (2013), 73-81.

[18] E. M. Elsayed, Dynamics of recursive sequence of order two, Kyungpook Math. J., 50(2010), 483-497.

[19] E. M. Elsayed, Solutions of rational difference system of order two, Math. Comp. Mod., 55 (2012), 378–384.

[20] E. M. Elsayed, Qualitative behavior of difference equation of order three, Acta Sci. Math. (Szeged), 75 (1-2), 113–129.

[21] E. M. Elsayed, On the Global attractivity and the solution of recursive sequence, Studia Sci. Math. Hung., 47 (3) (2010), 401-418.

[22] E. M. Elsayed, Solution and behavior of a rational difference equations, Acta Universitatis Apulensis, 23 (2010), 233–249.

[23] E. M. Elsayed, On the global attractivity and the periodic character of a recursive sequence, Opuscula Mathematica, 30 (4) (2010), 431–446.

[24] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Disc. Dyn. Nat. Soc., Volume 2011, Article ID 982309, 17 pages.

[25] E. M. Elsayed, On the solutions of a rational system of difference equations, Fasciculi Mathematici, 45 (2010), 25–36.

[26] E. M. Elsayed and M. M. El-Dessoky, Dynamics and behavior of a higher order rational recursive sequence, Adv. Differ. Equ., 2012, 2012:69.

[27] E. M. Elsayed and H. A. El-Metwally, On the solutions of some nonlinear systems of difference equations, Adv. Differ. Equ., 2013, 2013:16, doi:10.1186/1687-1847-2013-161.

[28] M. Mansour, M. M. El-Dessoky and E. M. Elsayed, On the solution of rational systems of difference equations, J. Comp. Anal. Appl. , 15 (5) (2013), 967-976.

[29] A. Gelisken, C. Cinar and I. Yalcinkaya, On a max-type difference equation, Adv. Differ. Equ., Vol. 2010, Article ID 584890, 2010, 6 pages.

[30] A. Gelişken, C. Cinar and I. Yalçınkaya, On the periodicity of a difference equation with maximum, Disc. Dyn. Nat. Soc., Vol. 2008, Article ID 820629, 11 pages, doi: 10.1155/2008/820629.

[31] R. Karatas and C. Cinar, On the solutions of the difference equation $x_{n+1} = \frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2} x_{n-i}}$, Int. J. Contemp. Math. Sciences, 2 (13) (2007), 1505-1509.

[32] R. Karatas, C. Cinar and D. Simsek, On positive solutions of the difference equation $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., 1(10) (2006), 495-500.

[33] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[34] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[35] V. Lakshmikantham and D. Trigiante, Theory of Difference Equations Numerical Methods and Applications, Monographs and textbooks in pure and applied mathematics, Marcel Dekker Inc, second edition, (2002).

[36] D. Simsek, C. Cinar and I. Yalcinkaya, On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 475-480.

[37] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comp. Mod., 55 (2012), 1987–1997.

[38] C. Wang, S. Wang and X. Yan, Global asymptotic stability of 3-species mutualism models with diffusion and delay effects, Disc. Dyn. Nat. Soc., Volume 2009, Article ID 317298, 20 pages.

[39] C. Wang, F. Gong, S. Wang, L. LI and Q. Shi, Asymptotic behavior of equilibrium point for a class of nonlinear difference equation, Adv. Differ. Equ., Volume 2009, Article ID 214309, 8 pages.

[40] I. Yalçınkaya, C. Cinar and M. Atalay, On the solutions of systems of difference equations, Adv. Differ. Equ., Vol. 2008, Article ID 143943, 9 pages.

[41] I. Yalçınkaya, C. Cinar and D. Simsek, Global asymptotic stability of a system of difference equations, Applicable Analysis, 87 (6) (2008), 689-699.

[42] I. Yalçınkaya, B. D. Iricanin and C. Cinar, On a max-type difference equation, Disc. Dyn. Nat. Soc., Volume 2007, Article ID 47264, 10 pages.

[43] I. Yalçınkaya, On the global asymptotic stability of a second-order system of difference equations, Disc. Dyn. Nat. Soc., Vol. 2008, Article ID 860152, 12 pages.

[44] I. Yalçınkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Disc. Dyn. Nat. Soc., Vol. 2008, Article ID 805460, 8 pages.

[45] I. Yalçınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Combinatoria, 95 (2010), 151-159.

[46] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$, Comm. Appl. Nonlin. Anal., 15 (2), (2008), 47-57.

[47] E. M. E. Zayed, Dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$, Eur. J. Pure Appl. Math., 3 (2) (2010), 254-268.g

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OSTROWSKI TYPE INEQUALITIES FOR $m-$ AND (α, m) −LOGARITMICALLY CONVEX FUNCTIONS

HAVVA KAVURMACI♣

Abstract. In this paper, we give some informations about the Ostrowski type inequality and (α, m) −logaritmically convex functions. And then, we establish some Ostrowski type inequalities for this class of functions.

1. INTRODUCTION

Let $f : I \subset [0, \infty] \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [1]).

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1.1}$$

This inequality is well known in the literature as Ostrowski inequality. For some results which generalize, improve and extend the inequality (1.1) see ([2]-[4]) and the references therein.

Let us recall some known definitions and results which will be used in this paper. The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [13], Toader defined $m-$convexity as following:

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

For recent results related to above definitions we refer interest of readers to [5]-[7], [9], [15] and [12].

In [11], Miheşan defined $(\alpha, m)-$convexity as following:

Definition 2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

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Corresponding Author.

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for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. For the recent results based on the above definition see the papers [3], [5]-[6], [8] and [10].

The $m-$ and $(\alpha, m)-$logaritmically convex functions were defined in [12] as following:

Definition 3. *A function* $f : [0, b] \to (0, \infty)$, *is said to be* $m-$*logaritmically convex if the inequality*

$$f(tx + m(1-t)y) \leq f(x)^t f(y)^{m(1-t)}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$ *and* $t \in [0, 1]$.

Obviously, if putting $m = 1$ in Definition 3, then f is just the ordinary logaritmically convex function on $[0, b]$.

Definition 4. *A function* $f : [0, b] \to (0, \infty)$, *is said to be* $(\alpha, m)-$*logaritmically convex if*

$$f(tx + m(1-t)y) \leq f(x)^{t^\alpha} f(y)^{m(1-t^\alpha)}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]^2$ *and* $t \in [0, 1]$.

Clearly, if taking $\alpha = 1$ in Definition 4, then f becomes the standart $m-$logaritmically convex function on $[0, b]$.

We will use the following lemma, [2], in proofs of our main results:

Lemma 1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L[a, b]$, *then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (a-b)\int_0^1 p(t)\, f'(ta + (1-t)b)\,dt \tag{1.2}$$

for each $t \in [0, 1]$, *where*

$$p(t) = \begin{cases} t & , \quad t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1 & , \quad t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases}$$

for all $x \in [a, b]$.

2. Ostrowski Type Inequalities

Theorem 1. *Let* $I \supset [0, \infty)$ *be an open real interval and let* $f : I \to (0, \infty)$ *be a differentiable function on* I *such that* $f' \in L[a, b]$ *for* $0 \leq a < b < \infty$. *If* $|f'(x)|^q$ *is*

$(\alpha, m)-$logaritmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1]^2$ and $|f'(x)| \leq M$, then

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq\ & \frac{(b-a)M^m}{(p+1)^{\frac{1}{p}}} \\
&\times \left[\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}} \left(\frac{\eta^{\frac{\alpha q(b-x)}{b-a}} - 1}{\alpha q \ln \eta} \right)^{\frac{1}{q}} + \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}} \left(\frac{\eta^{\alpha q} - \eta^{\frac{\alpha q(b-x)}{b-a}}}{\alpha q \ln \eta} \right)^{\frac{1}{q}} \right]
\end{aligned}$$

where $\eta = \frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Proof. By using Lemma, Definition 4 and Hölder integral inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq\ & (b-a)\int_0^{\frac{b-x}{b-a}} t\,|f'(ta+(1-t)b)|\,dt + \int_{\frac{b-x}{b-a}}^1 (1-t)\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & (b-a)\left\{ \left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q\,dt \right)^{\frac{1}{q}} \right\} \\
\leq\ & (b-a)\left\{ \left(\int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(a)|^{qt^\alpha} \left| f'\left(\frac{b}{m}\right) \right|^{mq(1-t^\alpha)} dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 |f'(a)|^{qt^\alpha} \left| f'\left(\frac{b}{m}\right) \right|^{mq(1-t^\alpha)} dt \right)^{\frac{1}{q}} \right\} \\
=\ & \frac{(b-a)\left|f'\left(\frac{b}{m}\right)\right|^m}{(p+1)^{\frac{1}{p}}} \left\{ \left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} \eta^{qt^\alpha} dt \right)^{\frac{1}{q}} + \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 \eta^{qt^\alpha} dt \right)^{\frac{1}{q}} \right. .
\end{aligned} \tag{2.1}$$

For $\eta < 1$, we have $\eta^{qt^\alpha} \leq \eta^{\alpha qt}$, thereby

$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} \eta^{qt^\alpha} dt &\leq \int_0^{\frac{b-x}{b-a}} \eta^{\alpha qt} dt \\
&= \frac{\eta^{\frac{\alpha q(b-x)}{b-a}} - 1}{\alpha q \ln \eta}
\end{aligned} \tag{2.2}$$

and

$$\int_{\frac{b-x}{b-a}}^{1} \eta^{qt^{\alpha}} dt \leq \int_{\frac{b-x}{b-a}}^{1} \eta^{\alpha q t} dt = \frac{\eta^{\alpha q} - \eta^{\frac{\alpha q(b-x)}{b-a}}}{\alpha q \ln \eta}. \tag{2.3}$$

If we write (2.2)-(2.3) in (2.1) and then use $|f'(x)| \leq M$, we get the desired result. □

Corollary 1. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L[a,b]$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is $m-$logaritmically convex on $\left[0, \frac{b}{m}\right]$ for $m \in (0,1]$ and $|f'(x)| \leq M$, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq\ & \frac{(b-a) M^m}{(p+1)^{\frac{1}{p}}} \\
&\times \left[\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}} \left(\frac{\eta^{\frac{q(b-x)}{b-a}} - 1}{q \ln \eta} \right)^{\frac{1}{q}} + \left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}} \left(\frac{\eta^{q} - \eta^{\frac{q(b-x)}{b-a}}}{q \ln \eta} \right)^{\frac{1}{q}} \right]
\end{aligned}$$

where η is in Theorem 1.

Corollary 2. *If in Theorem 1, we choose $x = \frac{a+b}{2}$, we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$

$$\leq \frac{(b-a) M^m}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{2}\right)^{1+\frac{1}{p}} \left[\left(\frac{\eta^{\frac{\alpha q}{2}} - 1}{\alpha q \ln \eta} \right)^{\frac{1}{q}} + \left(\frac{\eta^{\alpha q} - \eta^{\frac{\alpha q}{2}}}{\alpha q \ln \eta} \right)^{\frac{1}{q}} \right]$$

which is an Ostrowski type inequality.

Theorem 2. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L[a,b]$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is $(\alpha,m)-$logaritmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha,m) \in (0,1]^2$, $q \geq 1$, $|f'(x)| \leq M$, then*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
\leq\ & (b-a) M^m \left\{ \left[\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2 \right]^{1-\frac{1}{q}} \left[\frac{\eta^{\frac{\alpha q(b-x)}{b-a}} \left(\frac{\alpha q(b-x)}{b-a} \ln \eta - 1 \right) + 1}{\alpha^2 q^2 \ln^2 \eta} \right]^{\frac{1}{q}} \right. \\
&\left. + \left[\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2 \right]^{1-\frac{1}{q}} \left(\frac{\eta^{\alpha q} - \eta^{\frac{\alpha q(b-x)}{b-a}} \left(1 + \frac{\alpha q(x-a)\ln \eta}{b-a} \right)}{\alpha^2 q^2 \ln^2 \eta} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

where η is in Theorem 1.

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Proof. By using Lemma, Definition 4 and Power integral inequality, we have

(2.4)
$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\ & (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt \\
\leq\ & (b-a)\left\{\left(\int_0^{\frac{b-x}{b-a}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & (b-a)\left\{\left[\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\right]^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(a)\right|^{qt^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{mq(1-t^\alpha)}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\right]^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(a)\right|^{qt^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{mq(1-t^\alpha)}dt\right)^{\frac{1}{q}}\right\} \\
=\ & (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\right]^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t\eta^{qt^\alpha}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2\right]^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,\eta^{qt^\alpha}dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

For $\eta<1$, we have $\eta^{qt^\alpha}\leq\eta^{\alpha qt}$, thereby

(2.5)
$$\begin{aligned}
\int_0^{\frac{b-x}{b-a}} t\eta^{qt^\alpha}dt &\leq \int_0^{\frac{b-x}{b-a}} t\eta^{\alpha qt}dt \\
&= \frac{\eta^{\frac{\alpha q(b-x)}{b-a}}\left(\frac{\alpha q(b-x)}{b-a}\ln\eta-1\right)+1}{\alpha^2q^2\ln^2\eta}
\end{aligned}$$

and

(2.6)
$$\begin{aligned}
\int_{\frac{b-x}{b-a}}^1 (1-t)\,\eta^{qt^\alpha}dt &\leq \int_{\frac{b-x}{b-a}}^1 (1-t)\,\eta^{\alpha qt}dt \\
&= \frac{\eta^{\alpha q}-\eta^{\frac{\alpha q(b-x)}{b-a}}\left(1+\frac{\alpha q(x-a)\ln\eta}{b-a}\right)}{\alpha^2q^2\ln^2\eta}.
\end{aligned}$$

If we write (2.5)-(2.6) in (2.4) and then use $|f'(x)|\leq M$, we get the desired result. □

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Corollary 3. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L[a,b]$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is $m-$logaritmically convex on $\left[0, \frac{b}{m}\right]$ for $m \in (0,1]$, $q \ge 1$, $|f'(x)| \le M$, then*

$$\begin{aligned}&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \le\quad & (b-a) M^m \left\{ \left[\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2 \right]^{1-\frac{1}{q}} \left[\frac{\eta^{\frac{q(b-x)}{b-a}} \left(\frac{q(b-x)}{b-a}\ln\eta - 1\right) + 1}{q^2 \ln^2 \eta} \right]^{\frac{1}{q}} \right. \\ & \left. + \left[\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2 \right]^{1-\frac{1}{q}} \left(\frac{\eta^q - \eta^{\frac{q(b-x)}{b-a}} \left(1 + \frac{q(x-a)\ln\eta}{b-a}\right)}{q^2\ln^2\eta} \right)^{\frac{1}{q}} \right\}\end{aligned}$$

where η is in Theorem 1.

Corollary 4. *If in Thorem 2, we choose $x = \frac{a+b}{2}$, we get*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \le\quad & (b-a) M^m \left[\frac{1}{8}\right]^{1-\frac{1}{q}} \\ & \times \left(\left[\frac{\eta^{\frac{\alpha q}{2}} \left(\frac{\alpha q}{2}\ln\eta - 1\right) + 1}{\alpha^2 q^2 \ln^2\eta} \right]^{\frac{1}{q}} + \left(\frac{\eta^{\alpha q} - \eta^{\frac{\alpha q}{2}}\left(1 + \frac{\alpha q \ln\eta}{2}\right)}{\alpha^2 q^2 \ln^2 \eta} \right)^{\frac{1}{q}} \right)\end{aligned}$$

which is an Ostrowski type inequality.

References

[1] A. Ostrowski, *Über die Absolutabweichung einer differentierbaren Funktion von ihren Integralmittelwert*, Comment. Math. Helv., 10, 226-227, (1938).

[2] M. Alomari and M. Darus, Some Ostrowski Type Inequalities For Convex Functions With Applications, *RGMIA Res. Rep. Coll.* 13 (1) (2010), Article 3.

[3] E. Set, M. Sardari, M.E. Özdemir and J. Rooin, On generalizations of the Hadamard inequality for $(\alpha, m)-$convex functions, Kyungpook Math. J., 52 (3) (2012), 307—317.

[4] H. Kavurmacı, M. Avcı and M.E. Özdemir, *New Ostrowski type inequalities for $m-$convex functions and applications*, Hacettepe Journal of Mathematics and Statistics, Volume 40 (2) (2011), 135 – 145.

[5] M.K. Bakula, M. E Özdemir, J. Pečarić, Hadamard type inequalities for $m-$convex and $(\alpha, m)-$convex functions, *J. Inequal. Pure Appl. Math.* 9 (2008), Article 96.

[6] M.K. Bakula, J. Pečarić, M. Ribičić, Companion inequalities to Jensen's inequality for $m-$convex and $(\alpha, m)-$convex functions, *J. Inequal. Pure Appl. Math.* 7 (2006), Article 194.

[7] S.S. Dragomir and G. Toader, Some inequalities for $m-$convex functions, Studia Univ. Babes-bolyai. Mathematica, 38(1993), 1, 21-28.

[8] M.E. Özdemir, H. Kavurmaci, E. Set, Ostrowski's type inequalities for $(\alpha, m)-$convex functions, *Kyungpook Math. J.* 50 (2010) 371–378.

[9] M.E. Özdemir, M. Avcı and E. Set, *On some inequalities of Hermite-Hadamard type via $m-$convexity*, Applied Mathematics Letters, 23 (2010), 1065-1070.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 16, NO.5, 2014, COPYRIGHT 2014 EUDOXUS PRESS, LLC

[10] M.E. Özdemir, M. Avcı and H. Kavurmacı, Hermite–Hadamard-type inequalities via $(\alpha, m)-$convexity, *Computers and Mathematics with Applications*, 61 (2011), 2614–2620.
[11] V.G. Miheşan, A generalization of the convexity, *Seminar of Functional Equations, Approx. and Convex*, Cluj-Napoca (Romania) (1993).
[12] Rui-Fang Bai, Feng Qi, Bo-Yan Xi, Hermite-Hadamard type inequalities for the $m-$ and $(\alpha; m)-$logarithmically convex functions, Filomat 27(2013), 1-7; Available at: http://www.pmf.ni.ac.rs/filomat.
[13] G. Toader, *Some generalization of the convexity*, Proc. Colloq. Approx. *Opt.*, Cluj-Napoca, (1984), 329-338.
[14] G. Toader, *On a generalization of the convexity*, Mathematica, 30 (53) (1988), 83-87.
[15] S.S. Dragomir, *On some new inequalities of Hermite-Hadamard type for $m-$convex functions*, Tamkang Journal of Mathematics, 33 (1) (2002).

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A new version of Mazur-Ulam theorem under weaker conditions in linear n-normed spaces

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Abstract: The purpose of this paper is to prove a new result of Mazur-Ulam theorem for n-isometry without any other conditions in linear n-normed spaces.

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1 Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot, \cdot)$ and $d_Y(\cdot, \cdot)$ denote the metrics in the spaces X and Y, respectively. Two metric spaces X and Y are defined to be isometric if there exists an isometry of X onto Y. In 1932, Mazur and Ulam [1] proved the following theorem.

Mazur-Ulam Theorem. *Every isometry of a real normed linear space onto a real normed linear space is a linear mapping up to translation.*

Baker [2] showed an isometry from a real normed linear space into a strictly convex real normed linear space is affine. Also, Jian [3] investigated the generalizations of the Mazur-Ulam theorem in F^*-spaces. Th.M. Rassias and Wagner [4] described all volume preserving mappings from a real finite dimensional vector space into itself and Väisälä [5] gave a short and simple proof of the Mazur-Ulam theorem. Chu [6] proved that the Mazur-Ulam theorem holds when X is a linear 2-normed space. Chu et al. [7] generalized the Mazur-Ulam theorem when X is a linear n-normed space, that is, the Mazur-Ulam theorem holds, when the n-isometry mapped to a linear n-normed space is affine. They also obtained extensions of the Th.M. Rassias and Šemrl's theorem [8]. The Mazur-Ulam theorem has been extensively studied by many authors (see [9, 10]).

Recently, Moslehian and Sadeghi [11] investigated the Mazur-Ulam theorem in non-Archimedean spaces. Cho et al. [12] investigated the Mazur-Ulam theorem on probabilistic 2-normed spaces. Choy and Ku [13] proved that the barycenter of triangle carries the barycenter of corresponding triangle. They showed the Mazur-Ulam problem on non-Archimedean 2-normed spaces using

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the above statement. Chen and Song [14] introduced the concept of weak n-isometry and then they got under some conditions, a weak n-isometry is also an n-isometry. Alaca [15] gave the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, he gave a new generalization of the Mazur-Ulam theorem when X is a 2-fuzzy 2-normed linear space or $\Im(X)$ is a fuzzy 2-normed linear space. Choy et al. [16] proved the Mazur-Ulam theorem for the interior preserving mappings in linear 2-normed spaces and also proved the theorem on non-Archimedean 2-normed spaces over a linear ordered non-Archimedean field without the strict convexity assumption. Ren [17] showed that every generalized area n preserving mapping between real 2-normed linear spaces X and Y which is strictly convex is affine under some conditions.

In the present paper, we show that every n-isometry without any other conditions from a linear n-normed space to another linear n-normed space is affine. A new version of Mazur-Ulam theorem is proved under much weaker conditions.

2 A new result for Mazur-Ulam theorem

Definition 2.1 ([18]) Let $n \in \mathbb{N}$ and let X be a real linear space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\bullet, ..., \bullet\|$ on $\underbrace{X \times \cdots \times X}_{n}$ satisfying the following properties

(nN$_1$) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1,\ x_2, ..., x_n$ are linearly dependent,

(nN$_2$) $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation,

(nN$_3$) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha|\ \|x_1, x_2, ..., x_n\|$ for any $\alpha \in \mathbb{R}$,

(nN$_4$) $\|x + y, x_2, ..., x_{n-1}, x_n\| \leq \|x, x_2, ..., x_n\| + \|y, x_2, ..., x_n\|$,

is called an *n-norm* on X and the pair $(X, \|\bullet, ..., \bullet\|)$ is called an *linear n-normed space.*

From now on, let X and Y be linear n-normed spaces and $f : X \to Y$ a mapping without special statements.

Chu et al. [19] introduced the concept of n-isometry which is suitable to represent the notion of area preserving mappings in linear n-normed spaces as follows.

For $x_0, x_1, ..., x_n \in X$, $\|x_1 - x_0, ..., x_n - x_0\|$ is called an area of $x_0, x_1, ..., x_n$. We call f an n-isometry if $\|x_1 - x_0, ..., x_n - x_0\| = \|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\|$ for all $x_0, x_1, ..., x_n \in X$.

A version of Mazur-Ulam theorem has been obtained in [7] as follows.

Theorem 2.1 ([7]) Assume that X and Y are linear n-normed spaces. If $f : X \to Y$ is an n-isometry and satisfies $f(x_0), f(x_1), ..., f(x_n)$ are n-collinear when $x_0, x_1, ..., x_n$ are n-collinear, then f is affine.

A natural question is that whether the n-isometry in linear n-normed spaces is also affine without the condition of preserving n-collinearity. In this section, we will find a reply to this question in linear n-normed spaces.

Lemma 2.1 ([19]) Let x_i be an element of a linear n-normed spaces X for every $i \in \{1, ..., n\}$ and $\gamma \in \mathbb{R}$. Then

$$\|x_1, ..., x_i, ..., x_j, ..., x_n\| = \|x_1, ..., x_i, ..., x_j + \gamma x_i, ..., x_n\|$$

for all $1 \leq i \neq j \leq n$.

Lemma 2.2 Let x_0, x_1 be elements of X. Then $u = \frac{x_0+x_1}{2}$ is the unique element of X satisfying

$$\begin{aligned}&\|x_0 - u, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \|x_1 - u, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \frac{1}{2}\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\|\end{aligned}$$

with $\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \neq 0$ and $u \in \{tx_0 + (1-t)x_1 : t \in \mathbb{R}\}$.

Proof. From Lemma 2.1, it is obvious that $u = \frac{x_0+x_1}{2}$ satisfies

$$\begin{aligned}&\|x_0 - u, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \|x_1 - u, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \frac{1}{2}\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\|\end{aligned}$$

with $\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \neq 0$ and $u \in \{tx_0 + (1-t)x_1 : t \in \mathbb{R}\}$.

Now we prove the uniqueness of u. Assume that ν is an element of X satisfying the above properties.

$$\begin{aligned}&\|x_0 - \nu, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \|x_1 - \nu, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = \frac{1}{2}\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\|\end{aligned}$$

with $\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \neq 0$ and $\nu \in \{tx_0 + (1-t)x_1 : t \in \mathbb{R}\}$. Let $\nu = tx_0 + (1-t)x_1$ for some $t \in \mathbb{R}$. From Lemma 2.1, we get

$$\begin{aligned}&\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2\|x_0 - \nu, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2\|x_0 - (tx_0 + (1-t)x_1), x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2|1-t|\,\|x_0 - x_1, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2|1-t|\,\|x_1 - x_n, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\|\end{aligned}$$

and

$$\begin{aligned}&\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2\|x_1 - \nu, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2\|x_1 - (tx_0 + (1-t)x_1), x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2|t|\,\|x_1 - x_0, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \\ &\quad = 2|t|\,\|x_1 - x_n, x_0 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\|.\end{aligned}$$

Since $\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \neq 0$, we have $1 = 2|1-t| = 2|t|$. So $t = \frac{1}{2}$ and $\nu = u = \frac{x_0+x_1}{2}$.

Theorem 2.2 Let X and Y be linear n-normed spaces. If $f : X \to Y$ is an n-isometry, then f is affine.

Proof. Let $g(x) = f(x) - f(0)$. Then g is an n-isometry and $g(0) = 0$. For $x_0, x_1, ..., x_n \in X$ with $\|x_0 - x_n, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n\| \neq 0$,

$$\|g(x_0) - g(x_n), g(x_1) - g(x_n), g(x_2) - g(x_n), ..., g(x_{n-1}) - g(x_n)\| \neq 0.$$

From Lemma 2.1, we get

$$
\begin{aligned}
&\left\|g(x_0) - g\left(\frac{x_0+x_1}{2}\right), g(x_0)-g(x_n), g(x_2)-g(x_n), ..., g(x_{n-1})-g(x_n)\right\| \\
&= \left\|x_0 - \left(\frac{x_0+x_1}{2}\right), x_0-x_n, x_2-x_n, ..., x_{n-1}-x_n\right\| \\
&= \left\|\frac{x_0-x_1}{2}, x_0-x_n, x_2-x_n, ..., x_{n-1}-x_n\right\| \\
&= \frac{1}{2}\left\|x_0-x_1, x_0-x_n, x_2-x_n, ..., x_{n-1}-x_n\right\| \\
&= \frac{1}{2}\left\|x_1-x_n, x_0-x_n, x_2-x_n, ..., x_{n-1}-x_n\right\| \\
&= \frac{1}{2}\left\|g(x_1)-g(x_n), g(x_0)-g(x_n), g(x_2)-g(x_n), ..., g(x_{n-1})-g(x_n)\right\|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\|g(x_1) - g\left(\frac{x_0+x_1}{2}\right), g(x_1)-g(x_n), g(x_2)-g(x_n), ..., g(x_{n-1})-g(x_n)\right\| \\
&= \frac{1}{2}\left\|g(x_1)-g\left(x_n\right), g(x_0)-g(x_n), g(x_2)-g(x_n), ..., g(x_{n-1})-g(x_n)\right\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|g\left(\frac{x_0+x_1}{2}\right) - g(x_1), g(x_0)-g(x_1), g(x_2)-g(x_1), ..., g(x_{n-1})-g(x_1)\right\| \\
&= \left\|\left(\frac{x_0+x_1}{2}\right) - x_1, x_0-x_1, x_2-x_1, ..., x_{n-1}-x_1\right\| \\
&= \frac{1}{2}\left\|x_0-x_1, x_0-x_1, x_2-x_1, ..., x_{n-1}-x_1\right\| \\
&= 0.
\end{aligned}
$$

So

$$g\left(\frac{x_0+x_1}{2}\right) - g(x_1) = t\left(g(x_0)-g(x_1)\right)$$

for some $t \in \mathbb{R}$ by Definition 2.1. That is,

$$g\left(\frac{x_0+x_1}{2}\right) = tg(x_0) + (1-t)g(x_1).$$

By Lemma 2.2,

$$g\left(\frac{x_0+x_1}{2}\right) = \frac{g(x_0)+g(x_1)}{2}$$

for all $x_0, x_1 \in X$. Since $g(0) = 0$, we have

$$g\left(\frac{x_0}{2}\right) = g\left(\frac{x_0+0}{2}\right) = \frac{g(x_0)+g(0)}{2} = g\left(\frac{x_0}{2}\right)$$

and

$$g\left(x_0+x_1\right) = g\left(\frac{2x_0+2x_1}{2}\right) = \frac{g(2x_0)+g(2x_1)}{2} = \frac{g(2x_0)}{2} + \frac{g(2x_1)}{2} = g(x_0)+g(x_1).$$

It follows that g is additive.

Let $r \in \mathbb{R}^+$ and $x_0 \in X$. Since $g(0) = 0$ and g is an n-isometry, we get

$$\begin{aligned}
&\|g(rx_0) - 0, g(x_0) - 0, g(x_2) - 0, ..., g(x_{n-1}) - 0\| \\
&\quad = \|g(rx_0) - g(0), g(x_0) - g(0), g(x_2) - g(0), ..., g(x_{n-1}) - g(0)\| \\
&\quad = \|rx_0 - 0, x_0 - 0, x_2 - 0, ..., x_{n-1} - 0\| \\
&\quad = \|rx_0, x_0, x_2, ..., x_{n-1}\| \\
&\quad = 0.
\end{aligned}$$

So $g(rx_0) = sg(x_0)$ for some $s \in \mathbb{R}$ by Definition 2.1. Since dim $X > 1$, there exists an $x_1 \in X$ such that $\|x_0, x_1, x_2, ..., x_{n-1}\| \neq 0$. It is easy to see that

$$\begin{aligned}
r\|x_0, x_1, x_2, ..., x_{n-1}\| &= \|rx_0, x_1, x_2, ..., x_{n-1}\| \\
&= \|g(rx_0), g(x_1), g(x_2), ..., g(x_{n-1})\| \\
&= \|sg(x_0), g(x_1), g(x_2), ..., g(x_{n-1})\| \\
&= |s|\,\|g(x_0), g(x_1), g(x_2), ..., g(x_{n-1})\| \\
&= |s|\,\|x_0, x_1, x_2, ..., x_{n-1}\|.
\end{aligned}$$

So $s = r$ or $s = -r$. If $s = -r$, then

$$\begin{aligned}
&|r-1|\,\|x_0, x_1, x_2, ..., x_{n-1}\| \\
&\quad = \|(r-1)x_0, x_1, x_2, ..., x_{n-1}\| \\
&\quad = \|rx_0 - x_0, x_1 - 0, x_2 - 0, ..., x_{n-1} - 0\| \\
&\quad = \|(g(rx_0) - g(x_0), g(x_1) - g(0), g(x_2) - g(0), ..., g(x_{n-1}) - g(0)\| \\
&\quad = \|-rg(x_0) - g(x_0), g(x_1) - g(0), g(x_2) - g(0), ..., g(x_{n-1}) - g(0)\| \\
&\quad = (r+1)\,\|g(x_0), g(x_1), g(x_2), ..., g(x_{n-1})\| \\
&\quad = (r+1)\,\|x_0, x_1, x_2, ..., x_{n-1}\|.
\end{aligned}$$

So $|r-1| = r+1$. This is a contradiction since $r \in \mathbb{R}^+$. Thus, $g(rx_0) = rg(x_0)$ for every $r \in \mathbb{R}^+$ and $x_0 \in X$.

Similarly, we can prove $g(rx_0) = rg(x_0)$ for every $r \in \mathbb{R}^-$ and $x_0 \in X$.

Hence, g is linear and f is affine.

Remark 2.1 Theorem 2.1 has been substantially improved by Theorem 2.2.

Remark 2.2 It is clear that the Mazur-Ulam theorem has been proved under much weaker conditions than the main result of Chu et al. [7] in the framework of 2-fuzzy 2-normed linear spaces.

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References

[1] S. Mazur, S. Ulam, Sur les transformationes isométriques d'espaces vectoriels normés, C.R. Acad. Sci. Paris **194** (1932), 946–948.

[2] J.A. Baker, Isometries in normed spaces, Amer. Math. Monthly **78** (1971), 655–658.

[3] W. Jian, On the generations of the Mazur-Ulam isometric theorem, J. Math. Anal. Appl. **263** (2001), 510–521.

[4] Th.M. Rassias, P. Wagner, Volume preserving mappings in the spirit of the Mazur-Ulam theorem, Aequationes Math. **66** (2003), 85–89.

[5] J. Väisälä, A proof of the Mazur-Ulam theorem, Amer. Math. Monthly **110** (2003), 633–635.

[6] H. Chu, On the Mazur-Ulam problem in linear 2-normed spaces, J. Math. Anal. Appl. **327** (2007), 1041–1045.

[7] H. Chu, S. Choi, D. Kang, Mappings of conservative distances in linear n-normed spaces, Nonlinear Analysis–TMA **70** (2009), 1168–1174.

[8] Th.M. Rassias, P. Šemrl, On the Mazur-Ulam problem and the Aleksandrov problem for unit distance preserving mappings, Proc. Amer. Math. Soc. **118** (1993), 919–925.

[9] Th.M. Rassias, On the A.D. Aleksandrov problem of conservative distances and the Mazur-Ulam theorem, Nonlinear Analysis–TMA **47** (2001), 2597–2608.

[10] S. Xiang, Mappings of conservative distances and the Mazur-Ulam theorem, J. Math. Anal. Appl. **254** (2001), 262–274.

[11] M.S. Moslehian, Gh. Sadeghi, A Mazur-Ulam theorem in non-Archimedean normed spaces, Nonlinear Analysis–TMA **69** (2008), 3405–3408.

[12] Y. Cho, F. Rahbarnia, R. Saadati, Gh. Sadeghi, Isometries in probabilistic 2-normed spaces, J. Chungcheong Math. Soc. **22** (2009), 623–634.

[13] J. Choy, S. Ku, Characterization on 2-isometries in non-Archimedean 2-normed spaces, J. Chungcheong Math. Soc. **22** (2009), 65–71.

[14] X.Y. Chen, M.M. Song, Characterizations on isometries in linear n-normed spaces, Nonlinear Analysis–TMA **72** (2010), 1895–1901.

[15] C. Alaca, A new perspective to the Mazur-Ulam problem in 2-fuzzy 2-normed linear spaces, Iranian J. Fuzzy Systems **7** (2010), 109–119.

[16] J. Choy, H. Chu, S. Ku, Characterizations on Mazur-Ulam theorem, Nonlinear Analysis–TMA **72** (2010), 1291–1297.

[17] W. Ren, On the generalized 2-isometry, Rev. Mat. Complut. **23** (2010), 97–104.

[18] Y. Cho, P.C.S. Lin, S. Kim, A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publ., New York, 2001.

[19] H. Chu, K. Lee, C. Park, On the Aleksandrov problem in linear n-normed spaces, Nonlinear Analysis–TMA **59** (2004), 1001–1011.

ON UMBRAL CALCULUS INVOLVING SPECIAL POLYNOMIALS

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Abstract. In this paper, we give some new and interesting identities involving special polynomials which are derived from the transfer formula for the associated sequences.

1. Introduction

Let $\alpha \in \mathbb{R}$, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad \text{(see [1,11,12,18])}. \tag{1}$$

In the special case, $x=0$, $B_n^{(\alpha)}(0)=B_n^{(\alpha)}$ are called the n-th Bernoulli numbers of order α. From (1), we note that

$$B_n^{(\alpha)}(x)=\sum_{l=0}^{n}\binom{n}{l}B_l^{(\alpha)}x^{n-l}. \tag{2}$$

As is well known, the Euler polynomials of of order α are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad \text{(see [2,4,9,13,19,20])}. \tag{3}$$

In the special case, $x=0$, $E_n^{(\alpha)}(0)=E_n^{(\alpha)}$ are called the n-th Euler numbers of order α.

By (3), we get

$$E_n^{(\alpha)}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{n-l}^{(\alpha)}x^{l}, \quad \text{(see [3,7,10,17])}. \tag{4}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}, \quad \text{(see [5,8,16])}. \tag{5}$$

Let us assume that $\mathbf{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbf{P}^*$ be the vector space of all linear functionals on $\mathbf{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$. For $f(t)\in\mathcal{F}$, we define the linear functional $f(t)$ on $\mathbf{P}$ by

$$\langle f(t)|x^n\rangle=a_n, \quad (n\geq 0), \quad \text{(see [6,10,14,15])}. \tag{6}$$

Thus, by (5) and (6), we get

$$\langle t^k | x^n \rangle = n!\delta_{n,k}, \quad (n,k \geq 0), \quad \text{(see [8,15])}, \tag{7}$$

where $\delta_{n,k}$ is the Kronecker symbol.
Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. From (7), we note that

$$\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle. \tag{8}$$

Thus, by (8), we see that $f_L(t) = L$.
The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbf{P}^*$ onto $\mathcal{F}$.
Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional (see [9,15,16]). We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series (see [15,16]). Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k | S_n(x) \rangle = n!\delta_{n,k}$ where $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$. By (7), we easily see that $\langle e^{yt}|p(x)\rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbf{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x) \rangle}{k!} x^k, \quad \text{(see [10,15])}. \tag{9}$$

Thus, by (9), we get

$$p^{(k)}(0) = \langle t^k|p(x)\rangle, \quad \langle 1|p(x)\rangle = p^{(k)}(0). \tag{10}$$

From (10), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \quad \text{(see [9,15,16])}. \tag{11}$$

By (11), we easily get $e^{yt}p(x) = p(x+y)$.
Let $S_n(x) \sim (g(t), f(t))$. Then we see that

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all} \quad y \in \mathbb{C}, \quad \text{(see [8,9,15,16])}, \tag{12}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.
For $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$, we have

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x), \quad \text{(see [8,15,16])}. \tag{13}$$

Now, we introduce several important sequences which are used to derive our results in this paper (see [8,9,10,15,16]):
(The Poisson-Charlier sequences)

$$C_n(x;a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim (e^{a(e^t-1)},\ a(e^t-1)), \tag{14}$$

where $a \neq 0,\ (x)_n = x(x-1)\cdots(x-n+1)$,

$$\sum_{k=0}^{\infty} C_n(k;a)\frac{t^k}{k!}e^{-t} = \left(\frac{t-a}{a}\right)^n,\ (a \neq 0),\ n \in \mathbb{N}, \tag{15}$$

(The Abel sequences)

$$A_n(x;b) = x(x-bn)^{n-1} \sim (1,\ te^{bt}),\ (b \neq 0), \tag{16}$$

(The Mittag-Leffler sequences)

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k \sim \left(1,\ \frac{e^t-1}{e^t+1}\right), \tag{17}$$

(The Laguerre sequences)

$$L_n(x) = \sum_{k=1}^{n}\binom{n-1}{k-1}\frac{n!}{k!}(-x)^k \sim \left(1,\ \frac{t}{t-1}\right). \tag{18}$$

In this paper, we give some new and interesting identities involving special polynomials which are derived from the transfer formula for the associated sequences.

2. On Associated Sequences of Polynomials

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1,\ \frac{t}{(t-1)^a}\right),\quad q_n(x) \sim \left(1,\ \frac{t}{(t+1)^a}\right),\ (a \neq 0). \tag{19}$$

By (13), we easily see that

$$\begin{aligned} p_n(x) &= x(t-1)^{an}x^{n-1} = x\sum_{k=0}^{n-1} C_{an}(k;1)\frac{1}{k!}t^k(x-1)^{n-1} \\ &= x\sum_{k=0}^{n-1} C_{an}(k;1)\binom{n-1}{k}(x-1)^{n-1-k} \\ &= x\sum_{k=0}^{n-1} C_{an}(n-1-k;a)\binom{n-1}{k}(x-1)^k, \end{aligned} \tag{20}$$

and

(21)

$$\begin{aligned} q_n(x) &= x\left(\frac{t}{\frac{t}{(t+1)a}}\right)^n x^{-1}x^n = x(t+1)^{an}x^{n-1} = (-1)^{an}x\sum_{k=0}^{\infty} C_{an}(k;-1)\frac{t^k}{k!}e^{-t}x^{n-1} \\ &= x(-1)^{an}\sum_{k=0}^{n-1} C_{an}(n-1-k;-1)\binom{n-1}{k}(x-1)^k. \end{aligned}$$

From (12) and (17), we can derive the generating function of Mittag-Leffler sequences $M_n(x)$ as follows:

$$\sum_{k=0}^{\infty} M_k(x)\frac{t^k}{k!} = \left(\frac{1+t}{1-t}\right)^x. \tag{22}$$

By (13) and (19), we get

(23)

$$\begin{aligned}
q_n(x) &= x\left(\frac{t+1}{t-1}\right)^{an} x^{-1}p_n(x) = (-1)^{an}x\left(\frac{1+t}{1-t}\right)^{an} x^{-1}p_n(x) \\
&= (-1)^{an}x\left(\sum_{l=0}^{n-1}\frac{M_l(an)}{l!}t^l\right)\left(\sum_{k=0}^{n-1}C_{an}(n-k-1;1)\binom{n-1}{k}(x-1)^k\right) \\
&= (-1)^{an}x\sum_{l=0}^{n-1}\sum_{k=l}^{n-1}\binom{n-1}{k}\binom{k}{l}M_l(an)C_{an}(n-k-1;1)(x-1)^{k-l} \\
&= (-1)^{an}x\sum_{m=0}^{n-1}\sum_{l=0}^{n-1-m}\binom{n-1}{l+m}\binom{l+m}{l}M_l(an)C_{an}(n-1-l-m;1)(x-1)^m.
\end{aligned}$$

Therefore, by (21) and (23), we obtain the following lemma.

Lemma 2.1. *For $n \geq 1$, $0 \leq m \leq n-1$, and $a \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, we have*

$$\binom{n-1}{m}C_{an}(n-1-m;-1) = \sum_{l=0}^{n-1-m}\binom{n-1}{l+m}\binom{l+m}{l}M_l(an)C_{an}(n-1-l-m;1).$$

It is easy to show that

$$xB_{n-1}^{(an)}(x) \sim \left(1,\ \left(\frac{e^t-1}{t}\right)^a t\right), \quad (x)_n \sim (1, e^t-1). \tag{24}$$

For $n \geq 1$, by (13) and (24), we get

$$\begin{aligned}
(x)_n &= x\left(\frac{e^t-1}{t}\right)^{(a-1)n} B_{n-1}^{(an)}(x) \\
&= x(n-1)!\sum_{l=0}^{n-1}\frac{((a-1)n)!S_2(l+(a-1)n,(a-1)n)}{(l+(a-1)n)!(n-1-l)!}B_{n-1-l}^{(an)}(x),
\end{aligned} \tag{25}$$

where $S_2(n,k)$ is the Stiring number of the second kind.
It is known that

$$(x)_n = \sum_{k=0}^{n-1}S_1(n,k+1)x^{k+1},\ (n\in\mathbb{N}), \tag{26}$$

where $S_1(n,k)$ is the Stiring number of the first kind. Therefore, by (2), (25) and (26), we obtain the following theorem.

Theorem 2.2. *For $n \geq 1$, $0 \leq m \leq n-1$, and $a \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, we have*

$$S_1(n,m+1) = (n-1)!\sum_{l=0}^{n-1-m}\frac{((a-1)n)!S_2(l+(a-1)n,(a-1)n)}{(l+(a-1)n)!(n-1-l)!}\binom{n-1-l}{m}B_{n-1-l-m}^{(an)}.$$

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1, t\left(\frac{t}{e^t-1}\right)^a\right),\ (x)_k \sim \left(1, e^t-1\right),\ a\in\mathbb{Z}_+.$$

From (13) and (24), we can derive

$$p_n(x) = x\left(\frac{e^t-1}{t}\right)^{an} x^{n-1} = x\sum_{l=0}^{n-1}\frac{(an)!}{(l+an)!}S_2(l+an,an)(n-1)_l x^{n-1-l}, \tag{27}$$

and, for $n \geq 1$, we have

$$p_n(x) = x\left(\frac{e^t-1}{t}\right)^{(a+1)n} x^{-1}(x)_n = x\left(\frac{e^t-1}{t}\right)^{(a+1)n}(x-1)_{n-1}. \tag{28}$$

By (27) and (28), we get

$$\begin{aligned}&\frac{(an)!(n-1)!}{((a+1)n-m-1)!m!}S_2((a+1)n-m-1,an)\\ &=\sum_{l=0}^{n-1-m}\sum_{k=l+m}^{n-1}(-1)^{k-l-m}\frac{((a+1)n)!l!}{(l+(a+1)n)!}\binom{k}{l}\binom{k-l}{m}\\ &\quad\times S_2(l+(a+1)n,(a+1)n)S_1(n-1,k),\end{aligned} \tag{29}$$

where $n \geq 1$, $0 \leq m \leq n-1$, and $a \in \mathbb{Z}_+$.
Let us consider the following associated sequences:

$$(x)_n \sim (1,\ e^t-1), \quad p_n(x) \sim \left(1,\ t\left(\frac{e^t+1}{2}\right)^a\right),\ a \in \mathbb{Z}_+. \tag{30}$$

From (13) and (30), we can prove the following Exercise.
Exercise. (I) For $n \geq 1$, we have

$$E_{n-1}^{(an)}(x+1) = \sum_{m=0}^{n-1}\sum_{k=0}^{n-1-m}\frac{n!(k+m)!}{(k+n)!m!}S_1(n-1,k+m)S_2(k+n,n)E_m^{(an)}(x).$$

(II) For $n \geq 1$, $0 \leq j \leq n-1$, we have

$$\begin{aligned}&\binom{n-1}{j}\sum_{k=0}^{an}\binom{an}{k}k^{n-1-j}\\ &=\sum_{l=0}^{an}\sum_{m=j}^{n-1}\sum_{k=0}^{m-j}\binom{an}{l}\binom{m-k}{j}(l-1)^{m-k-j}\frac{n!m!}{(k+n)!(m-k)!}S_1(n-1,m)S_2(k+n,n).\end{aligned}$$

(III) For $n \geq 1$, $a \in \mathbb{N}$, $0 \leq j \leq n-1$, we have

$$\begin{aligned}\binom{n-1}{j}\sum_{k=0}^{an}k^{n-1-j} &= \sum_{l=0}^{(a-1)n}\sum_{m=j}^{n-1}\sum_{k=0}^{m-j}\binom{(a-1)n}{l}\binom{m-k}{j}(l-1)^{m-k-j}\\ &\quad\times\frac{n!2^{n+k}m!}{(k+n)!(m-k)!}S_1(n-1,m)S_2(k+n,n).\end{aligned}$$

Let us consider the following associated sequences:

$$(x)_n \sim (1,\ e^t-1), \quad xB_{n-1}^{(an)}(x) \sim \left(1,\ \left(\frac{e^t-1}{t}\right)^a t\right),\ (n,a \in \mathbb{N}). \tag{31}$$

By (13) and (31), we get

(32)
$$
\begin{aligned}
(x)_n &= x\left(\frac{e^t-1}{t}\right)^{(a-1)n} B_{n-1}^{(an)}(x)\\
&= x\sum_{k=0}^{n-1}\frac{((a-1)n)!}{(k+(a-1)n)!}S_2(k+n(a-1),(a-1)n)t^k B_{n-1}^{(an)}(x)\\
&= x\sum_{k=0}^{n-1}\frac{(n-1)!((a-1)n)!}{(k+(a-1)n)!(n-k-1)!}S_2(k+n(a-1),(a-1)n)B_{n-1-k}^{(an)}(x)\\
&= x\sum_{l=0}^{n-1}\Bigg\{\sum_{k=0}^{n-1-l}\binom{n-1-k}{l}B_{n-1-k-l}^{(an)}\frac{(n-1)!((a-1)n)!}{(k+(a-1)n)!(n-k-1)!}\\
&\qquad\qquad \times S_2(k+n(a-1),(a-1)n)\Bigg\}x^l,
\end{aligned}
$$

and, for $n\in\mathbb{N}$, we have

$$(x)_n = x\sum_{k=0}^{n-1}S_1(n,k+1)x^k. \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 2.3. *For $n\geq 1$, $0\leq l\leq n-1$, we have*

$$
\begin{aligned}
&S_1(n,l+1)\\
&=\sum_{k=0}^{n-1-l}\binom{n-1-k}{l}B_{n-1-k-l}^{(an)}\frac{(n-1)!((a-1)n)!}{(k+(a-1)n)!(n-k-1)!}S_2(k+(a-1)n,(a-1)n).
\end{aligned}
$$

In particular, $a=1$,

$$S_1(n,l+1)=\binom{n-1}{l}B_{n-1-l}^{(n)}.$$

From (13) and (31), we can easily derive the following equation:

$$B_{n-1}^{(an)}(x+1)=\sum_{k=0}^{n-1}S_1(n-1,k)B_k^{n(a-1)}(x). \tag{34}$$

Let us consider the following associated sequences:

$$xB_{n-1}^{(an)}(x)\sim\left(1,\ t\left(\frac{e^t-1}{t}\right)^a\right),\quad A_n(x;b)=x(x-bn)^{n-1}\sim(1,te^{bt}), \tag{35}$$

where $n,a\in\mathbb{N}$ and $b\neq 0$. By (13) and (35), we get

(36)
$$
\begin{aligned}
A_n(x;b)=x(x-bn)^{n-1}&=xe^{-bnt}\left(\frac{e^t-1}{t}\right)^{an}B_{n-1}^{(an)}(x)\\
&=xe^{-bnt}\sum_{l=0}^{n-1}\frac{(an)!}{(l+an)!}S_2(l+an,an)(n-1)_l B_{n-1-l}^{(an)}(x)\\
&=x\sum_{l=0}^{n-1}\frac{(an)!(n-1)!}{(l+an)!(n-1-l)!}S_2(l+an,an)B_{n-1-l}^{(an)}(x-nb)
\end{aligned}
$$

Thus, by (36), we get

$$(x-bn)^{n-1}=\sum_{l=0}^{n-1}\frac{(an)!(n-1)!}{(l+an)!(n-1-l)!}S_2(l+an,an)B_{n-1-l}^{(an)}(x-nb).$$

It is not difficult to show that

$$\frac{(e^t-1)^n}{e^{tx}t^n}=\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{n}}x^{k-j}\right\}\frac{t^k}{k!}. \tag{37}$$

By (13), (35) and (37), we see that

(38)

$$\begin{aligned}
A_n(x;b)&=x(x-bn)^{n-1}=x\frac{(e^t-1)^{an}}{e^{nbt}t^{an}}B_{n-1}^{(an)}(x)\\
&=x\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+an,an)}{\binom{j+an}{j}}(nb)^{k-j}\right\}\frac{t^k}{k!}B_{n-1}^{(an)}(x)\\
&=x\sum_{k=0}^{n-1}\left\{\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+an,an)}{\binom{j+an}{j}}(nb)^{k-j}\right\}\frac{(n-1)_k}{k!}B_{n-1-k}^{(an)}(x)\\
&=x\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{\binom{k}{j}\binom{n-1}{k}S_2(j+an,an)}{\binom{j+an}{j}}(-nb)^{k-j}\ B_{n-1-k}^{(an)}(x).
\end{aligned}$$

Therefore, by (38), we obtain the following theorem.

Theorem 2.4. *For $n\geq 1$, $a\in\mathbb{N}$ and $b\neq 0$, we have*

$$(x-bn)^{n-1}=\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{\binom{k}{j}\binom{n-1}{k}}{\binom{j+an}{j}}S_2(j+an,an)(-nb)^{k-j}\ B_{n-1-k}^{(an)}(x).$$

Let

$$xE_{n-1}^{(an)}(x)\sim\left(1,\ t\left(\frac{e^t+1}{2}\right)^a\right),\quad A_n(x;b)=x(x-bn)^{n-1}\sim(1,te^{bt}), \tag{39}$$

where $n,a\in\mathbb{N}$ and $b\neq 0$.
By (13) and (39), we get

$$\begin{aligned}
A_n(x;b)&=xe^{-bnt}\left(\frac{e^t+1}{2}\right)^{an}E_{n-1}^{(an)}(x)\\
&=2^{-an}x\sum_{l=0}^{an}\binom{an}{l}e^{(l-nb)t}E_{n-1}^{(an)}(x)\\
&=2^{-an}x\sum_{l=0}^{an}\binom{an}{l}E_{n-1}^{(an)}(x+l-nb).
\end{aligned}\tag{40}$$

Thus, from (40), we have

$$(x-bn)^{n-1}=2^{-an}\sum_{l=0}^{an}\binom{an}{l}E_{n-1}^{(an)}(x+l-nb). \tag{41}$$

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Let us assume that

$$xB_{n-1}^{(an)}(x) \sim \left(1,\ t\left(\frac{e^t-1}{t}\right)^a\right), \quad M_n(x) \sim \left(1,\ \frac{e^t-1}{e^t+1}\right), \tag{42}$$

where $n, a \in \mathbb{N}$.
By (13) and (42), we get

(43)

$$\begin{aligned} M_n(x) &= x\left(\frac{e^t-1}{t}\right)^{(a-1)n}(e^t+1)^n B_{n-1}^{(an)}(x) \\ &= x\sum_{l=0}^{n}\binom{n}{l}\left(\frac{e^t-1}{t}\right)^{(a-1)n} e^{lt} B_{n-1}^{(an)}(x) \\ &= x\sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}^{n-1}\frac{((a-1)n)!}{(k+(a-1)n)!}S_2(k+(a-1)n,(a-1)n)\frac{(n-1)!}{(n-k-1)!} \\ &\times B_{n-1-k}^{(an)}(x+l). \end{aligned}$$

Therefore, by (43), we obtain the following proposition.

Proposition 2.5. *For $n, a \in \mathbb{N}$, we have*

$$M_n(x) = x\sum_{l=0}^{n}\sum_{k=0}^{n-1}\binom{n}{l}\frac{((a-1)n)!S_2(k+(a-1)n,(a-1)n)(n-1)!}{(k+(a-1)n)!(n-k-1)!}B_{n-1-k}^{(an)}(x+l).$$

From (13) and (42), we note that

$$xB_{n-1}^{(an)}(x) = x\left(\frac{t}{e^t-1}\right)^{(a-1)n}\left(\frac{1}{e^t+1}\right)^n x^{-1}M_n(x), \tag{44}$$

and, by (17), we get

$$\begin{aligned} x^{-1}M_n(x) &= x^{-1}\sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k \\ &= \sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x-1)_{k-1} \\ &= \sum_{k=1}^{n}\binom{n}{k}(n-1)_{n-k}2^k\sum_{l=0}^{k-1}S_1(k-1,l)(x-1)^l. \end{aligned} \tag{45}$$

From (44), we have

$$\left(\frac{e^t-1}{t}\right)^{(a-1)n}B_{n-1}^{(an)}(x) = 2^{-n}\left(\frac{2}{e^t+1}\right)^n x^{-1}M_n(x). \tag{46}$$

$$\begin{aligned} \text{RHS of (46)} &= \sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)\left(\frac{2}{e^t+1}\right)^n(x-1)^l \\ &= \sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)E_l^{(n)}(x-1), \end{aligned} \tag{47}$$

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and

(48)

$$\begin{aligned}\text{LHS of (46)} &= \sum_{k=0}^{n-1} \frac{((a-1)n)!}{(k+(a-1)n)!} S_2(k+(a-1)n,(a-1)n) t^k B_{n-1}^{(an)}(x) \\ &= \sum_{k=0}^{n-1} \frac{((a-1)n)!(n-1)!}{(k+(a-1)n)!(n-1-k)!} S_2(k+(a-1)n,(a-1)n) B_{n-1-k}^{(an)}(x).\end{aligned}$$

Therefore, by (46), (47) and (48), we obtain the following proposition.

Proposition 2.6. *For $n \geq 1$, we have*

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{((a-1)n)!(n-1)!}{(k+(a-1)n)!(n-1-k)!} S_2(k+(a-1)n,(a-1)n) B_{n-1-k}^{(an)}(x) \\ &= \sum_{k=1}^{n}\sum_{l=0}^{k-1} \binom{n}{k} (n-1)_{n-k} 2^{k-n} S_1(k-1,l) E_l^{(n)}(x-1).\end{aligned}$$

Let

(49) $$p_n(x) \sim \left(1,\ t\left(\frac{t}{e^t-1}\right)^a\right), \quad M_n(x) \sim \left(1,\ \frac{e^t-1}{e^t+1}\right).$$

For $n \geq 1$, by (13) and (49), we get

(50) $$M_n(x) = x\left(\frac{t}{e^t-1}\right)^{(a+1)n} (e^t+1)^n x^{-1} p_n(x),$$

and

(51) $$\begin{aligned}p_n(x) &= x\left(\frac{t}{t\left(\frac{t}{e^t-1}\right)^a}\right)^n x^{-1}x^n = x\left(\frac{e^t-1}{t}\right)^{an} x^{n-1} \\ &= x\sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!} S_2(l+an,an)(n-1)_l x^{n-1-l}.\end{aligned}$$

From (50) and (51), we can derive

$$M_n(x) = \sum_{k=0}^{n}\sum_{l=0}^{n-1} \binom{n}{k} \frac{(an)!(n-1)!}{(l+an)!(n-1-l)!} S_2(l+an,an) x B_{n-1-l}^{((a+1)n)}(x+k).$$

Let us assume that

(52) $$M_n(x) \sim \left(1,\ \frac{e^t-1}{e^t+1}\right), \quad p_n(x) \sim \left(1,\ t\left(\frac{2}{e^t+1}\right)^a\right),\ (a \in \mathbb{N}).$$

It is easy to see that

(53) $$\begin{aligned}p_n(x) &= x\left(\frac{t}{t\left(\frac{2}{e^t+1}\right)^a}\right)^n x^{-1}x^n = x2^{-an}(e^t+1)^{an}x^{n-1} \\ &= x2^{-an}\sum_{l=0}^{an}\binom{an}{l} e^{lt}x^{n-1} = x2^{-an}\sum_{l=0}^{an}\binom{an}{l}(x+l)^{n-1}.\end{aligned}$$

For $n \geq 1$, by (13) and (52), we get

$$p_n(x) = 2^{-an} x \left(\frac{e^t-1}{t}\right)^n (e^t+1)^{(a-1)n} x^{-1} M_n(x). \tag{54}$$

By (53) and (54), we get

$$\sum_{k=0}^{an} \binom{an}{k} (x+k)^{n-1} = \left(\frac{e^t-1}{t}\right)^n (e^t+1)^{(a-1)n} x^{-1} M_n(x). \tag{55}$$

Thus, from (55), we have

$$\begin{aligned}
&\sum_{k=0}^{an} \binom{an}{k} \left(\frac{t}{e^t-1}\right)^n (x+k)^{n-1} = (e^t+1)^{(a-1)n} x^{-1} M_n(x) \\
&= (e^t+1)^{(a-1)n} \sum_{k=1}^{n} \binom{n}{k} (n-1)_{n-k} 2^k x^{-1} (x)_k \\
&= \sum_{m=0}^{(a-1)n} \sum_{k=1}^{n} \sum_{l=0}^{k-1} \binom{n}{k} \binom{(a-1)n}{m} (n-1)_{n-k} 2^k S_1(k-1,l) e^{mt} (x-1)^l.
\end{aligned} \tag{56}$$

Therefore, by (56), we obtain the following proposition.

Proposition 2.7. *For $n \geq 1$, $a \in \mathbb{N}$ we have*

$$\begin{aligned}
&\sum_{k=0}^{an} \binom{an}{k} B_{n-1}^{(n)}(x+k) \\
&= \sum_{k=1}^{n} \sum_{l=0}^{k-1} \sum_{m=0}^{(a-1)n} \binom{n}{k} \binom{(a-1)n}{m} (n-1)_{n-k} 2^k S_1(k-1,l)(x+m-1)^l.
\end{aligned}$$

From (45) and (55), we can also derive the following identity:

$$\begin{aligned}
&\sum_{k=0}^{an} \binom{an}{k} (x+k)^{n-1} = \left(\frac{e^t-1}{t}\right)^n (e^t+1)^{(a-1)n} \sum_{k=1}^{n} \binom{n}{k} (n-1)_{n-k} 2^k (x-1)_{k-1} \\
&= \left(\frac{e^t-1}{t}\right)^n (e^t+1)^{(a-1)n} \sum_{k=1}^{n} \sum_{l=0}^{k-1} \binom{n}{k} (n-1)_{n-k} 2^k S_1(k-1,l)(x-1)^l \\
&= \left(\frac{e^t-1}{t}\right)^n \sum_{k=1}^{n} \sum_{l=0}^{k-1} \sum_{m=0}^{(a-1)n} \binom{n}{k} \binom{(a-1)n}{m} (n-1)_{n-k} 2^k S_1(k-1,l)(x+m-1)^l \\
&= n!(n-1)! \sum_{k=1}^{n} \sum_{l=0}^{k-1} \sum_{m=0}^{(a-1)n} \sum_{r=0}^{l} \frac{\binom{n}{k}\binom{(a-1)n}{m} 2^k S_1(k-1,l)}{(r+n)!(k-1)!(l-r)!} S_2(r+n,n) l! (x+m-1)^{l-r}.
\end{aligned} \tag{57}$$

For $n \geq 1$, by (13), (52) and (53), we get

$$M_n(x) = 2^n x \left(\frac{t}{e^t-1}\right)^n \left(\frac{2}{e^t+1}\right)^{(a-1)n} x^{-1} p_n(x). \tag{58}$$

Thus, by (53) and (58), we see that

$$2^{-n}\left(\frac{e^t-1}{t}\right)^n x^{-1}M_n(x)=\left(\frac{2}{e^t+1}\right)^{(a-1)n}2^{-an}\sum_{k=0}^{an}\binom{an}{k}(x+k)^{n-1}. \tag{59}$$

By (59), we get

(60)

$$\sum_{k=0}^{an}\binom{an}{k}E_{n-1}^{((a-1)n)}(x+k)=2^{(a-1)n}\left(\frac{e^t-1}{t}\right)^n x^{-1}M_n(x)$$
$$=2^{(a-1)n}\sum_{k=1}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^kS_1(k-1,l)\left(\frac{e^t-1}{t}\right)^n(x-1)^l$$
$$=n!(n-1)!2^{(a-1)n}\sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{m=0}^{l}\frac{\binom{n}{k}2^kl!S_2(m+n,n)}{(k-1)!(m+n)!(l-m)!}S_1(k-1,l)(x-1)^{l-m}.$$

Therefore, by (60), we obtain the following proposition.

Proposition 2.8. *For $a,n\in\mathbb{N}$, we have*

$$\sum_{k=0}^{an}\binom{an}{k}E_{n-1}^{((a-1)n)}(x+k)$$
$$=n!(n-1)!2^{(a-1)n}\sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{m=0}^{l}\frac{\binom{n}{k}2^kl!S_2(m+n,n)}{(k-1)!(m+n)!(l-m)!}S_1(k-1,l)(x-1)^{l-m}.$$

For $\lambda\neq 0$, let us consider the following associated sequences:

$$p_n(x)\sim\left(1,\ \frac{t}{1-\lambda(e^t-1)}\right),\quad x^n\sim(1,t). \tag{61}$$

From (61), we can derive

$$p_n(x)=x(1-\lambda(e^t-1))^nx^{n-1}=x\sum_{k=0}^{n}\binom{n}{k}(1+\lambda)^{n-k}(-\lambda)^k(x+k)^{n-1}. \tag{62}$$

By (13) and (61), we get

$$x^n=x\left(\frac{1}{1-\lambda(e^t-1)}\right)^n x^{-1}p_n(x). \tag{63}$$

From Boyadzhiev, we get

$$\frac{1}{1-\lambda(e^t-1)}=\sum_{l=0}^{\infty}W_l(\lambda)\frac{t^l}{l!}, \tag{64}$$

where $W_n(x)=\sum_{k=0}^{n}k!S_2(n,k)x^k$, (see [15,16]).
Thus, by (64), we get

$$\left(\frac{1}{1-\lambda(e^t-1)}\right)^n=\sum_{l=0}^{\infty}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}W_{l_1}(\lambda)\cdots W_{l_n}(\lambda)\right\}\frac{t^l}{l!}. \tag{65}$$

For $n \geq 1$, by (63) and (65), we see that

(66)
$$x^{n-1} = \sum_{l=0}^{n-1}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}\left(\prod_{i=1}^{n}W_{l_i}(\lambda)\right)\frac{t^l}{l!}\right\}\sum_{k=0}^{n}\binom{n}{k}(1+\lambda)^{n-k}(-\lambda)^k(x+k)^{n-1}$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}\binom{n}{k}\binom{n-1}{l}(1+\lambda)^{n-k}(-\lambda)^k\left(\prod_{i=1}^{n}W_{l_i}(\lambda)\right) \times (x+k)^{n-1-l}.$$

Therefore, by (60), we obtain the following theorem.

Theorem 2.9. *For $n \geq 1$, we have*

$$x^{n-1} = \sum_{k=0}^{n}\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}\binom{n}{k}\binom{n-1}{l}(1+\lambda)^{n-k}(-\lambda)^k\left(\prod_{i=1}^{n}W_{l_i}(\lambda)\right) \times (x+k)^{n-1-l}.$$

The Boole polynomials $Bl_n(x;\lambda)$ are Sheffer sequences for $(1+e^{\lambda t},\ e^t-1)$. That is

$$Bl_n(x;\lambda) \sim (1+e^{\lambda t},\ e^t-1), \quad \text{(see [15,16])}. \tag{67}$$

From (12) and (67), we can derive the generating function of Boole sequences as follows:

$$\sum_{k=0}^{\infty}\frac{Bl_k(x;\lambda)}{k!}t^k = \frac{(1+t)^x}{(1+t)^\lambda+1}. \tag{68}$$

For $\lambda = 0$, we have $Bl_n(x;0) = (x)_n$.
Let us consider the following associated sequences:

$$S_n^\mu(x) \sim \left(1,\ \frac{t}{(1+t)^\mu}\right). \tag{69}$$

By (13) and (69), we get

$$\begin{aligned} S_n^\mu(x) &= x\left(\frac{t}{\left(\frac{t}{(1+t)^\mu}\right)}\right)^n x^{-1}x^n = x(1+t)^{\mu n}x^{n-1} \\ &= \sum_{k=0}^{n-1}\binom{\mu n}{k}(n-1)_k x^{n-k} = \sum_{k=1}^{n}\binom{\mu n}{n-k}\frac{(n-1)!}{(k-1)!}x^k. \end{aligned} \tag{70}$$

For $\lambda = 1$, we note that $S_n^1(x) = S_n(x) = L_n(-x)$.
Let us assume that

$$t_n(x) \sim \left(1,\ \frac{t}{1+(1+t)^\lambda}\right). \tag{71}$$

By (13) and (71), we get

(72)

$$\begin{aligned} t_n(x) &= x\left(\frac{t}{\left(\frac{t}{1+(1+t)^{\lambda}}\right)}\right)^n x^{-1}x^n = x(1+(1+t)^{\lambda})^n x^{n-1} \\ &= x\sum_{a=0}^{n}\binom{n}{a}(1+t)^{\lambda a}x^{n-1} = x\sum_{a=0}^{n}\binom{n}{a}\sum_{b=0}^{n-1}\binom{\lambda a}{b}t^b x^{n-1} \\ &= \sum_{a=0}^{n}\sum_{b=1}^{n}\binom{n}{a}\binom{\lambda a}{n-b}(n-1)_{n-b}x^b = \sum_{a=0}^{n}\sum_{b=1}^{n}\binom{n}{a}\binom{\lambda a}{n-b}\frac{(n-1)!}{(b-1)!}x^b. \end{aligned}$$

From (13), (69) and (71), we can derive

(73)

$$\begin{aligned} &S_n^{\mu}(x) = x\left(\frac{(1+t)^{\mu}}{1+(1+t)^{\lambda}}\right)^n x^{-1}t_n(x) = x\left(\sum_{l=0}^{\infty}\frac{Bl_l(\mu;\lambda)}{l!}t^l\right)^n x^{-1}t_n(x) \\ &= x\sum_{l=0}^{\infty}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}\left(\prod_{i=1}^{n}Bl_{l_i}(\mu;\lambda)\right)\sum_{a=0}^{n}\sum_{b=1}^{n}\binom{n}{a}\binom{\lambda a}{n-b}\frac{(n-1)!}{(b-1)!}\right\}\frac{1}{l!}t^l x^{b-1} \\ &= \sum_{a=0}^{n}\left\{\sum_{b=1}^{n}\sum_{l=0}^{b-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\cdots,l_n}\left(\prod_{i=1}^{n}Bl_{l_i}(\mu;\lambda)\right)\binom{n}{a}\binom{\lambda a}{n-b}\binom{b-1}{l}\frac{(n-1)!}{(b-1)!}\right\}x^{b-l} \\ &= \sum_{k=1}^{n}\left\{\sum_{a=0}^{n}\sum_{b=k}^{n}\sum_{l_1+\cdots+l_n=b-k}\binom{n}{a}\binom{\lambda a}{n-b}\binom{b-k}{l_1,\cdots,l_n}\binom{b-1}{k-1}\frac{(n-1)!}{(b-1)!}\left(\prod_{i=1}^{n}Bl_{l_i}(\mu;\lambda)\right)\right\}x^k \end{aligned}$$

Therefore, by (70) and (73), we obtain the following proposition.

Proposition 2.10. *For $n \in \mathbb{N}$, $1 \le k \le n$ we have*

$$\begin{aligned} &\binom{\mu n}{n-k}\frac{(n-1)!}{(k-1)!} \\ &= \sum_{a=0}^{n}\sum_{b=k}^{n}\sum_{l_1+\cdots+l_n=b-k}\binom{n}{a}\binom{\lambda a}{n-b}\binom{b-k}{l_1,\cdots,l_n}\binom{b-1}{k-1}\frac{(n-1)!}{(b-1)!}\left(\prod_{i=1}^{n}Bl_{l_i}(\mu;\lambda)\right). \end{aligned}$$

Remark Let

$$x^{(n)} = x(x+1)\cdots(x+n-1) \sim (1,\ 1-e^{-t}). \tag{74}$$

Then the generating function of $x^{(n)}$ is given by

$$\sum_{k=0}^{\infty}x^{(k)}\frac{t^k}{k!} = \left(\frac{1}{1-t}\right)^x. \tag{75}$$

The Stirling polynomials $St_n(x)$ are Sheffer sequences for $(e^{-t},\ f(t))$, where $\bar{f}(t) = \log(\frac{t}{1-e^{-t}})$.

$$St_n(x) \sim (e^{-t},\ f(t)), \quad \text{where } \bar{f}(t) = \log\left(\frac{t}{1-e^{-t}}\right). \tag{76}$$

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Thus, from (12) and (76), we have

$$\sum_{k=0}^{\infty} St_k(x)\frac{t^k}{k!} = \left(\frac{t}{1-e^{-t}}\right)^{x+1}. \tag{77}$$

Indeed, we note that

$$St_k(x) = B_k^{(x+1)}(x+1), \quad \text{for} \ \ k \geq 0. \tag{78}$$

For $n \geq 1$, we have

$$\begin{aligned} x^{(n)} &= x\left(\frac{t}{1-e^{-t}}\right)^n x^{-1}x^n = x\left(\frac{t}{1-e^{-t}}\right)^n x^{n-1} \\ &= x\sum_{k=0}^{n-1}\frac{St_k(n-1)}{k!}(n-1)_k x^{n-1-k} = \sum_{k=1}^{n}\binom{n-1}{k-1}St_{n-k}(n-1)x^k. \end{aligned} \tag{79}$$

As $x^{(n)} = \sum_{k=1}^{n} |S_1(n,k)|\, x^k$, we also have : for $1 \leq k \leq n$,

$$|S_1(n,k)| = \binom{n-1}{k-1}St_{n-k}(n-1).$$

Let us consider the following associated sequences:

$$\begin{aligned} A_n(x;a) = x(x-an)^{n-1} \ &\sim \ (1, \ te^{at}), \\ \left(\frac{x}{a}\right)_n \ &\sim \ (1, \ e^{at}-1), \ (a \neq 0). \end{aligned} \tag{80}$$

For $n \geq 1$, by (13) and (80), we get

$$\left(\frac{x}{a}\right)_n = \sum_{k=0}^{n-1} a^{k-n}\binom{n-1}{k}St_k(n-1)x(x-an)^{n-1-k}. \tag{81}$$

Let

$$p_n(x) \ \sim \ \left(1, \ t\left(\frac{t-b}{b}\right)\right), \quad x^{(n)} \ \sim \ (1, \ 1-e^{-t}), \quad (b \neq 0). \tag{82}$$

From (13) and (82), we have

$$\begin{aligned} p_n(x) &= x\left(\frac{b}{t-b}\right)^n x^{-1}x^n = (-1)^n x\left(1-\frac{t}{b}\right)^{-n} x^{n-1} \\ &= (-1)^n x\sum_{l=0}^{n-1}\binom{-n}{l}(-1)^l b^{-l}t^l x^{n-1} \\ &= (-1)^n x\sum_{l=0}^{n-1}\binom{n+l-1}{l}b^{-l}\frac{(n-1)!}{(n-l-1)!}x^{n-1-l} \\ &= (-1)^n x\sum_{l=0}^{n-1}\binom{2n-l-2}{n-1}b^{-(n-1-l)}\frac{(n-1)!}{l!}x^l. \end{aligned} \tag{83}$$

By (13) and (81), we get

$$\begin{aligned} x^{(n)} &= x\left(\frac{t(\frac{t-b}{b})}{1-e^{-t}}\right)^n x^{-1}p_n(x) \\ &= x\left(\frac{t-b}{b}\right)^n\left(\frac{t}{1-e^{-t}}\right)^n(-1)^n\sum_{l=0}^{n-1}\binom{2n-l-2}{n-1}\frac{(n-1)!}{l!b^{n-1-l}}x^l \\ &= (-1)^n\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{2n-l-2}{n-1}\binom{l}{k}\frac{(n-1)!}{l!b^{n-1-l}}St_k(n-1)x\left(\frac{t-b}{b}\right)^n x^{l-k}. \end{aligned} \tag{84}$$

From (15) and (84), we can derive

$$\begin{aligned} x^{(n)} =&(-1)^n\sum_{l=0}^{n-1}\sum_{k=0}^{l}\sum_{r=0}^{l-k}\binom{2n-l-2}{n-1}\binom{l}{k}\binom{l-k}{r}\frac{(n-1)!}{l!b^{n-1-l}}St_k(n-1)C_n(l-k-r;b) \\ &\times x(x-1)^r. \end{aligned} \tag{85}$$

Let us assume that

$$q_n(x) \sim \left(1,\ t\left(\frac{t-b}{b}\right)e^{at}\right), \quad \left(\frac{x}{a}\right)_n \sim (1,\ e^{at}-1), \quad (a,b\neq 0). \tag{86}$$

We easily see that

$$\begin{aligned} q_n(x) &= x\left(\frac{t}{t(\frac{t-b}{b})e^{at}}\right)^n x^{-1}x^n = xe^{-nat}\left(\frac{t-b}{b}\right)^{-n}x^{n-1} \\ &= xe^{-nat}(-1)^n\left(1-\frac{t}{b}\right)^{-n}x^{n-1} = xe^{-ant}(-1)^n\sum_{l=0}^{n-1}\binom{n+l-1}{l}b^{-l}(n-1)_l x^{n-1-l} \\ &= (-1)^n\sum_{l=0}^{n-1}\binom{2n-l-2}{n-1}\frac{(n-1)!}{b^{n-1-l}l!}x(x-an)^l. \end{aligned} \tag{87}$$

From (13), (86) and (87), we can prove the following exercise.

Exercise For $n \geq 1$, $a, b \neq 0$, we have

$$\begin{aligned} \left(\frac{x}{a}\right)_n =&(-a)^{-n}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\sum_{r=0}^{l-k}\binom{2n-l-2}{n-1}\binom{l}{k+r}\binom{k+r}{k}\frac{(n-1)!}{l!b^{n-1-l}}St_{l-k-r}(n-1) \\ &\times C_n(k;b)x(x-na-1)^r. \end{aligned}$$

References

[1] S. Araci, M. Acikgoz, A. Esi, *A note on the q-Dedekind-type Daehee-Changhee sums with weight alpha arising from modified q-Genocchi polynomials with weight alpha*, arXiv:1211.2350.
[2] S. Araci, M. Acikgoz, *Extended q-Dedekind-type Daehee-Changhee sums associated with Extended q-Euler polynomials*, arXiv:1211.1233.
[3] S. Araci, E Sen, M. Acikgoz, *A note on the modified q-Dedekind sums*, arXiv:1212.5837.

[4] L. Carlitz, *Some generating functions for Laguerre polynomials*, Duke Math. J. **35** (1968) 825-827.

[5] D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. **20** (2010), no. 1, 7-21.

[6] Q. Fang, T. Wang, *Umbral calculus and invariant sequences*, Ars Combin. **101** (2011), 257–264.

[7] T. Kim, *On the von Staudt-Clausen's theorem for the q-Euler numbers*, Russ. J. Math. Phys. **20** (2013), no. 1, 34-39.

[8] D. S. Kim, T.Kim, S.-H. Lee, S.-H. Rim, *Frobenius-Euler polynomials and umbral calculus in the p-adic case,* Adv. Stud. Contemp. Math. 2012, 2012:222.

[9] D. S. Kim, T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, Journal of Inequalities and Applications 2012, 2012:307.

[10] D. S. Kim, T. Kim, *Applications of Umbral Calculus Associated with p-Adic Invariant Integrals on* $\mathbb{Z}_p$, Abstract and Applied Analysis 2012 (2012), Article ID **865721**, 12 pages.

[11] D. S. Kim, T. Kim, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Advances in Difference Equations 2012, 2012:196.

[12] T. Kim, S.-H. Rim, D. V. Dolgy, S.-H. Lee, *Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials*, Advances in Difference Equations 2012, 2012:201.

[13] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16** (2009), no. 4, 484-491.

[14] T. J. Robinson, *Formal calculus and umbral calculus*, Electron. J. Combin. 17 (2010), no. **1** , Research Paper 95, 31 pp.

[15] S. Roman, *More on the umbral calculus, with emphasis on the q-umbral calculus*, J. Math. Anal. Appl.107 (1985), 222-254. MR0786026 (86h:05024)

[16] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

[17] C. S. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math. **21** (2011), no. 2, 217–223.

[18] C. S. Ryoo, T. Kim, R. P. Agarwal, *Exploring the multiple Changhee q-Bernoulli polynomials*, Int. J. Comput. Math. **82** (2005), no. 4, 483-493.

[19] J. Sandor, *An additive analogue of the Euler minimum function*, Adv. Stud. Contemp. Math. **10** (2005), no. 1, 53-62.

[20] Y. Simsek, S.-H. Rim, L.-C. Jang, D.-J. Kang, J.-J. Seo, *A note on q-Daehee sums*, J. Anal. Comput. **1** (2005), no. 2, 151-160.

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Semilocal convergence theorem by using majorizing functions for Harmonic mean Newton's method in Banach spaces

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Abstract: In this paper, we study the semilocal convergence of the Harmonic mean Newton's method for solving nonlinear equations in Banach spaces. We establish the Newton-Kantorovich-type convergence theorem for the method by using majorizing functions. We obtain an existence-uniqueness theorem and an error estimate. In comparison with the results obtained in Chen et al., we can provide a larger convergence radius. Finally, some numerical applications is presented to demonstrate our approach.

Keywords: Nonlinear equation; Banach spaces; Majorizing functions; Semilocal convergence; Newton's method; A priori error bounds

AMS classifications: 65D10; 65D99; 47H17

1 Introduction

Solving nonlinear operator equation is an important issue in the engineering and technology field. In this study, we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice-order Fréchet differentiable operator defined on an open convex subset Ω of a Banach space X with values in a Banach space Y. This equation can represent differential equations, integral equations or a system of nonlinear equations in the simplest case.

There are kinds of methods to find a solution of equation (1.1). Iterative methods are often used to solve this problem [10]. The Newton's method which has quadratic convergence is the most well known iterative method. Recently, a lot of research has been carried out to provide improvements in these methods. Third-order iterative methods such as Halley's method, Chebyshev's method, super-Halley's method and Newton-like methods [3, 4, 8, 12, 13] are used to solve equation (1.1). The convergence of these iterative methods in Banach spaces is established by using recurrence relations. An alternative approach is developed to establish the convergence by using majorizing functions. The approach is also a very popular technique to establish the convergence of iterative methods. For example, it has been successfully applied to the convergence analysis of Newton's method and some high-order methods [1, 2, 5, 6, 7, 14, 15].

Our goals in this paper is to increase the speed of convergence of Newton's method and not to increase its operational cost very much. Taking into account these goals, we consider a multipoint Newton-type method called the Harmonic mean Newton's method studied by Özban [11] and Homeier [9]. This method

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is defined for all $n \geq 0$ by

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}[\Gamma_n + \overline{\Gamma}_n]F(x_n), \end{aligned} \tag{1.2}$$

where $\Gamma_n = F^{'}(x_n)^{-1}$ and $\overline{\Gamma}_n = F^{'}(y_n)^{-1}$.

Recently, Chen et al. [6] use recurrence relations to establish the convergence of third-order Harmonic mean Newton's method for solving nonlinear operator equations (1.1). In this paper, we apply majorizing functions to establish the semilacal convergence of the method to solve nonlinear equations in Banach spaces. We prove the Newton-Kantorovich-type convergence theorem, along with a priori error bounds, which demonstrates the R-order convergence of the method. In comparison with the results obtained in [6], we can provide a larger convergence radius. And in the process of the proving, we find that the R-order of Harmonic mean Newton's method can be reached at four. It is an interesting discover.

The paper is organized as follows. Section 1 is the introduction. The convergence analysis based on majorizing functions is given in Section 2. In Section 3, some numerical examples are worked out and some simple comparisons are made. Finally, conclusions form Section 4.

2 Analysis of convergence

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator, where Ω is an open convex domain. The Harmonic mean Newton's method to solve the equation (1.1) given by (1.2) can be written in the following form:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ H(x_n, y_n) &= \overline{\Gamma}_n[F^{'}(y_n) - F^{'}(x_n)], \\ x_{n+1} &= y_n - \frac{1}{2}H(x_n, y_n)(y_n - x_n), \quad n = 0, 1, 2, \cdots, \end{aligned} \tag{2.3}$$

where $\Gamma_n = F^{'}(x_n)^{-1}$ and $\overline{\Gamma}_n = F^{'}(y_n)^{-1}$.

Let $x_0 \in \Omega$, we assume that

(C1) $\|\Gamma_0\| \leq \beta$,

(C2) $\|\Gamma_0 F(x_0)\| \leq \eta$,

(C3) $\|F^{''}(x)\| \leq M, \quad x \in \Omega$,

(C4) there exists a positive real number N such that

$$\|F^{''}(x) - F^{''}(y)\| \leq N\|x - y\|, \quad \forall x, y \in \Omega.$$

We denote

$$g(t) = \frac{1}{2}Kt^2 - \frac{t}{\beta} + \frac{\eta}{\beta},$$

where K, β, η, M and N are positive real numbers and

$$K \geq M + \frac{5N}{3M\beta}. \tag{2.4}$$

Let $h = K\beta\eta$. When $h \leq \frac{1}{2}$, $t^* = \frac{\eta}{h}(1-\sqrt{1-2h})$ and $t^{**} = \frac{\eta}{h}(1+\sqrt{1-2h})$ are two positive roots of $g(t)$. Let

$$\begin{aligned} s_n &= t_n - \frac{g(t_n)}{g'(t_n)}, \quad t_0 = 0, \\ h_n &= g'(s_n)^{-1}[g'(s_n) - g'(t_n)], \\ t_{n+1} &= s_n + \frac{1}{2}h_n\frac{g(t_n)}{g'(t_n)}. \end{aligned} \tag{2.5}$$

First, we have the following lemmas.

Lemma 2.1. *Suppose the sequences $\{t_n\}$ and $\{s_n\}$ are generated by Equation (2.5). If $h \leq \frac{1}{2}$, then the sequences $\{t_n\}$ and $\{s_n\}$ increase monotonically and converge to t^*. Let $a_n = t^* - t_n$, $b_n = t^{**} - t_n$. Then, for all natural numbers n,*

$$\begin{gathered} t^* - t_{n+1} = \frac{a_n^4}{(a_n+b_n)(a_n^2+b_n^2)}, \\ t^{**} - t_{n+1} = \frac{b_n^4}{(a_n+b_n)(a_n^2+b_n^2)}, \\ 0 \leq t_n \leq s_n \leq t_{n+1} < t^*. \end{gathered} \tag{2.6}$$

Proof. By direct calculating, we have

$$g(t_n) = \frac{K}{2}(t^*-t_n)(t^{**}-t_n) = \frac{K}{2}a_nb_n,$$

$$g'(t_n) = -\frac{K}{2}[(t^*-t_n)+(t^{**}-t_n)] = -\frac{K}{2}(a_n+b_n).$$

We also know $g'(t_n) = Kt_n - \frac{1}{\beta}$, so it follows that

$$-\frac{K}{2}(a_n+b_n) = Kt_n - \frac{1}{\beta}.$$

Therefore,

$$s_n - t_n = -\frac{g(t_n)}{g'(t_n)} = \frac{a_nb_n}{a_n+b_n}, \tag{2.7}$$

$$h_n = \frac{g'(s_n)-g'(t_n)}{g'(s_n)} = \frac{K(s_n-t_n)}{Kt_n - \frac{1}{\beta} + K(s_n-t_n)} = -\frac{2a_nb_n}{a_n^2+b_n^2},$$

$$t_{n+1} - s_n = \frac{1}{2}h_n\frac{g(t_n)}{g'(t_n)} = \frac{a_nb_n}{a_n+b_n}\cdot\frac{a_nb_n}{a_n^2+b_n^2}, \tag{2.8}$$

and

$$t_{n+1} - t_n = t_{n+1} - s_n + s_n - t_n = \frac{a_n^2+a_nb_n+b_n^2}{a_n^2+b_n^2}\cdot\frac{a_nb_n}{a_n+b_n}. \tag{2.9}$$

Thus

$$a_{n+1} = t^* - t_{n+1} = a_n - (t_{n+1}-t_n) = \frac{a_n^4}{(a_n+b_n)(a_n^2+b_n^2)}, \tag{2.10}$$

$$b_{n+1} = t^{**} - t_{n+1} = b_n - (t_{n+1}-t_n) = \frac{b_n^4}{(a_n+b_n)(a_n^2+b_n^2)}. \tag{2.11}$$

By equations (2.7)-(2.11), $t_0 = 0 < t^*$, and by induction, we know that equation (2.6) holds. So we get t_n and s_n increase and converge to t^* (see the proof of Theorem 2.2). $\square$

Lemma 2.2. *Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is continuously second-order Fréchet differentiable, where Ω is an open set, X and Y are Banach spaces. The sequences $\{x_n\}$ and $\{y_n\}$ are generated by iterations (2.3), then we have*

$$\begin{aligned} F(x_{n+1}) = &\int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)](1-t)dt(y_n - x_n)^2 \\ &- \frac{1}{2}\int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)]dt(y_n - x_n)^2 \\ &+ \int_0^1 F^{''}(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2. \end{aligned} \tag{2.12}$$

Proof. By Taylor expansion, we have

$$\begin{aligned} F(x_{n+1}) = &F(y_n) + F^{'}(y_n)(x_{n+1} - y_n) \\ &+ \int_0^1 F^{''}(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2. \end{aligned} \tag{2.13}$$

$$\begin{aligned} F^{'}(y_n) - F^{'}(x_n) = &F^{''}(x_n)(y_n - x_n) \\ &+ \int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)]dt(y_n - x_n), \end{aligned} \tag{2.14}$$

From iterations (2.3), we note that

$$F^{'}(y_n)(x_{n+1} - y_n) = -\frac{1}{2}[F^{'}(y_n) - F^{'}(x_n)](y_n - x_n), \tag{2.15}$$

$$F(x_n) + F^{'}(x_n)(y_n - x_n) = 0. \tag{2.16}$$

By Taylor expansion and equation (2.16), we obtain

$$\begin{aligned} F(y_n) &= F(x_n) + F^{'}(x_n)(y_n - x_n) + \frac{1}{2}F^{''}(x_n)(y_n - x_n)^2 \\ &\quad + \int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)](1-t)dt(y_n - x_n)^2 \\ &= \frac{1}{2}F^{''}(x_n)(y_n - x_n)^2 + \int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)](1-t)dt(y_n - x_n)^2, \end{aligned} \tag{2.17}$$

Substituting equation (2.14) into equation (2.15), we have

$$\begin{aligned} F^{'}(y_n)(x_{n+1} - y_n) = &-\frac{1}{2}F^{''}(x_n)(y_n - x_n)^2 \\ &- \frac{1}{2}\int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)]dt(y_n - x_n)^2, \end{aligned} \tag{2.18}$$

Then, substituting equations (2.17) and (2.18) into equation (2.13), we can obtain

$$\begin{aligned}
F(x_{n+1}) &= \frac{1}{2}F^{''}(x_n)(y_n - x_n)^2 + \int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)](1-t)dt(x_{n+1} - y_n)^2 \\
&\quad - \frac{1}{2}F^{''}(x_n)(y_n - x_n)^2 - \frac{1}{2}\int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)]dt(y_n - x_n)^2 \\
&\quad + \int_0^1 F^{''}(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\
&= \int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)](1-t)dt(y_n - x_n)^2 \\
&\quad - \frac{1}{2}\int_0^1 [F^{''}(x_n + t(y_n - x_n)) - F^{''}(x_n)]dt(y_n - x_n)^2 \\
&\quad + \int_0^1 F^{''}(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2.
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Under the conditions (C1)-(C4), equation (2.4), and* $h = K\beta\eta \leq \frac{1}{2}$*, considering the sequences* $\{t_n\}$ *and* $\{s_n\}$ *generated by iterations (2.5), the following items are verified for all* $n \geq 0$*:*
(I) $\|x_n - x_0\| \leq t_n$*,*
(II) $\|F^{'}(x_n)^{-1}\| \leq -g^{'}(t_n)^{-1}$*,*
(III) $\|y_n - x_n\| \leq s_n - t_n$*,*
(IV) $\|x_{n+1} - y_n\| \leq t_{n+1} - s_n$*,*
(V) $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$*.*

Proof. It can be easily proved that when $n = 0$ the above formula holds. Suppose that (I)-(V) are true for all integers $k \leq n$.
(I). From the above assumptions, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \|x_n - x_0\| \leq t_{n+1} - t_n + t_n = t_{n+1}.$$

(II). By conditions (C3) and (2.4), we can obtain

$$\begin{aligned}
\|F^{'}(x_{n+1}) - F^{'}(x_0)\| &\leq M\|x_{n+1} - x_0\| \leq Mt_{n+1} < Kt^* \\
&= K\eta\frac{1 - \sqrt{1-2h}}{h} \leq \frac{1}{\beta} \leq \frac{1}{\|F'(x_0)^{-1}\|}.
\end{aligned}$$

By perturbation lemma see [10], page45, we get that $F^{'}(x_{n+1})^{-1}$ exists, and

$$\begin{aligned}
\|F^{'}(x_{n+1})^{-1}\| &\leq \frac{\|F^{'}(x_0)^{-1}\|}{1 - \|F'(x_0)^{-1}\|\|F'(x_{n+1}) - F'(x_0)\|} \leq \frac{\beta}{1 - \beta M\|x_{n+1} - x_0\|} \\
&\leq \frac{\beta}{1 - \beta K t_{n+1}} = \frac{1}{\frac{1}{\beta} - Kt_{n+1}} = -g^{'}(t_{n+1})^{-1}.
\end{aligned}$$

(III). By lemma 2.2, and using induction hypothesis, one has

$$\begin{aligned}
\|F(x_{n+1})\| &\leq \int_0^1 \|[F^{''}(x_n+t(y_n-x_n))-F^{''}(x_n)]\|(1-t)dt\|y_n-x_n\|^2\\
&+\frac{1}{2}\int_0^1 \|[F^{''}(x_n+t(y_n-x_n))-F^{''}(x_n)]\|dt\|y_n-x_n\|^2\\
&+\int_0^1 \|F^{''}(y_n+t(x_{n+1}-y_n))\|(1-t)dt\|x_{n+1}-y_n\|^2\\
&\leq \frac{N}{6}\|y_n-x_n\|^3+\frac{N}{4}\|y_n-x_n\|^3+\frac{M}{2}\|x_{n+1}-y_n\|^2\\
&\leq \frac{5N}{12}(s_n-t_n)^3+\frac{M}{2}(t_{n+1}-s_n)^2\\
&= \frac{5N}{12}\frac{(s_n-t_n)^2}{t_{n+1}-s_n}(t_{n+1}-s_n)(s_n-t_n)+\frac{M}{2}(t_{n+1}-s_n)^2.
\end{aligned}$$

Since

$$\frac{(s_n-t_n)^2}{t_{n+1}-s_n}=\frac{(a_nb_n)^2}{(a_n+b_n)^2}\cdot\frac{(a_n+b_n)(a_n^2+b_n^2)}{(a_nb_n)^2}\leq a_n+b_n=t^*+t^{**}=\frac{2}{K\beta}\leq\frac{2}{M\beta},$$

using lemma 2.1, we know

$$\begin{aligned}
\|F(x_{n+1})\| &\leq \frac{5N}{3M\beta}\frac{1}{2}(t_{n+1}-s_n)(s_n-t_n)+\frac{M}{2}(t_{n+1}-s_n)^2\\
&\leq \frac{K}{2}(s_n-t_{n+1})(s_n-t_{n+1})\\
&\leq \frac{K}{2}(t^*-t_{n+1})(t^*-t_{n+1})=g(t_{n+1}).
\end{aligned}$$

Hence, we get

$$\|y_{n+1}-x_{n+1}\|=\|-F^{'}(x_{n+1})^{-1}F(x_{n+1})\|\leq -g^{'}(t_{n+1})^{-1}g(t_{n+1})=s_{n+1}-t_{n+1}. \tag{2.19}$$

(IV). By the assumption (C3), we get

$$\begin{aligned}
\|F^{'}(y_{n+1})-F^{'}(x_0)\| &\leq M\|y_{n+1}-x_0\|\leq M(\|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\|)\leq Ms_{n+1}\\
&< Kt^*\leq \frac{1}{\|F'(x_0)^{-1}\|}.
\end{aligned}$$

So, $F^{'}(y_{n+1})^{-1}$ exists, and

$$\begin{aligned}
\|F^{'}(y_{n+1})^{-1}\| &\leq \frac{\|F^{'}(x_0)^{-1}\|}{1-\|F'(x_0)^{-1}\|\|F'(y_{n+1})-F'(x_0)\|}\\
&\leq \frac{\beta}{1-\beta Ms_{n+1}}=\frac{1}{\frac{1}{\beta}-Ks_{n+1}}=-g^{'}(s_{n+1})^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+2}-y_{n+1}\| &\leq \frac{1}{2}\|F^{'}(y_{n+1})^{-1}\|\|F^{'}(y_{n+1})-F^{'}(x_{n+1})\|\|F^{'}(x_{n+1})^{-1}F(x_{n+1})\|\\
&\leq -\frac{1}{2}g^{'}(s_{n+1})^{-1}\cdot M(s_{n+1}-t_{n+1})^2\\
&\leq -\frac{1}{2}g^{'}(s_{n+1})^{-1}\cdot K(s_{n+1}-t_{n+1})(s_{n+1}-t_{n+1})\\
&= \frac{1}{2}g^{'}(s_{n+1})^{-1}(g^{'}(s_{n+1})-g^{'}(t_{n+1}))g^{'}(t_{n+1})^{-1}g(t_{n+1})\\
&= t_{n+2}-s_{n+1}.
\end{aligned} \tag{2.20}$$

(V). From inequalities (2.19) and (2.20), we can obtain

$$\|x_{n+2} - x_{n+1}\| \leq \|x_{n+2} - y_{n+1}\| + \|y_{n+1} - x_{n+1}\| \leq t_{n+2} - t_{n+1}.$$

This completes the proof. □

Theorem 2.1. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a third-order Fréchet differentiable on a non-empty open convex subset Ω. Assume that all conditions (C1)-(C4) hold and $x_0 \in \Omega$, $h = K\beta\eta \leq \frac{1}{2}$, $\overline{B(x_0, t^*)} \subset \Omega$, where $B(x_0, t) = \{x | \|x - x_0\| < t, x \in \Omega\}$, $\overline{B(x_0, t)}$ is the closed domain of $B(x_0, t)$. Then, the sequence $\{x_n\}$ generated by iterations (2.3) is well defined, $x_n \in \overline{B(x_0, t^*)}$ and $\|x_n - x^*\| \leq t^* - t_n$, and $\{x_n\}$ converges to the unique solution $x^* \in \overline{B(x_0, t^{**})}$ of (1.1).*

Proof. From Lemma 2.3, we can obtain that the sequence $\{x_n\}$ generated by iterations (2.3) is well defined, $x_n \in \overline{B(x_0, t^*)}$ and converges to the unique solution $x^* \in \overline{B(x_0, t^{**})}$ of (1.1).

Now, we prove the uniqueness. Also suppose y^* is the solution of (1.1) on $B(x_0, t^{**})$. Then, we have

$$\begin{aligned}
&\|F'(x_0)^{-1} \int_0^1 F'(x^* + t(y^* - x^*))dt - I\| \\
&\leq \|F'(x_0)^{-1}\| \cdot \| \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| \\
&\leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\| dt \\
&\leq M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|y^* - x_0\|]dt \\
&\leq \frac{M\beta}{2}(t^* + t^{**}) < 1.
\end{aligned}$$

By perturbation lemma, we get that the inverse of $\int_0^1 F'(x^* + t(y^* - x^*))dt$ exists. Because

$$F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*),$$

$y^* = x^*$. This completes the proof of the unique solution. Moreover, when $m > n$,

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \leq t_m - t_n,$$

and let $m \to \infty$, we get

$$\|x_n - x^*\| \leq t^* - t_n.$$

This completes the proof. □

Theorem 2.2. *Suppose F satisfies the conditions of Theorem 2.1. Then,*
(I). when $h < \frac{1}{2}$,

$$\|x_n - x^*\| \leq t^* - t_n = \frac{(1-\theta^2)\eta}{1-\theta^{4^n}} \theta^{4^n - 1},$$

(II). when $h = \frac{1}{2}$,

$$\|x_n - x^*\| \leq t^* - t_n = \frac{2\eta}{4^n},$$

where $\theta = \frac{t^*}{t^{**}} = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}}$.

Proof. When $h < \frac{1}{2}$, by lemma 2.1, we get

$$\frac{t^* - t_n}{t^{**} - t_n} = (\frac{t^* - t_{n-1}}{t^{**} - t_{n-1}})^4 = (\frac{t^* - t_{n-2}}{t^{**} - t_{n-2}})^{4^2} = \cdots (\frac{t^* - t_0}{t^{**} - t_0})^{4^n} = \theta^{4^n}.$$

Because $t^{**} = \frac{\eta}{h}(1 + \sqrt{1-2h}) = \eta(\frac{1}{\theta} + 1)$ and $t^{**} - t^* = \eta(\frac{1}{\theta} + 1)(1-\theta) = \frac{\eta}{\theta}(1-\theta^2)$, we get

$$t^* - t_n = \frac{(1-\theta^2)\eta}{1-\theta^{4^n}}\theta^{4^n - 1}.$$

When $h = \frac{1}{2}$, $t^* = t^{**} = 2\eta$. By lemma 2.1, we get

$$t^* - t_n = \frac{t^* - t_{n+1}}{4} = \cdots = \frac{t^* - t_0}{4^n} = \frac{2\eta}{4^n}.$$

This completes the proof. □

Remark 1. *From Theorem 2.2, we note that the R-order of convergence of method (2.3) is at least four when $h < \frac{1}{2}$, and for $h = \frac{1}{2}$, the order of convergence goes down to one. In [6], Chen obtain the R-order of convergence of the Harmonic mean Newton's method is third by using recurrence relations, while, we obtain the fourth-order convergence in this paper, it's the interesting thing.*

3 Numerical examples

In this section, we illustrate the previous study with applications to the following two nonlinear equations.

Example 1. Consider the root of the equation $F(x) = \frac{1}{10}x^3 - \frac{1}{5}x - \frac{1}{2} = 0$ on $[1, 3]$. If we select the initial point $x_0 = 2$, then we easily get

$$\|F^{'}(x_0)^{-1}\| = 1 = \beta, \quad \|F^{'}(x_0)^{-1}F(x_0)\| = 0.1 = \eta, \quad \|F^{''}(x)\| \leq \frac{9}{5} = M, \quad x \in \Omega,$$

and the Lipschitz condition with $N = \frac{3}{5}$,

$$\|F^{''}(x) - F^{''}(y)\| \leq \frac{3}{5}\|x - y\|, \quad x, y \in \Omega.$$

Note that $K = 2.3556$, and $h = K\beta\eta = 0.2356 \leq \frac{1}{2}$, therefor $t^* = 0.1158$, $t^{**} = 0.7333$. This means that the hypotheses of Theorem 2.1 are satisfied. Hence, the solution of (1.1) exists in $\overline{B(2, 0.1158)} \subseteq \Omega$, and the unique solution exists in the ball $B(2, 0.7333) \cap \Omega$.

However, by the convergence method given in [6] or [14], the solution of $F(x)$ exists in $\overline{B(2, 0.1115)} \subseteq \Omega$, which is inferior to our result.

We apply the method given by the iterations (2.3) to compute the solution of example 1, and then compare it with Newton's method. The initial values $x_0 = 2$ and $x_0 = 2.5$ are considered, respectively. In Table 1, the value of $|x_n - x_{n-1}|$ at each iterative step is displayed. The stopping criterion that we consider is $|F(x_n)| \leq 1e - 15$.

Table 1. Example 1

step	$x_0 = 2$		$x_0 = 2.5$	
	Method (2.3)	Newton's method	Method (2.3)	Newton's method
1	0.0945	0.1	0.1944	0.2198
2	2.7884e-5	0.0054	1.1035e-4	0.0249
3	8.8818e-16	1.6639e-5	5.9952e-14	3.5217e-4
4		1.5587e-10		6.9829e-8

From the numerical results, we can see that the R-order of Harmonic mean Newton's method achieves four, which is coincided with Theorem 2.2.

Example 2. We now consider the nonlinear integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - \frac{7}{5} + \frac{1}{2}\int_0^1 s \cdot cos(x(t))dt,$$

where $s \in [0,1]$ and $x \in \Omega = B(0,2) \subset X$. Here, $X = C[0,1]$ is the space of continuous functions on $[0,1]$ with the max-norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

We can obtain the derivatives of F given by

$$F^{'}(x)y(s) = y(s) - \frac{1}{2}\int_0^1 s \cdot sin(x(t))y(t)dt, \quad y \in \Omega,$$
$$F^{''}(x)yz(s) = -\frac{1}{2}\int_0^1 s \cdot cos(x(t))y(t)z(t)dt, \quad y,z \in \Omega,$$

Furthermore, we have $\|F^{''}(x)\| \leq \frac{1}{2} = M, \quad x \in \Omega$, and the Lipschitz condition with $N = \frac{1}{2}$,

$$\|F^{''}(x) - F^{''}(y)\| \leq \frac{1}{2}\|x-y\|, \quad x,y \in \Omega.$$

A constant function, that is, $x_0(t) = \frac{7}{5}$, is chosen as the initial approximate solution. It follows that

$$\|F(x_0)\| \leq \frac{1}{2}cos\frac{7}{5}.$$

In this case, we have

$$\|I - F^{'}(x_0)\| \leq \frac{1}{2}sin\frac{7}{5},$$

and then by the perturbation lemma, we include that Γ_0 exists and obtain

$$\|\Gamma_0\| \leq \frac{2}{2 - sin(7/5)} = \beta, \quad \|\Gamma_0 F(x_0)\| \leq \frac{cos(7/5)}{2 - sin(7/5)} = \eta \quad and \quad K = 1.34545855834295.$$

Note that $h = 0.44434280650540 \leq \frac{1}{2}$, therefore $t^* = 0.25123688628137$, $t^{**} = 0.50281849782578$ and $\theta = 0.49965720705927$. This means that the hypotheses of Theorem 2.1 are satisfied. Hence, the solution of $F(x)$ exists in $\overline{B(\frac{7}{5}, 0.25123688628137)} \subseteq \Omega$, and the unique solution exists in the ball $B(\frac{7}{5}, 0.50281849782578) \cap \Omega$.

However, by the convergence method given in [6], the solution of $F(x)$ exists in $\overline{B(\frac{7}{5}, 0.23567274887651)} \subseteq \Omega$, which is inferior to our result.

4 Conclusions

This paper is devoted to a fourth-order variant of the Newton's method for solving nonlinear equations in Banach spaces. We establish the Newton-Kantorovich-type convergence theorem for this method by using majorizing functions and get the error estimate. This approach is simple and efficient in comparison with the approach using recurrence relations in [6]. Numerical examples are worked out to demonstrate our approach.

References

[1] I. K. Argyros, *On the Newton-Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., 169: 315–332, 2004.

[2] I. K. Argyros and S. Hilout, *Majorizing sequences for iterative methods*, J. Comput. Appl. Math., 236: 1947–1960, 2012.

[3] V. Candela and A. Marquina, *Recurrence relations for rational cubic methods I: The Halley method*, Computing, 44: 169–184, 1990.

[4] V. Candela and A. Marquina, *Recurrence relations for rational cubic methods II: The Chebyshev method*, Computing, 45: 355–367, 1990.

[5] J.-H. Chen, I. K. Argyros, and R. P. Agarwal, *Majorizing function and two-point Newton-type method*, J. Comput. Appl. Math., 234: 1473–1484, 2010.

[6] L. Chen, C.-Q. Gu, and Y.-F. Ma, *Recurrence relations for the Harmonic mean Newton's method in Banach Spaces*, J. Comput. Anal. Appl., 14(6): 1154–1164, 2012.

[7] X.-P. Guo, *On semilocal convergence of inexact newton methods*, J. Comput. Math., 25(2): 231–242, 2007.

[8] J. M. Gutiérrez and M. A. Hernndez, *Recurrence relations for the super-Halley method*, Comput. Math. Appl., 36: 1–8, 1998.

[9] H. Homeier, *On Newton-type methods with cubic convergence*, J. Comput. Appl. Math., 176: 425–432, 2005.

[10] J. M. Ortega and W. C. Rheinbolt, *Iterative solution of nonlinear equations in several variables*, Academic Press, NewYork, 1970.

[11] A. Y. Özban, *Some new variants of Newton's method*, Appl. Math. Lett., 17: 677–682, 2004.

[12] P. K. Parida and D. K. Gupta, *Recurrence relations for a Newton-like method in Banach spaces*, J. Comput. Appl. Math., 206: 873–887, 2007.

[13] P. K. Parida and D. K. Gupta, *Semilocal convergence of a family of a third-order Chebyshev-type methods under a mild differentiability condition*, Int. J. Comput. Math., 87: 3405–3419, 2010.

[14] Q.-B. Wu and Y.-Q. Zhao, *Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space*, Appl. Math. Comput., 175: 1515–1524, 2006.

[15] L. Zheng and C.-Q. Gu, *Fourth-order convergence theorem by using majorizing functions for super-Halley method in Banach spaces*, Int. J. Comput. Math., 1: 1–12, 2012.

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FERMIONIC p-ADIC INTEGRALS ON $\mathbb{Z}_p$ AND UMBRAL CALCULUS

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Abstract. In this paper we study some properties of the fermionic p-adic integrals on $\mathbb{Z}_p$ arising from the umbral calculus.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [1,2,11]).} \end{aligned} \tag{1}$$

For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{2}$$

In the special case, $n = 1$, we note that

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0), \text{ (see [11]).} \tag{3}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}_p$ with

$$\mathcal{F} = \Big\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \Big| a_k \in \mathbb{C}_p \Big\}.$$

Let $\mathbb{P} = \mathbb{C}_p[x]$ and let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$. The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}. \tag{4}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t) | x^n \rangle = a_n \text{ for all } n \geq 0, \text{ (see [7,14]).} \tag{5}$$

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Thus, by (4) and (5), we get

$$\langle t^k | x^n \rangle = n! \delta_{n,k}, \ (n, \ k \geq 0), \tag{6}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [7,14]). Here, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [7,14]).
The order $O(f(t))$ of power series $f(t)(\neq 0)$ is the smallest integer k for which a_k does not vanish (see [7,4]). The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $O(f(t)) = 0$. Such series is called an invertible series. A series $f(t)$ for which $O(f(t)) = 1$ is called a delta series (see [7,14]). For $f(t), \ g(t) \in \mathcal{F}$, we have $\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$. By (6), we get

$$\langle e^{yt} | x^n \rangle = y^n, \quad \langle e^{yt} | p(x) \rangle = p(y), \ \text{(see [7,14])}. \tag{7}$$

Let $f(t) \in \mathcal{F}$. Then we note that

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \tag{8}$$

and

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k, \ \text{for } p(x) \in \mathbb{P}, \ \text{(see [14])}. \tag{9}$$

Let $f_1(t), \ f_2(t), \ \cdots, \ f_m(t) \in \mathcal{F}$. It is known in [7,14] that

$$\langle f_1(t) \cdots f_m(t) | x^n \rangle = \sum \binom{n}{i_1, \cdots, i_m} \langle f_1(t)|x^{i_1}\rangle \cdots \langle f_m(t)|x^{i_m}\rangle, \tag{10}$$

where the sum is over all nonnegative integers $i_1, \ \cdots, \ i_m$ such that $i_1 + i_2 + \cdots + i_m = n$ (see [7,14]).

By (9), we get

$$\begin{aligned} p^{(k)}(x) = \frac{d^k p(x)}{dx^k} &= \sum_{l=k}^{\infty} \frac{\langle t^l|p(x)\rangle}{l!} l(l-1)\cdots(l-k+1)x^{l-k} \\ &= \sum_{l=k}^{\infty} \langle t^l|p(x)\rangle \binom{l}{k} \frac{k!}{l!} x^{l-k}. \end{aligned} \tag{11}$$

Thus, from (11), we have

$$p^{(k)}(0) = \langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle, \tag{12}$$

and

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [6,7,14])}. \tag{13}$$

From (13), we note that

$$e^{yt} p(x) = p(x+y) \quad \text{(see [7,14])}. \tag{14}$$

In this paper, $s_n(x)$ denotes a polynomial of degree n. Let us assume that $f(t), \ g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 1$. Then there exists a unique sequence $s_n(x)$

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of polynomials satisfying $\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}$ for all $n,\ k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t),\ f(t))$, which is denoted by $s_n(x) \sim (g(t),\ f(t))$. If $s_n(x) \sim (g(t),\ t)$, then $s_n(x)$ is called the Appell sequence for $g(t)$ (see [6,7,14]).

Let $p(x) \in \mathbb{P}$. Then we note that

$$\langle f(t)|xp(x)\rangle = \langle \partial_t f(t)|p(x)\rangle = \langle f'(t)|p(x)\rangle, \tag{15}$$

and

$$\langle e^{yt} - 1|p(x)\rangle = p(y) - p(0), \quad \text{(see [7,14])}. \tag{16}$$

Let us assume that $s_n(x) \sim (g(t),\ f(t))$. Then we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|s_k(x)\rangle}{k!} g(t)f(t)^k, \quad h(t) \in \mathcal{F}, \tag{17}$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x)\rangle}{k!} s_k(x), \quad p(x) \in \mathbb{P}, \tag{18}$$

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C}_p, \tag{19}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, and

$$f(t)s_n(x) = ns_{n-1}(x), \quad \text{(see [7,14])}. \tag{20}$$

As is well known, the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1-19])}, \tag{21}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers

Let $s_n(x) \sim (g(t),\ t)$. Then Appell identity is known to be

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(x) y^k = \sum_{k=0}^{n} \binom{n}{k} s_k(x) y^{n-k}. \tag{22}$$

From (21), we note that the recurrence relation of the Euler numbers is given by

$$E_0 = 1, \quad (E+1)^n + E_n = E_n(1) + E_n = 2\delta_{0,n}. \tag{23}$$

By (1) and (21), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(y) = E_n, \tag{24}$$

where $n \geq 0$ (see [1,11,16]).

Recently, D. S. Kim and T. Kim have studied applications of umbral calculus associated with p-adic invariant integrals on $\mathbb{Z}_p$ (see [7]). In this paper we study some properties of the fermionic p-adic integrals on $\mathbb{Z}_p$ arising from the umbral calculus.

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2. Umbral calculus and fermionic p-adic integrals on $\mathbb{Z}_p$

Let $s_n(x) \sim (g(t),\ t)$. Then, by (19), we get

$$\frac{1}{g(t)}x^n = s_n(x) \quad \text{if and only if} \quad x^n = g(t)s_n(x). \tag{25}$$

Let us assume that $g(t) = \frac{e^t+1}{2}$. Then we note that $g(t)$ is an invertible functional. By (21), we get

$$\frac{1}{g(t)}e^{xt} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!}. \tag{26}$$

Thus, from (26), we have

$$\frac{1}{g(t)}x^n = E_n(x), \quad tE_n(x) = \frac{n}{g(t)}x^{n-1} = nE_{n-1}(x). \tag{27}$$

By (19), (20) and (27), we see that $E_n(x)$ is an Appell sequence for $g(t) = \frac{e^t+1}{2}$.

It is easy to show that

$$E_{n+1}(x) = \Big(x - \frac{g'(t)}{g(t)}\Big)E_n(x), \quad (n \geq 0). \tag{28}$$

From (2), (21) and (24), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y+1)t}d\mu_{-1}(y) + \int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = 2e^{xt}. \tag{29}$$

Thus, by (29), we get

$$\int_{\mathbb{Z}_p} (x+y+1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = 2x^n. \tag{30}$$

From (24) and (30), we have

$$E_n(x+1) + E_n(x) = 2x^n, \quad (n \geq 0). \tag{31}$$

By (28), we see that

$$g(t)E_{n+1}(x) = g(t)xE_n(x) - g'(t)E_n(x), \quad (n \geq 0). \tag{32}$$

Thus, we have

$$(e^t+1)E_{n+1}(x) = (e^t+1)xE_n(x) - e^tE_n(x). \tag{33}$$

By (33), we get

$$E_{n+1}(x+1) + E_{n+1}(x) = (x+1)E_n(x+1) + xE_n(x) - E_n(x+1). \tag{34}$$

Thus, from (34) and (31), we have

$$E_{n+1}(x+1) + E_{n+1}(x) = x(E_n(x+1) + E_n(x)). \tag{35}$$

By (35), we get

$$\begin{aligned} E_n(x+1) + E_n(x) &= x(E_{n-1}(x+1) + E_{n-1}(x)) = x^2(E_{n-2}(x+1) + E_{n-2}(x)) \\ &= \cdots = x^n(E_0(x+1) + E_0(x)) = 2x^n. \end{aligned}$$

Let us consider the functional $f(t)$ such that

$$\langle f(t)|p(x)\rangle = \int_{\mathbb{Z}_p} p(u)d\mu_{-1}(u), \tag{36}$$

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for all polynomials $p(x)$. It can be determined from (8) to be

$$f(t)=\sum_{k=0}^{\infty}\frac{\langle f(t)|x^k\rangle}{k!}t^k=\sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}u^k d\mu_{-1}(u)\frac{t^k}{k!}=\int_{\mathbb{Z}_p}e^{ut}d\mu_{-1}(u). \tag{37}$$

By (29) and (37), we get

$$f(t)=\int_{\mathbb{Z}_p}e^{ut}d\mu_{-1}(u)=\frac{2}{e^t+1}. \tag{38}$$

Therefore, by (38), we obtain the following theorem.

Theorem 2.1. *For $p(x)\in\mathbb{P}$, we have*

$$\langle\int_{\mathbb{Z}_p}e^{yt}d\mu_{-1}(y)|p(x)\rangle=\int_{\mathbb{Z}_p}p(u)d\mu_{-1}(u).$$

That is,

$$\langle\frac{2}{e^t+1}|p(x)\rangle=\int_{\mathbb{Z}_p}p(u)d\mu_{-1}(u).$$

Also, the n-th Euler number is given by

$$E_n=\langle\int_{\mathbb{Z}_p}e^{yt}d\mu_{-1}(y)|x^n\rangle.$$

By (3) and (30), we get

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}e^{(x+y)t}d\mu_{-1}(y)=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}e^{yt}d\mu_{-1}(y)x^n\frac{t^n}{n!}. \tag{39}$$

From (24) and (39), we have

$$E_n(x)=\int_{\mathbb{Z}_p}e^{yt}d\mu_{-1}(y)x^n=\frac{2}{e^t+1}x^n, \tag{40}$$

where $n\geq 0$.

Therefore, by (40), we obtain the following theorem.

Theorem 2.2. *For $p(x)\in\mathbb{P}$, we have*

$$\int_{\mathbb{Z}_p}p(x+y)d\mu_{-1}(y)=\int_{\mathbb{Z}_p}e^{yt}d\mu_{-1}(y)p(x)=\frac{2}{e^t+1}p(x).$$

From (22), we note that

$$E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}.$$

The Euler polynomials of order r are defined by the generating function to be

$$\underbrace{\left(\frac{2}{e^t+1}\right)\times\left(\frac{2}{e^t+1}\right)\times\cdots\times\left(\frac{2}{e^t+1}\right)}_{r\text{ times}}e^{xt}=\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}. \tag{41}$$

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In the special case, $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the n-th Euler numbers of order r $(r \geq 0)$, (see [1-19]). Let us take $g^r(t) = \left(\frac{e^t+1}{2}\right)^r$. Then we see that $g^r(t)$ is an invertible functional in $\mathcal{F}$. By (41), we get

$$\frac{1}{g^r(t)}e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}. \tag{42}$$

Thus, we have

$$\frac{1}{g^r(t)}x^n = E_n^{(r)}(x), \quad tE_n^{(r)}(x) = \frac{n}{g^r(t)}x^{n-1} = nE_{n-1}^{(r)}(x). \tag{43}$$

So, by (42), we see that $E_n^{(r)}(x)$ is the Appell sequence for $\left(\frac{e^t+1}{2}\right)^r$. From (22), we have

$$E_n^{(r)}(x+y) = \sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{(r)}(x)y^k. \tag{44}$$

It is easy to show that

(45)
$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}.$$

By (6) and (45), we get

$$E_n^{(r)} = \langle \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)|x^n\rangle, \quad (n \geq 0), \tag{46}$$

and, by (10),

$$\begin{aligned}
&\langle \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)|x^n\rangle \\
&= \sum_{n=i_1+\cdots+i_r}\binom{n}{i_1,\cdots,i_r}\langle\int_{\mathbb{Z}_p} e^{x_1t}d\mu_{-1}(x_1)|x^{i_1}\rangle\cdots\langle\int_{\mathbb{Z}_p} e^{x_rt}d\mu_{-1}(x_r)|x^{i_r}\rangle \\
&= \sum_{n=i_1+\cdots+i_r}\binom{n}{i_1,\cdots,i_r}E_{i_1}E_{i_2}\cdots E_{i_r}.
\end{aligned} \tag{47}$$

From (46) and (47), we have

$$E_n^{(r)} = \sum_{n=i_1+\cdots+i_r}\binom{n}{i_1,\cdots,i_r}E_{i_1}\cdots E_{i_r}. \tag{48}$$

By (44) and (48), we see that $E_n^{(r)}(x)$ is a monic polynomial of degree n with coefficients in $\mathbb{Q}$. Let $r \in \mathbb{N}$. Then we note that

$$g^r(t) = \frac{1}{\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)} = \left(\frac{e^t+1}{2}\right)^r. \tag{49}$$

By (49), we get

$$\frac{1}{g^r(t)}e^{xt} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}. \tag{50}$$

From (50), we have

$$\begin{aligned} E_n^{(r)}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \frac{1}{\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}x^n \\ &= \frac{1}{g^r(t)}x^n. \end{aligned} \tag{51}$$

Therefore, by (51), we obtain the following theorem.

Theorem 2.3. *For $p(x) \in \mathbb{P}$ and $r \in \mathbb{N}$. Then we have*

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} p(x_1+\cdots x_r+x)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \Big(\frac{2}{e^t+1}\Big)^r p(x).$$

In particular,

$$E_n^{(r)}(x) = \Big(\frac{2}{e^t+1}\Big)^r x^n = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)x^n.$$

That is,

$$E_n^{(r)}(x) \sim \Big(\frac{1}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}, t\Big).$$

Let us take the functional $f^r(t)$ such that

$$\langle f^r(t)|p(x)\rangle = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} p(x_1+\cdots x_r)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r), \tag{52}$$

for all polynomials $p(x)$. It can be determined from (8) to be

$$\begin{aligned} f^r(t) &= \sum_{k=0}^{\infty}\frac{\langle f^r(t)|x^k\rangle}{k!}t^k \\ &= \sum_{k=0}^{\infty}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}}(x_1+\cdots x_r)^k d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^k}{k!} \\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-\text{times}} e^{(x_1+\cdots x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \end{aligned} \tag{53}$$

Therefore, by (52) and (53) , we obtain the following theorem.

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Theorem 2.4. *For $p(x) \in \mathbb{P}$, we have*

$$\langle \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-times} e^{(x_1+\cdots x_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)|p(x)\rangle = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-times} p(x_1+\cdots x_r) d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).$$

Moveover,

$$\langle \Big(\frac{2}{e^t+1}\Big)^r |p(x)\rangle = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-times} p(x_1+\cdots x_r) d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).$$

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References

[1] S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. 22 (2012), no. 3, 399–406.

[2] S. Araci, M. Acikgöz, H. Jolany, J. J. Seo, *A unified generating function of the q-Genocchi polynomials with their interpolation functions*, Proc. Jangjeon Math. Soc. **15** (2012), no. 2, 227–233.

[3] L. Carlitz, *Some remarks on the multiplication theorems for the Bernoulli and Euler polynomials*, Glas. Mat. Ser. III **16(36)** (1981), no. 1, 3–23.

[4] L. Carlitz, *Some polynomials related to the Bernoulli and Euler polynomials*, Utilitas Math. **19** (1981), 81–127.

[5] M. Can, M. Cenkci, V. Kurt, Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. **19** (2009), no. 1, 39–57.

[6] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. 14 (2008), no. 12, 1267–1277.

[7] T. Kim, *Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math. 20 (2010), no. 1, 23–28.

[8] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$*, Rocky Mountain J. Math. 41 (2011), no. 1, 239–247

[9] T. Kim, *A note on q-Bernstein polynomials*, Russ. J. Math. Phys. 18 (2011), no. 1, 73–82.

[10] T. Kim, *Identities involving Frobenius?Euler polynomials arising from nonlinear differential equations*, J. Number Theory **132** (2012), no. 12, 2854–2865.

[11] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **16**(2009), no. 1, 93-96 .

[12] S.-H. Rim, J. Jeong, *On the modified q-Euler numbers of higher order with weight*, Adv. Stud. Contemp. Math. **22** (2012), no. 1, 93-98.

[13] T, Kim, S.-H. Rim, D.V. Dolgy, S.-H. Lee, *Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials*, Adv. Difference Equ. 2012, 2012:201.
[14] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.
[15] T. Kim, D. S. Kim, A. Bayad, S.-H. Rim *Identities on the Bernoulli and the Euler numbers and polynomials*, Ars Combin. **107** (2012), 455–463.
[16] C. S. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math. **21** (2011), no. 2, 217–223.
[17] T. Kim, J. Choi, Y. H. Kim, C. S. Ryoo, *On the fermionic p-adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials*, J. Inequal. Appl. 2010, Art. ID 864247, 12 pp.
[18] K. Shiratani, *On Euler numbers*, Mem. Fac. Sci., Kyushu Univ., Ser. A **27**(1973), 1-5 .
[19] K. Shiratani, S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A **39** (1985), no. 1, 113–125.

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Soft set theory and $\mathcal{N}$-structures applied to BCH-algebras

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Abstract

The notions of (closed) $\mathcal{N}$-filters, regular $\mathcal{N}$-subalgebras and $\mathcal{N}$-soft BCH-algebras are introduced, and related properties are investigated. Conditions for an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-closed ideal) to be an $\mathcal{N}$-closed ideal (resp. closed $\mathcal{N}$-filter) are provided. Characterizations of an $\mathcal{N}$-structure with $\mathcal{N}$-regularity are considered. A condition for an $\mathcal{N}$-closed ideal to satisfy the $\mathcal{N}$-structure is discussed. The union (resp. intersection) of two $\mathcal{N}$-soft BCH-algebras are discussed.

Keywords: $\mathcal{N}$-closed ideal, $\mathcal{N}$-subalgebra, (closed) $\mathcal{N}$-filter, $\mathcal{N}$-transfer principle, $\mathcal{N}$-regularity, regular subset, $\mathcal{N}$-soft BCH-algebra,

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1 Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on

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spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [7] introduced and used a new function which is called negative-valued function. They studied the ideal theory in BCK/BCI-algebras based on $\mathcal{N}$-structures.

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [13]. In response to this situation Zadeh [14] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [15]. To solve complicated problem in economics, engineering, and environment, we can not successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can not be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [12]. Maji et al. [11] and Molodtsov [12] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [11] described the application of soft set theory to a decision making problem.

In [4, 5], Hu and Li introduced the notion of BCH–algebras which are a generalization of BCK/BCI-algebras. Ahmad [1] classified BCH–algebras, and decompositions of BCH–algebras are considered by Dudek and Thomys [3]. Chaudhry et al. studied closed ideals and filters in BCH–algebras. In this paper, we apply the $\mathcal{N}$-structures and soft set theory to BCH–algebras. We introduce the notions of (closed) $\mathcal{N}$-filters, regular $\mathcal{N}$-subalgebras and $\mathcal{N}$-soft BCH-algebras, and investigate related properties. We provide conditions for

an $\mathcal{N}$-subalgebra (resp. $\mathcal{N}$-closed ideal) to be an $\mathcal{N}$-closed ideal (resp. closed $\mathcal{N}$-filter). We consider characterizations of an $\mathcal{N}$-structure with $\mathcal{N}$-regularity. We also discuss a condition for an $\mathcal{N}$-closed ideal to satisfy the $\mathcal{N}$-structure, and deal with the union (resp. intersection) of two $\mathcal{N}$-soft BCH-algebras.

2 Preliminaries

By a *BCH–algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(H1) $x * x = 0$,

(H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(H3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

In a BCH–algebra X, the following conditions are valid (see [3, 4]).

(a1) $x * 0 = x$,

(a2) $x * 0 = 0$ implies $x = 0$,

(a3) $0 * (x * y) = (0 * x) * (0 * y)$,

(a4) $0 * (0 * (0 * x)) = 0 * x$.

A BCH–algebra X is said to be *medial* if it satisfies:

$$(\forall x, y, a, b \in X)((x * y) * (a * b) = (x * a) * (y * b)). \tag{2.1}$$

A subset R of a BCH–algebra X is said to be *regular* if it satisfies:

$$(\forall x \in R)(\forall y \in X)(x * y \in R \;\Rightarrow\; y \in R). \tag{2.2}$$

A nonempty subset S of a BCH–algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset A of a BCH–algebra X is called a *closed ideal* of X (see [2]) if it satisfies:

(1) $(\forall x \in X)\ \big(x \in A \;\Rightarrow\; 0 * x \in A\big)$,

(2) $(\forall y \in X)\ (\forall x \in A)\ \big(y * x \in A \;\Rightarrow\; y \in A\big)$.

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Note that every closed ideal is a subalgebra, but the converse is not true (see [2]). Since every closed ideal is a subalgebra, we know that any closed ideal contains the element 0. A *filter* of a BCH–algebra X is a nonempty subset F of X satisfying the following conditions:

(1) $(\forall x, y \in X)\ (x \in F,\ y \in F\ \Rightarrow\ x * (x * y) \in F,\ y * (y * x) \in F)$,

(2) $(\forall x, y \in X)\ (x \in F,\ x \leq y\ \Rightarrow\ y \in F)$.

A filter F of a BCH–algebra X is said to be *closed* if $0 * x \in F$ for all $x \in F$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a BCH–algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0]$, the set

$$C(f; \alpha) := \{x \in X \mid f(x) \leq \alpha\}$$

is called a *closed* (f, α)*-cut* of (X, f).

Using the similar method to the transfer principle in fuzzy theory (see [6, 10]), we can consider transfer principle in $\mathcal{N}$-structures. Let A be a subset of X and satisfy the following property $\mathcal{P}$ expressed by a first-order formula:

$$\mathcal{P} : \tfrac{t_1(x,\cdots,y)\in A, \cdots, t_n(x,\cdots,y)\in A}{t(x,\cdots,y)\in A},$$

where $t_1(x, \cdots, y), \cdots, t_n(x, \cdots, y)$ and $t(x, \cdots, y)$ are terms of X constructed by variables $x, \cdots, y$. We note that the subset A satisfies the property $\mathcal{P}$ if, for all elements $a, \cdots, b \in X$, $t(a, \cdots, b) \in A$ whenever $t_1(a, \cdots, b), \cdots, c_n(a, \cdots, b) \in A$. For the subset A we define an $\mathcal{N}$-structure (X, f_A) which satisfies the following property

$$\bar{\mathcal{P}} : f_A(t(x, \cdots, y) \leq \bigvee\{f_A(t_1(x, \cdots, y)), \cdots f_A(t_n(x, \cdots, y))\}.$$

Theorem 2.1 ([8]). ($\mathcal{N}$-transfer principle) *An $\mathcal{N}$-structure (X, f) satisfies the property $\bar{\mathcal{P}}$ if and only if for all $\alpha \in [-1, 0]$,*

$$C(f;\alpha) \neq \emptyset \;\Rightarrow\; C(f;\alpha) \text{ satisfies the property } \mathcal{P}.$$

Definition 2.2 ([8]). By an $\mathcal{N}$-*subalgebra* of X we mean an $\mathcal{N}$-structure (X, f) in which f satisfies:

$$(\forall x, y \in X)\ \big(f(x * y) \leq \bigvee\{f(x), f(y)\}\big). \tag{2.3}$$

Definition 2.3 ([8]). By an $\mathcal{N}$-*closed ideal* of X we mean an $\mathcal{N}$-structure (X, f) in which f satisfies:

$$(\forall x, y \in X)\ \big(f(0 * x) \leq f(x) \leq \bigvee\{f(x * y), f(y)\}\big). \tag{2.4}$$

3 Closed $\mathcal{N}$-filters of $BCH-$algebras

In what follows let X denote a $BCH-$algebra unless otherwise specified. For any $\mathcal{N}$-structure (X, f), consider the following conditions:

(b1) f is order reversing.

(b2) $(\forall x \in X)\ \big(f(0 * x) \leq f(x)\big)$.

(b3) $(\forall x, y \in X)\ \big(\bigvee\{f(x * (x * y)), f(y * (y * x))\} \leq \bigvee\{f(x), f(y)\}\big)$.

Definition 3.1. By an $\mathcal{N}$-*filter* of X we mean an $\mathcal{N}$-structure (X, f) in which f satisfies (b1) and (b3).

If an $\mathcal{N}$-filter (X, f) of X satisfies the condition (b2), then we say (X, f) is *closed*.

Theorem 3.2. *For any $\mathcal{N}$-structure (X, f), thew following are equivalent:*

(1) *(X, f) is a (closed) $\mathcal{N}$-filter of X.*

(2) $(\forall \alpha \in [-1, 0])$ $(C(f;\alpha) \neq \emptyset \Rightarrow C(f;\alpha)$ *is a (closed) filter of X*$)$.

Proof. It follows from the $\mathcal{N}$-transfer principle. $\square$

Table 1: $*$-operation

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set with the $*$-operation given by Table 1. Then $(X; *, 0)$ is a BCH–algebra. Define an $\mathcal{N}$-function f on X by

X	0	1	2	3
f	-0.8	-0.8	-0.5	-0.2

Then (X, f) is an $\mathcal{N}$-filter of X. Since $f(0 * 2) = f(3) = -0.2 > -0.5 = f(2)$, (X, f) is not closed.

Proposition 3.4. *Let* (X, f) *be an* $\mathcal{N}$*-structure satisfying conditions* (b1) *and* (b2)*. Then*

(1) $(\forall x, y \in X)\ (f(y * x) \leq f(x * y))$.

(2) $(\forall x, y \in X)\ (f(x * (x * y)) \leq f(y))$.

(3) f *is order preserving.*

Proof. (1) Using (a3), (H3) and (H1), we have $(0 * (x * y)) * (y * x) = 0$ for all $x, y \in X$. It follows from (b1) and (b2) that

$$f(y * x) \leq f(0 * (x * y)) \leq f(x * y)$$

for all $x, y \in X$. This proves (1).

(2) For any $x, y \in X$, we get

$$f(x * (x * y)) \leq f((x * y) * x) = f((x * x) * y) = f(0 * y) \leq f(y).$$

(3) Let $x, y \in X$ be such that $x * y = 0$. Then

$$0 * x = (x * y) * x = (x * x) * y = 0 * y,$$

and so $f(x) = f(x * 0) \leq f(0 * x) = f(0 * y) \leq f(y * 0) = f(y)$. Hence f is order preserving. □

We provide conditions for an $\mathcal{N}$-subalgebra to be an $\mathcal{N}$-closed ideal.

Theorem 3.5. *Let (X, f) be an $\mathcal{N}$-subalgebra of X. If (X, f) satisfies two conditions* (b1) *and* (b2)*, then (X, f) is an $\mathcal{N}$-closed ideal of X.*

Proof. It is sufficient to show that $f(y) \leq \bigvee \{f(y * x), f(x)\}$ for all $x, y \in X$. Let $x, y \in X$. Then

$$\begin{aligned} f(y) &= f(y * 0) \leq f(0 * y) \\ &= f((x * x) * y) = f((x * y) * x) \\ &\leq \bigvee \{f(x * y), f(x)\} \\ &\leq \bigvee \{f(y * x), f(x)\}. \end{aligned}$$

This completes the proof. □

We provide conditions for an $\mathcal{N}$-closed ideal to be a closed $\mathcal{N}$-filter.

Theorem 3.6. *Let (X, f) be an $\mathcal{N}$-closed ideal of X. If (X, f) satisfies the condition* (1) *in Proposition 3.4, then (X, f) is a closed $\mathcal{N}$-filter of X.*

Proof. Let $x, y \in X$ be such that $x * y = 0$. Then $0 * x = 0 * y$, and so

$$f(y) = f(y * 0) \leq f(0 * y) = f(0 * x) \leq f(x * 0) = f(x),$$

i.e., f is order reversing. Note that every $\mathcal{N}$-closed ideal is an $\mathcal{N}$-subalgebra (see [8, Theorem 3.5]). Hence

$$\begin{aligned} f(x * (x * y)) &\leq \bigvee \{f(x), f(x * y)\} \\ &\leq \bigvee \left\{f(x), \bigvee\{f(x * y)\}\right\} = \bigvee \{f(x), f(y)\}. \end{aligned}$$

Similarly, we have $f(y * (y * x)) \leq \bigvee \{f(x), f(y)\}$. Therefore

$$\bigvee \{f(x * (x * y)), f(y * (y * x))\} \leq \bigvee \{f(x), f(y)\}$$

for all $x, y \in X$. Consequently, (X, f) is a closed $\mathcal{N}$-filter of X. □

Corollary 3.7. *Let (X, f) be an $\mathcal{N}$-structure of X that satisfies* (b1), (b2) *and the condition* (1) *in Proposition 3.4. Then (X, f) is a closed $\mathcal{N}$-filter of X.*

Definition 3.8. An $\mathcal{N}$-structure (X, f) is said to satisfy the $\mathcal{N}$-*regularity* if it satisfies:

$$(\forall x, y \in X) \left(f(y) \leq \bigvee \{f(x * y), f(x)\}\right). \tag{3.1}$$

Table 2: $*$-operation

$*$	0	1	2	3
0	0	3	0	3
1	1	0	3	2
2	2	3	0	1
3	3	0	3	0

Table 3: $*$-operation

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

An $\mathcal{N}$-subalgebra (X, f) satisfying the $\mathcal{N}$-regularity is called a *regular $\mathcal{N}$-subalgebra* of X.

Example 3.9. Let $X = \{0, 1, 2, 3\}$ be a set with the $*$-operation given by Table 2. Then $(X; *, 0)$ is a BCH–algebra. Define an $\mathcal{N}$-function f on X by

X	0	1	2	3
f	-0.8	-0.2	-0.8	-0.2

Then (X, f) is a regular $\mathcal{N}$-subalgebra of X.

Example 3.10. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the $*$-operation given by Table 3. Then $(X; *, 0)$ is a BCH–algebra. Define an $\mathcal{N}$-function f on X by

X	0	1	2	3	4
f	α_1	α_2	α_3	α_2	α_3

where $\alpha_1 < \alpha_2 < \alpha_3$ in $[-1, 0]$. Then (X, f) is an $\mathcal{N}$-subalgebra of X. Since $f(1) = \alpha_2 > \alpha_1 = \bigvee \{f(0 * 1), f(0)\}$, we know that (X, f) does not satisfy the $\mathcal{N}$-regularity.

Lemma 3.11. *If an $\mathcal{N}$-structure (X, f) of X satisfies the $\mathcal{N}$-regularity, then f assigns 0 the least value of the image set of f.*

Proof. Taking $y = 0$ in (3.1) and using (a1) induce the desired result. □

Proposition 3.12. *If an $\mathcal{N}$-structure (X, f) of X satisfies the $\mathcal{N}$-regularity and the following inequality:*

$$(\forall x, y \in X)\,(f(x * y) \leq f(y))\,, \tag{3.2}$$

then f is a constant mapping.

Proof. Using (a1) and (3.2), we have $f(x) = f(x * 0) \leq f(0)$ for all $x \in X$. It follows from Lemma 3.11 that $f(x) = f(0)$ for all $x \in X$. □

Proposition 3.13. *Every $\mathcal{N}$-structure (X, f) of X with the $\mathcal{N}$-regularity satisfies:*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; f(y) \leq f(x))\,. \tag{3.3}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$f(y) \leq \bigvee \{f(x * y), f(x)\} = \bigvee \{f(0), f(x)\} = f(x)$$

by (3.1) and Lemma 3.11. □

Proposition 3.14. *Let (X, f) be an $\mathcal{N}$-structure of X satisfying the $\mathcal{N}$-regularity. If X satisfies the following assertion:*

$$(\forall x, y, z \in X)\,(z \leq x * y)\,, \tag{3.4}$$

then $f(y) \leq \bigvee \{f(x), f(z)\}$ for all $x, y, z \in X$.

Proof. Assume that (3.4) is valid. Then

$$f(x * y) \leq \bigvee \{f(z * (x * y)), f(z)\} = \bigvee \{f(0), f(z)\} = f(z)$$

for all $x, y, z \in X$. It follows that

$$f(y) \leq \bigvee \{f(x * y), f(x)\} \leq \bigvee \{f(x), f(z)\}$$

for all $x, y, z \in X$. □

Theorem 3.15. *For any $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) *(X, f) satisfies the $\mathcal{N}$-regularity.*

(2) $(\forall \alpha \in [-1, 0])\,(C(f;\alpha) \neq \emptyset \;\Rightarrow\; C(f;\alpha)$ *is a regular subset of* $X)$.

Proof. Assume that (X, f) satisfies the $\mathcal{N}$-regularity. Let $\alpha \in [-1, 0]$ be such that $C(f;\alpha) \neq \emptyset$. Let $x, y \in X$ be such that $x \in C(f;\alpha)$ and $x * y \in C(f;\alpha)$. Then $f(x) \leq \alpha$ and $f(x * y) \leq \alpha$, which imply from (3.1) that

$$f(y) \leq \bigvee \{f(x * y), f(x)\} \leq \alpha.$$

Hence $y \in C(f;\alpha)$, and therefore $C(f;\alpha)$ is a regular subset of X.

Conversely suppose that (2) is valid. Assume that there exist $x, y \in X$ such that

$$f(y) > \bigvee \{f(x * y), f(x)\} = \beta.$$

Then $x * y \in C(f;\beta)$ and $x \in C(f;\beta)$, but $y \notin C(f;\beta)$. This is a contradiction, and so

$$f(y) \leq \bigvee \{f(x * y), f(x)\}$$

for all $x, y \in X$. Therefore (X, f) satisfies the $\mathcal{N}$-regularity. □

Corollary 3.16. *If an $\mathcal{N}$-structure (X, f) satisfies the $\mathcal{N}$-regularity, then the set*

$$X_w := \{x \in X \mid f(x) \leq f(w)\}$$

is a regular subset of X for all $w \in X$.

Proposition 3.17. *If an $\mathcal{N}$-structure (X, f) satisfies the $\mathcal{N}$-regularity, then the following implication is valid:*

$$(\forall x, y, z \in X)\left(f(x) \geq \bigvee \{f(y * z), f(y)\} \;\Rightarrow\; f(z) \leq f(x)\right). \tag{3.5}$$

Proof. Let $x, y, z \in X$ be such that $f(x) \geq \bigvee \{f(y * z), f(y)\}$. Then $y * z \in X_x$ and $y \in X_x$. Since X_x is a regular subset of X by Corollary 3.16, it follows that $z \in X_x$, that is, $f(z) \leq f(x)$. □

Theorem 3.18. *If an $\mathcal{N}$-structure (X, f) satisfies the condition (3.5), then the set X_w is a regular subset of X for all $w \in X$.*

Proof. Let $x, y \in X$ be such that $x \in X_w$ and $x*y \in X_w$ for all $w \in X$. Then $f(x) \leq f(w)$ and $f(x*y) \leq f(w)$, which imply that

$$f(w) \geq \bigvee \{f(x*y), f(x)\}.$$

It follows from (3.5) that $f(y) \leq f(w)$. Hence $y \in X_w$, and so X_w is a regular subset of X for all $w \in X$. □

Corollary 3.19. *If an $\mathcal{N}$-structure (X, f) satisfies the $\mathcal{N}$-regularity, then the set X_w is a regular subset of X for all $w \in X$.*

Theorem 3.20. *If an $\mathcal{N}$-structure (X, f) of X satisfies the $\mathcal{N}$-regularity and the condition*

$$(\forall x, y \in X) \left(f(x) \leq \bigvee \{f(x*y), f(y)\} \right), \tag{3.6}$$

then (X, f) is an $\mathcal{N}$-closed ideal of X.

Proof. For any $x \in X$, we have

$$\begin{aligned} f(0*x) &\leq \bigvee \{f(0*(0*x)), f(0)\} = f(0*(0*x)) \\ &\leq \bigvee \{f((0*(0*x))*x), f(x)\} \\ &= \bigvee \{f((0*x)*(0*x)), f(x)\} \\ &= \bigvee \{f(0), f(x)\} = f(x) \end{aligned}$$

by using (3.1), Lemma 3.11, (3.6), (a3) and (H1). Therefore (X, f) is an $\mathcal{N}$-closed ideal of X. □

We provide a condition for an $\mathcal{N}$-closed ideal to satisfy the $\mathcal{N}$-regularity.

Proposition 3.21. *If X is medial, then every $\mathcal{N}$-closed ideal of X satisfies the $\mathcal{N}$-regularity.*

Proof. Let (X, f) be an $\mathcal{N}$-closed ideal of a medial BCH-algebra X. Then $f(0*(x*y)) \leq f(x*y)$ for all $x, y \in X$. Note from [3, Lemma 1] that a medial BCH-algebra X satisfies the equality $x*y = 0*(y*x)$, It follows from (2.4) that

$$f(x) \leq \bigvee \{f(x*y), f(y)\} = \bigvee \{f(0*(y*x)), f(y)\} \leq \bigvee \{f(y*x), f(y)\}.$$

Therefore (X, f) satisfies the $\mathcal{N}$-regularity. □

4 $\mathcal{N}$-soft BCH-algebras

Definition 4.1 ([9]). Let X be an initial universe set and E a set of attributes. By an *$\mathcal{N}$-soft set* over X we mean a pair $(\tilde{f}, A)$ where $A \subset E$ and $\tilde{f}$ is a mapping from A to $\mathcal{F}(X, [-1, 0])$, i.e., for each $a \in A$, $\tilde{f}(a) := \tilde{f}_a$ is an $\mathcal{N}$-function on X.

Denote by $\mathcal{N}(X, E)$ the collection of all $\mathcal{N}$-soft sets over X with attributes from E and we call it an *$\mathcal{N}$-soft class*.

Definition 4.2 ([9]). Let $(\tilde{f}, A)$ and $(\tilde{g}, B)$ be $\mathcal{N}$-soft sets in $\mathcal{N}(X, E)$. Then $(\tilde{f}, A)$ is called an *$\mathcal{N}$-soft subset* of $(\tilde{g}, B)$, denoted by $(\tilde{f}, A) \tilde{\subseteq} (\tilde{g}, B)$, if it satisfies:

(i) $A \subseteq B$,

(ii) $(\forall e \in A) \left(\tilde{f}_e \subseteq \tilde{g}_e, \text{ i.e., } \tilde{f}_e(x) \le \tilde{g}_e(x) \text{ for all } x \in X \right)$.

Definition 4.3. Let $(\tilde{f}, A)$ be an $\mathcal{N}$-soft set over a BCH-algebra X where A is a subset of E. If there exists an attribute $u \in A$ for which the $\mathcal{N}$-structure $(X, \tilde{f}_u)$ is an $\mathcal{N}$-subalgebra of X, then we say that $(\tilde{f}, A)$ is an *$\mathcal{N}$-soft BCH-algebra* related to the attribute u (briefly, *$\mathcal{N}_u$-soft BCH-algebra*). If $(\tilde{f}, A)$ is an $\mathcal{N}_u$-soft BCH-algebra for all $u \in A$, we say that $(\tilde{f}, A)$ is an *$\mathcal{N}$-soft BCH-algebra*.

Example 4.4. Let $X := \{\text{apple}, \text{banana}, \text{carrot}, \text{peach}, \text{radish}\}$ be a universe, and consider a soft machine \$ which makes X into a BCH-algebra as follows:

$$x \,\$\, x = \text{ apple } \text{ for all } x \in X,$$

$$x \,\$\, \text{apple} = x \text{ for all } x \in X,$$

$$x \,\$\, \text{radish} = \text{radish} \quad \text{for all } x\,(\neq \text{radish}) \in X,$$

$$\text{apple} \,\$\, y = \text{apple} \quad \text{if } y \in \{\text{banana}, \text{carrot}, \text{peach}\},$$

$$\text{banana} \,\$\, y = \begin{cases} \text{apple} & \text{if } y = \text{carrot}, \\ \text{banana} & \text{if } y = \text{peach}, \end{cases}$$

$$\text{carrot} \,\$\, y = \begin{cases} \text{carrot} & \text{if } y = \text{banana}, \\ \text{apple} & \text{if } y = \text{peach}, \end{cases}$$

$$\text{peach} \,\$\, y = \text{peach} \quad \text{if } y \in \{\text{banana}, \text{carrot}\},$$

$$\text{radish} \,\$\, y = \text{radish} \quad \text{if } y \in \{\text{banana}, \text{carrot}, \text{peach}\}.$$

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Table 4: Tabular representation of $(\tilde{f}, A)$

$(\tilde{f}, A)$	apple	banana	carrot	peach	radish
cat	-0.8	-0.3	-0.6	-0.3	-0.8
cow	-0.7	-0.6	-0.5	-0.4	-0.3
horse	-0.5	-0.6	-0.2	-0.1	-0.3

Consider a set of attributes

$$A := \{\text{cat}, \text{cow}, \text{horse}\},$$

and let $(\tilde{f}, A)$ be an $\mathcal{N}$-soft set over X with the tabular representation which is given by Table 4. Then $(\tilde{f}, A)$ is an $\mathcal{N}$-soft BCH-algebra over X related to attributes "cat" and "cow". But it is not an $\mathcal{N}$-soft BCH-algebra over X related to the attribute "horse" since

$$\tilde{f}_{\text{hourse}}(\text{apple}) = -0.5 > -0.6 = \bigvee \left\{ \tilde{f}_{\text{hourse}}(\text{banana}), \tilde{f}_{\text{hourse}}(\text{banana}) \right\}.$$

Proposition 4.5. *Every $\mathcal{N}$-soft BCH-algebra $(\tilde{f}, A)$ over a BCH-algebra X satisfies the following inequality:*

$$(\forall x \in X)(\forall u \in A) \left(\tilde{f}_u(0) \leq \tilde{f}_u(x) \right). \tag{4.1}$$

Proof. For any $x \in X$ and $u \in A$, we have

$$\tilde{f}_u(0) = \tilde{f}_u(x * x) \leq \bigvee \left\{ \tilde{f}_u(x), \tilde{f}_u(x) \right\} = \tilde{f}_u(x).$$

This completes the proof. □

The problem we now discuss is:
If $(\tilde{g}, B)$ is an $\mathcal{N}$-soft BCH-algebra over a BCH-algebra X, then is every $\mathcal{N}$-soft subset of $(\tilde{g}, B)$ an $\mathcal{N}$-soft BCH-algebra over X?

Unfortunately this is not true as seen in the following example.

Example 4.6. Suppose there are four colors in the universe X, that is,

$$X := \{\text{white}, \text{blackish}, \text{reddish}, \text{green}\}$$

and $E := \{\text{beautiful}, \text{fine}, \text{moderate}, \text{delicate}, \text{elegant}, \text{smart}, \text{chaste}\}$ be a set of attributes. Let $\heartsuit$ be a soft machine to mix two colors according to order in such a way that we have the following results.

Table 5: Tabular representation of $(\tilde{g}, B)$

$(\tilde{g}, B)$	white	blackish	reddish	green
beautiful	-0.9	-0.4	-0.7	-0.4
fine	-0.8	-0.5	-0.8	-0.5
moderate	-0.7	-0.3	-0.3	-0.5
smart	-0.6	-0.4	-0.4	-0.6

$x \heartsuit \text{white} = x$ for all $x \in X$,

$y \heartsuit y = \text{white}$ for all $y \in X$,

$$\text{white} \heartsuit z = \begin{cases} \text{white} & \text{if } z = \text{reddish}, \\ \text{green} & \text{if } z \in \{\text{blackish}, \text{green}\}, \end{cases}$$

$$\text{blackish} \heartsuit w = \begin{cases} \text{green} & \text{if } w = \text{reddish}, \\ \text{redish} & \text{if } w = \text{green}, \end{cases}$$

$$\text{reddish} \heartsuit u = \begin{cases} \text{green} & \text{if } u = \text{blackish}, \\ \text{blackish} & \text{if } u = \text{green}, \end{cases}$$

$$\text{green} \heartsuit v = \begin{cases} \text{green} & \text{if } v = \text{reddish}, \\ \text{white} & \text{if } v = \text{blackish}. \end{cases}$$

Then $(X, \heartsuit, \text{white})$ is a BCH-algebra. Take

$$B = \{\text{beautiful}, \text{fine}, \text{moderate}, \text{smart}\}$$

and let $(\tilde{g}, B)$ be an $\mathcal{N}$-soft set over X with the tabular representation which is given by Table 5. Then $(\tilde{g}, B)$ is an $\mathcal{N}$-soft BCH-algebra over X. Now let $(\tilde{f}, A)$ be an $\mathcal{N}$-soft subset of $(\tilde{g}, B)$, where

$$A = \{\text{beautiful}, \text{fine}, \text{smart}\} \subset B$$

and the tabular representation of $(\tilde{f}, A)$ is given by Table 6. Then

$$\begin{aligned} &\tilde{f}_{\text{fine}}(\text{redish} \heartsuit \text{blackish}) = \tilde{f}_{\text{fine}}(\text{green}) = -0.55 \\ &> -0.65 = \bigvee \left\{ \tilde{f}_{\text{fine}}(\text{redish}), \tilde{f}_{\text{fine}}(\text{blackish}) \right\}, \end{aligned}$$

and so $(\tilde{f}, A)$ is not an $\mathcal{N}$-soft BCH-algebra over X related to the attribute "fine". Hence $(\tilde{f}, A)$ is not an $\mathcal{N}$-soft BCH-algebra over X.

Table 6: Tabular representation of $(\tilde{f}, A)$

$(\tilde{f}, A)$	white	blackish	reddish	green
beautiful	-0.99	-0.44	-0.77	-0.44
fine	-0.88	-0.65	-0.88	-0.55
smart	-0.66	-0.44	-0.44	-0.66

But, we have the following theorem.

Theorem 4.7. *For any subset A of E, let $(\tilde{f}, A)$ be an $\mathcal{N}$-soft BCH-algebra over a BCH-algebra X. If B is a subset of A, then $(\tilde{f}|_B, B)$ is an $\mathcal{N}$-soft BCH-algebra over X.*

Proof. Straightforward. □

Definition 4.8 ([9]). For any $(\tilde{f}, A), (\tilde{g}, B) \in \mathcal{N}(X, E)$, the *union* of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is defined to be the $\mathcal{N}$-soft set $(\tilde{h}, C)$ in (X, E) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $x \in C$,

$$\tilde{h}_x = \begin{cases} \tilde{f}_x & \text{if } x \in A \setminus B, \\ \tilde{g}_x & \text{if } x \in B \setminus A, \\ \tilde{f}_x \cup \tilde{g}_x & \text{if } x \in A \cap B. \end{cases}$$

In this case, we write $(\tilde{f}, A) \tilde{\cup} (\tilde{g}, B) = (\tilde{h}, C)$.

Lemma 4.9. *If $(X, \tilde{f})$ and $(X, \tilde{g})$ are $\mathcal{N}$-subalgebras of a BCH-algebra X, then the union $(X, \tilde{f} \cup \tilde{g})$ of $(X, \tilde{f})$ and $(X, \tilde{g})$ is an $\mathcal{N}$-subalgebra of X.*

Proof. Straightforward. □

Theorem 4.10. *If $(\tilde{f}, A)$ and $(\tilde{g}, B)$ are $\mathcal{N}$-soft BCH-algebras over a BCH-algebra X, then the union of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is an $\mathcal{N}$-soft BCH-algebra over X.*

Proof. Let $(\tilde{f}, A) \tilde{\cup} (\tilde{g}, B) = (\tilde{h}, C)$ be the union of $(\tilde{f}, A)$ and $(\tilde{g}, B)$. Then $C = A \cup B$. For any $x \in C$, if $x \in A \setminus B$ (resp. $x \in B \setminus A$) then $(X, \tilde{h}_x) = (X, \tilde{f}_x)$ (resp. $(X, \tilde{h}_x) = (X, \tilde{g}_x)$) is an $\mathcal{N}$-subalgebra of X. If $A \cap B \neq \emptyset$, then $(X, \tilde{h}_x) = (X, \tilde{f}_x \cup \tilde{g}_x)$ is an $\mathcal{N}$-subalgebra of X for all $x \in A \cap B$ by Lemma 4.9. Therefore $(\tilde{h}, C)$ is an $\mathcal{N}$-soft BCH-algebra over a BCH-algebra X. □

Table 7: Tabular representation of $(\tilde{g}, B)$

$(\tilde{f}, A)$	white	blackish	reddish	green
moderate	-0.6	-0.1	-0.6	-0.1
elegant	-0.4	-0.2	-0.2	-0.3

Definition 4.11 ([9]). Let $(\tilde{f}, A)$ and $(\tilde{g}, B)$ be two $\mathcal{N}$-soft sets in (X, E). The *intersection* of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is the $\mathcal{N}$-soft set $(\tilde{h}, C)$ in (X, E) where $C = A \cup B$ and for every $x \in C$,

$$\tilde{h}_x = \begin{cases} \tilde{f}_x & \text{if } x \in A \setminus B, \\ \tilde{g}_x & \text{if } x \in B \setminus A, \\ \tilde{f}_x \cap \tilde{g}_x & \text{if } x \in A \cap B. \end{cases}$$

In this case, we write $(\tilde{f}, A) \,\tilde{\cap}\, (\tilde{g}, B) = (\tilde{h}, C)$.

Theorem 4.12. *Let $(\tilde{f}, A)$ and $(\tilde{g}, B)$ be $\mathcal{N}$-soft BCH-algebras over a BCH-algebra X. If A and B are disjoint, then the intersection of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is an $\mathcal{N}$-soft BCH-algebra over X.*

Proof. Let $(\tilde{f}, A) \,\tilde{\cap}\, (\tilde{g}, B) = (\tilde{h}, C)$ be the intersection of $(\tilde{f}, A)$ and $(\tilde{g}, B)$. Then $C = A \cup B$. Since $A \cap B = \emptyset$, if $x \in C$ then either $x \in A \setminus B$ or $x \in B \setminus A$. If $x \in A \setminus B$, then $(X, \tilde{h}_x) = (X, \tilde{f}_x)$ is an $\mathcal{N}$-subalgebra of X. If $x \in B \setminus A$, then $(X, \tilde{h}_x) = (X, \tilde{g}_x)$ is an $\mathcal{N}$-subalgebra of X. Hence $(\tilde{h}, C)$ is an $\mathcal{N}$-soft BCH-algebra over a BCH-algebra X. □

The following example shows that Theorem 4.12 is not valid if A and B are not disjoint.

Example 4.13. Let X and $(\tilde{g}, B)$ be the BCH-algebra and the $\mathcal{N}$-soft BCH-algebra, respectively, in Example 4.6. Take $A = \{\text{moderate}, \text{elegant}\}$ and let $(\tilde{f}, A)$ be an $\mathcal{N}$-soft set over X with the tabular representation which is given by Table 7. Then $(\tilde{f}, A)$ is an $\mathcal{N}$-soft BCH-algebra over X. Note that A and B are not disjoint. The intersection $(\tilde{f}, A) \,\tilde{\cap}\, (\tilde{g}, B) = (\tilde{h}, C)$ of $(\tilde{f}, A)$ and $(\tilde{g}, B)$ is not an $\mathcal{N}$-soft BCH-algebra over X since

$$\begin{aligned} &\left(\tilde{f}_{\text{moderate}} \cap \tilde{f}_{\text{moderate}}\right)(\text{blackish} \heartsuit \text{reddish}) = \tilde{f}_{\text{moderate}}(\text{green}) = -05 > -0.6 \\ &= \bigvee \left\{ \left(\tilde{f}_{\text{moderate}} \cap \tilde{f}_{\text{moderate}}\right)(\text{blackish}), \left(\tilde{f}_{\text{moderate}} \cap \tilde{f}_{\text{moderate}}\right)(\text{reddish}) \right\}. \end{aligned}$$

References

[1] B. Ahmad, *On classification of BCH–algebras,* Math. Japon. **35** (1990), no. 5, 801–804.

[2] M. A. Chaudhry, *On BCH–algebras,* Math. Japon. **36** (1991), no. 4, 665–676.

[3] W. A. Dudek and J. Thomys, *On decompositions of BCH–algebras,* Math. Japon. **35** (1990), no. 6, 1131–1138.

[4] Q. P. Hu and X. Li, *On BCH–algebras,* Math. Sem. Notes Kobe Univ. **11** (1983), no. 2, part 2, 313–320.

[5] Q. P. Hu and X. Li, *On proper BCH–algebras,* Math. Japon. **30** (1985), no. 4, 659–661.

[6] Y. B. Jun and M. Kondo, *On transfer principle of fuzzy BCK/BCI-algebras,* Sci. Math. Jpn. **59** (2004), no. 1, 35–40.

[7] Y. B. Jun, K. J. Lee and S. Z. Song, *$\mathcal{N}$-ideals of BCK/BCI-algebras,* J. Chungcheong Math. Soc. **22** (2009), 417–437.

[8] Y. B. Jun, M. A. Öztürk and E. H. Roh, *$\mathcal{N}$-structures applied to closed ideals in BCH–algebras,* Int. J. Math. Math. Sci. **2010** Article ID 943565, 9 pages.

[9] Y. B. Jun, S. Z. Song and K. J. Lee, The combination of soft sets and $\mathcal{N}$-structures with applications, Commun. Korean Math. Soc. (submitted).

[10] M. Kondo and W. A. Dudek, *On the transfer principle in fuzzy theory,* Mathware Soft Comput. **12** (2005), no. 1, 41–55.

[11] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.

[12] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. 37 (1999) 19–31.

[13] L. A. Zadeh, From circuit theory to system theory, Proc. Inst. Radio Eng. 50 (1962) 856–865.

[14] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

[15] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) - an outline, Inform. Sci. 172 (2005) 1–40.

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Construction of Orthogonal Shearlet Tight Frames with Symmetry *

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Abstract

Shearlet frames play an important role in describing the singularities of multidimensional data. In this paper, we present a simple but complete method for constructing symmetric orthogonal shearlet tight frames from any given shearlet tight frames. This includes a family of cone-adapted ones. Finally, two examples are given.

Keywords: Shearlets, Tight frame, Orthogonality, Symmetry,

2000 MR Subject Classification: 42C15, 94A12

1 introduction

It is well known that the traditional theory of wavelet is based on the use of isotropic dilations. It can capture a point singularity of a function or distribution $f \in \mathbb{R}$, effectively. However, it is unable to describe the geometric regularity along the singularities of surfaces and lacks directional sensitivity(see [1]). These limitations have led to several new schemes, such as curvelets, contourlets and shearlets. Comparing to this methods, the shearlets stands out since it is based on a simple and rigorous mathematical framework which not only provides a more flexible theoretical tool for the geometric representation of multidimensional data, but also is more natural for implementation. Moveover, it can provide optimally sparse representations(see [2]). As a consequence, it can be associated to a multiresolution analysis and then this leads to a unified treatment of both the continuous and discrete world, i.e., allowing a digital theory to be a natural digitization of the continuum theory. It therefore has become popular in many applications, such as image denoising, enhancement, edge analysis and detection and separation(see [1], [3-7]).

In general, the construction of shearlet systems can be come in two classes today: One class is constructed through bandlimited functions on the space of $\mathbb{R}^2$, this is generated by a unitary representation of the shearlet group and equipped with a particulary 'nice' mathematical structure. For more details, we refer the reader to [1] and references therein. The other is constructed based on the cone, referred as cone-adapted shearlet. It is restricted to a horizontal and vertical cone in frequency domain, thereby ensuring an equal treatment of all direction. The interest reader is referred to [19] and references therein for more details. Both have their particular advantages and disadvantages.

Orthogonal frames, introduced by Weber in [8], is useful in multiple access communication systems, and has received much attention recently(see [9-12]). Just as the traditional theory of wavelets and frames, shearlet frames, as general frames, with symmetry is very desirable in various applications, since it can preserver linear phase properties and also allow symmetric boundary conditions

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in shearlet algorithms. Wu et al.(see [13]) constructed (anti)symmetric composite dilation multiwavelet frames with any symmetric points. However, the shearlet is the special case of composite dilation wavelet. In [14], in order to provide the parametrization of orthogonal and symmetric multiwavelets, Li and Yang presented a algorithms for constructing paraunitary symmetric matrices. Inspired by [13] and [14], we give a simple but complete method for constructing symmetric shearlet tight frames for $L_2(\mathbb{R}^2)$ and for the cone from any given shearlet tight frames, respectively.

2 Notations and lemmas

In this section, let us introduce some notations and lemmas. The Fourier transform of a function $f \in L_1(\mathbb{R}^2)$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^2} f(x)e^{-i\xi\cdot x}dx$ and can be extended to $L_2(\mathbb{R}^2)$ functions and tempered distributions, naturally, where $\cdot$ denotes the standard inner product in $\mathbb{R}^2$. An $r \times r$ integer matrix A is called a dilation matrix if $\lim_{n\to\infty} A^{-n} = 0$, i.e., all eigenvalues of A are greater than one in modulus. For a matrix $A(z)$, we denote $A^*(z)$ its transpose conjugate. We say that a matrix $A(z)$ is paraunitary symmetric if its entries are all (anti)symmetric Laurent polynomials and $A(z)A^*(z) = I_d$.

Let $A_a, B \in GL_2(\mathbb{R})$, where $a > 0$, and $GL_2(\mathbb{R})$ denotes the set of all 2×2 invertible matrices with real entries, which are defined by

$$A = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

where A and B denote a parabolic scaling matrix and a shear matrix, respectively. A shearlet system for $L_2(\mathbb{R}^2)$ is given by

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, ..., L\},$$

where $\psi \in L_2(\mathbb{R}^2)$, D_A is the dilation operator, defined by $D_A f(x) = |detA|^{-1/2} f(A^{-1}x)$, T_n is the translation operator, defined by $T_n f(x) = f(x - n)$. In this paper, we are interested in the special case where $a = 4$. It is interesting to observe that this choice gives a special case of the affine systems with composite dilations introduced in [15]. In this case, the shearlet system associated with shearlets $\psi_\ell \in L_2(\mathbb{R}^2)$ is given by the following expression

$$\{D_A^k D_B^m T_n \psi_\ell(x) := 2^{-3k/2}\psi_\ell(B_{-m}A^{-k}x - n),\ k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, ..., L\}, \tag{2.2}$$

where k denotes the scale, and m, n denote the direction and position of singularities, respectively.

Recall that a countable collection $\{\psi_\ell\}_{\ell\in\Gamma} \in L_2(\mathbb{R}^2)$ is a tight frame for $L_2(\mathbb{R}^2)$ if

$$\sum_{\ell\in\Gamma} |\langle f, \psi_\ell\rangle|^2 = \|f\|^2, \text{ for all } f \in L_2(\mathbb{R}^2).$$

This is equivalent to the reproducing formula $f = \sum_{\ell\in\Gamma}\langle f, \psi_\ell\rangle\psi_\ell$, for all $f \in L_2(\mathbb{R}^2)$, Γ is a countable index set.

With respect to characterizations of shearlet tight frames, there is the following Lemma, which is adapted from Theorem 5.5 in [15].

Lemma 2.1. *Let $A, B \subset GL_2(\mathbb{Z})$ be given by (2.1) and $\Psi = \{\psi_\ell\}_{\ell=1}^L \subset L_2(\mathbb{R}^2)$. Suppose that*

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^2}\int_{supp\ \hat{f}} |\hat{f}(\xi + nBA^k)|^2 |\hat{\psi}_\ell(\xi A^{-k}B^{-1})|^2 d\xi < \infty,$$

for all $f \in \mathcal{D}$, where $\mathcal{D}$ is a dense subspace of $L_2(\mathbb{R}^2)$ contained in the set

$$\{f \in L_2(\mathbb{R}^2) : \hat{f} \in L_\infty(\mathbb{R}^2) \text{ and } supp\ \hat{f} \text{ is compact}\}.$$

Then shearlet system $\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, ..., L\}$ is a shearlet tight frame for $L_2(\mathbb{R}^2)$ if and only if

$$\begin{cases} \sum\limits_{\ell=1}^{L}\sum\limits_{k\in\mathbb{Z}} |\hat{\psi}_\ell(\xi A^k B)|^2 = 1, \\ \sum\limits_{\ell=1}^{L}\sum\limits_{k\geq 0} \hat{\psi}_\ell(\xi A^k B)\overline{\hat{\psi}_\ell((\xi+\gamma)A^k B)} = 0, \quad for\ \gamma \in \mathbb{Z}^2 \backslash (\mathbb{Z}^2 A), \\ \sum\limits_{\ell=1}^{L}\sum\limits_{k\in\mathbb{Z}} \hat{\psi}_\ell(\xi A^k B)\overline{\hat{\psi}_\ell((\xi+\gamma)A^k B)} = 0, \quad for\ \gamma \in \bigcap_{k\in\mathbb{Z}} (\mathbb{Z}^2 A^k)\backslash\{0\}. \end{cases} \tag{2.3}$$

In [13], Wu et al. constructed the symmetric composite dilation multiwavelet frames from given dilation multiwavelet frames. For some functions $\{\psi_\ell\}_{\ell=1}^L \in L_2(\mathbb{R}^2)$ and points $\{x_\ell\}_{\ell=1}^L \in \mathbb{R}^2$, defining new functions through

$$\begin{cases} \psi_\ell^1(x) = \frac{\psi_\ell(x+x_\ell)+\psi_\ell(x-x_\ell)}{2}, \\ \psi_\ell^2(x) = \frac{\psi_\ell(x+x_\ell)-\psi_\ell(x-x_\ell)}{2}. \end{cases} \tag{2.4}$$

It is obvious that functions defined by (2.4) are (anti)symmetric with respect to points $\{x_\ell\}_{\ell=1}^L$.

As a special case of Theorem3.1 in [13], we have the following Lemma

Lemma 2.2. *Suppose that shearlet system*

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

defined by (2.2) is a shearlet tight frame for $L_2(\mathbb{R}^2)$. Then shearlet system

$$\{D_A^k D_B^m T_n \psi_\ell^1 \bigcup D_A^k D_B^m T_n \psi_\ell^2 : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

is a (anti)symmetric shearlet tight frame for $L_2(\mathbb{R}^2)$ with symmetric points $\{x_\ell\}_{\ell=1}^L$, where functions $\psi_\ell^1(x)$ and $\psi_\ell^2(x)$ are defined by (2.4).

Orthogonal frames plays an important role in multiple access communication systems and characterizations of superframes, the interested reader can find more details in [8], [9], [11].

Definition 2.3. Two Bessel sequence $\{\psi_\ell\}_{\ell\in\Gamma}$ and $\{\widetilde{\psi}_\ell\}_{\ell\in\Gamma}$ in $L_2(\mathbb{R}^2)$ are said to be orthogonal, if, for any function $f \in L_2(\mathbb{R}^2)$,

$$\sum_{\ell\in\Gamma} \langle f, \psi_\ell\rangle \widetilde{\psi}_\ell = 0.$$

Suppose that two sequence $\{\psi_\ell\}_{\ell\in\Gamma}$ and $\{\widetilde{\psi}_\ell\}_{\ell\in\Gamma}$ are both tight frames and are orthogonal to each other. Then for any functions $f, g \in L_2(R^2)$, we have

$$f = \sum_{\ell\in\Gamma} (\langle f, \psi_\ell\rangle + \langle g, \tilde{\psi}_\ell\rangle)\psi_\ell \quad \text{and} \quad g = \sum_{\ell\in\Gamma} (\langle f, \psi_\ell\rangle + \langle g, \tilde{\psi}_\ell\rangle)\tilde{\psi}_\ell.$$

That is, the frames can be used to encode two signals f and g, which can be sent over a single communications channel. Moreover, sequence $\{\psi_\ell, \tilde{\psi}_\ell\}_{\ell\in\Gamma}$ is a tight frame of the space $L_2(\mathbb{R}^2) \bigoplus L_2(\mathbb{R}^2)$, also it is referred as superframes, where $\bigoplus$ denotes the direct sum of $L_2(\mathbb{R}^2)$ 2 times. For more details see [8-11].

The following Lemma, which be trivial deduced from Theorem 1.5 in [8], describes the orthogonality of shearlet tight frames.

Lemma 2.4. *Let $\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ...L\}$ and $\{D_A^k D_B^m T_n \widetilde{\psi_\ell} : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ...L\}$ be two shearlet tight frames for $L_2(\mathbb{R}^2)$. Then they are a pair of orthogonal frames for $L_2(\mathbb{R}^2)$ if and only if, for a.e. $\xi \in \mathbb{R}^2$, the following equations hold*

$$\begin{cases} \sum\limits_{\ell=1}^{L}\sum\limits_{k\in\mathbb{Z}} \overline{\widehat{\psi}_\ell(\xi A^k B)}\widehat{\widetilde{\psi_\ell}}(\xi A^k B) = 0, \\ \sum\limits_{\ell=1}^{L}\sum\limits_{k\geq 0} \overline{\widehat{\psi}_\ell(\xi A^k B)}\widehat{\widetilde{\psi_\ell}}((\xi+\gamma)A^k B) = 0, \quad for\ \gamma \in \mathbb{Z}^2 \setminus (\mathbb{Z}^2 A), \\ \sum\limits_{\ell=1}^{L}\sum\limits_{k\in\mathbb{Z}} \overline{\widehat{\psi}_\ell(\xi A^k B)}\widehat{\widetilde{\psi_\ell}}((\xi+\gamma)A^k B) = 0, \quad for\ \gamma \in \bigcap\limits_{k\in\mathbb{Z}} (\mathbb{Z}^2 A^k) \setminus \{0\}. \end{cases} \tag{2.5}$$

3

3 Construction of symmetric orthogonal shearlet tight frames

In this section, we present a method for the construction of symmetric orthogonal shearlet tight frames of $L_2(\mathbb{R}^2)$. This also includes a family of cone-adapted shearlet system.

Theorem 3.1. *Suppose that shearlet system*

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

is a shearlet tight frame for $L_2(\mathbb{R}^2)$. Let $M(z)$ be a $Q \times P$ matrix with $\frac{\pi}{4}$-periodic entries $M_{i,j}(e^{-i\xi})$ and satisfy $M(e^{-i\xi}) \times M^*(e^{-i\xi}) = I_Q$, where $\xi \in \mathbb{R}^2$. Construct LQP functions $\psi_{\ell;i,j}(x)$ through

$$\widehat{\psi_{\ell;i,j}}(\xi) = M_{i,j}(e^{-i\xi})\widehat{\psi_\ell}(\xi), \ell = 1, ..., L; i = 1, ..., Q; j = 1, ..., P.$$

Then, for any integer $i \in \{1, ..., Q\}$, $\{D_A^k D_B^m T_n \psi_{\ell;i,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L\}$ is also a shearlet tight frames for $L_2(\mathbb{R}^2)$. Moreover, for any two different integers $i_1, i_2 \in \{1, ..., Q\}$, $\{D_A^k D_B^m T_n \psi_{\ell;i_1,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L; j = 1, ..., P\}$ and $\{D_A^k D_B^m T_n \psi_{\ell;i_2,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L; j = 1, ..., P\}$ are orthogonal to each other.

Proof.

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\sum_{\ell=1}^{L}\left|\widehat{\psi_{\ell;i,j}}(\xi A^k B)\right|^2 &= \sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\sum_{\ell=1}^{L}\left|M_{i,j}(e^{-i\xi A^k B})\right|^2\left|\widehat{\psi_\ell}(\xi A^k B)\right|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{L}\sum_{j=1}^{P}\left[\left|M_{i,j}(e^{-i\xi A^k B})\right|^2\right]\left|\widehat{\psi_\ell}(\xi A^k B)\right|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{L}\left|\widehat{\psi_\ell}(\xi A^k B)\right|^2 = 1,
\end{aligned}$$

where we use the property of $M(e^{-i\xi})M^*(e^{-i\xi}) = I_Q$ in the last equality. For a.e. $\xi \in \mathbb{R}^2$ and $\gamma \in \mathbb{Z}^2 \setminus (\mathbb{Z}^2 A)$, we obtain

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\geq 0}\sum_{j=1}^{P}\widehat{\psi}_{\ell;i,j}(\xi A^k B)\overline{\widehat{\psi}_{\ell;i,j}((\xi+\gamma)A^k B)} \\
&= \sum_{\ell=1}^{L}\sum_{k\geq 0}\sum_{j=1}^{P}\left[\overline{M_{i,j}(e^{-i\xi A^k B})} \times M_{i,j}\left(e^{-i(\xi+\gamma)A^k B}\right)\overline{\hat{\psi}_\ell(\xi A^k B)}\hat{\psi}_\ell(\xi+\gamma)A^k B\right] \\
&= \sum_{\ell=1}^{L}\sum_{k\geq 0}\hat{\psi}_\ell(\xi A^k B)\overline{\hat{\psi}_\ell((\xi+q)a^k b)} = 0,
\end{aligned}$$

where in the above equality, we use the periods of the components of $M(z)$ and the condition $M(e^{-i\xi})M^*(e^{-i\xi}) = I_Q$.

If $\gamma \in \bigcap_{k\in\mathbb{Z}}(\mathbb{Z}^2 A^k) \setminus \{0\}$, we also deduce that

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\widehat{\psi}_{\ell;i,j}(\xi A^k B)\overline{\widehat{\psi}_{\ell;i,j}((\xi+\gamma)A^k B)} \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\left[\overline{M_{i,j}(e^{-i\xi A^k B})} \times M_{i,j}(e^{-i(\xi+\gamma)A^k B})\overline{\hat{\psi}_\ell(\xi A^k B)}\hat{\psi}_\ell(\xi+\gamma)A^k B\right] \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\hat{\psi}_\ell(\xi A^k B)\overline{\hat{\psi}_\ell((\xi+\gamma)A^k B)} = 0.
\end{aligned}$$

Therefore, $\{D_A^k D_B^m T_n \psi_{\ell;i,j} : \ell = 1, ..., L; i = 1, ..., Q; j = 1, ..., P\}$ is a shearlet tight frame from Lemma 2.1.

In the following, we prove that they are orthogonal for any different integers $i_1, i_2 \in \{1, ..., Q\}$. In fact,

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\overline{\widehat{\psi}_{\ell;i_1,j}(\xi A^k B)}\widehat{\psi}_{\ell;i_2,j}(\xi A^k B)\\
&\quad=\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\left[\overline{M_{i_1,j}(e^{-i\xi A^k B})\hat{\psi}_{\ell;i_1,j}(\xi A^k B)}\times M_{i_2,j}(e^{-i\xi A^k B})\hat{\psi}_{\ell;i_2,j}(\xi A^k B)\right]\\
&\quad=\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\left[\sum_{j=1}^{P}\left(\overline{M_{i_1,j}(e^{-i\xi A^k B})}M_{i_2,j}(e^{-i\xi A^k B})\right)\times\overline{\hat{\psi}_{\ell;i_1,j}(\xi A^k B)}\hat{\psi}_{\ell;i_2,j}(\xi A^k B)\right]=0.
\end{aligned}$$

where in the second equality we use the periods of $M_{i,j}(e^{-i\xi})$ and the condition $M(e^{-i\xi})\times M^*(e^{-i\xi}) = I_Q$.

If $\gamma \in \mathbb{Z}^2 \setminus (\mathbb{Z}^2 A)$, we can deduce that

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\overline{\widehat{\psi}_{\ell;i_1,j}(\xi A^k B)}\widehat{\psi}_{\ell;i_2,j}((\xi+\gamma) A^k B)\\
&\quad=\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\left[\overline{M_{i_1,j}(e^{-i\xi A^k B})\hat{\psi}_{\ell;i_1,j}(\xi A^k B)}\times M_{i_2,j}(e^{-i(\xi+\gamma) A^k B})\hat{\psi}_{\ell;i_2,j}((\xi+\gamma) A^k B)\right]\\
&\quad=\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\left[\sum_{j=1}^{P}\left(\overline{M_{i_1,j}(e^{-i\xi A^k B})}M_{i_2,j}(e^{-i(\xi+\gamma) A^k B})\right)\times\overline{\hat{\psi}_{\ell;i_1,j}(\xi A^k B)}\hat{\psi}_{\ell;i_2,j}((\xi+\gamma) A^k B)\right]=0.
\end{aligned}$$

Similarly, for $\gamma \in \bigcap_{k\in\mathbb{Z}}(\mathbb{Z}^2 A^k) \setminus \{0\}$, we can have that

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{P}\overline{\widehat{\psi}_{\ell;i_1,j}(\xi A^k B)}\widehat{\psi}_{\ell;i_2,j}((\xi+\gamma) A^k B)=0$$

Thus we obtain the desired result. □

The following Corollary provides the construction of symmetric orthogonal shearlet tight frames from any symmetric shearlet tight frames.

Corollary 3.2. *Suppose that shearlet system*

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

is a symmetric shearlet tight frame for $L_2(\mathbb{R}^2)$. Let $M(e^{-i\xi})$ be a $Q \times P$ paraunitary symmetric matrix with $\frac{\pi}{4}$-periodic entries $M_{i,j}(e^{-i\xi})$, where $\xi \in \mathbb{R}^2$. Construct LQP functions $\psi_{\ell;i,j}$ through

$$\widehat{\psi_{\ell;i,j}}(\xi) = M_{i,j}(e^{-i\xi})\widehat{\psi_\ell}(\xi), \ell = 1, ..., L; i = 1, ..., Q; j = 1, ..., P.$$

Then, for any integer $i \in \{1, ..., Q\}$, $\{D_A^k D_B^m T_n \psi_{\ell;i,j} : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$ is also a symmetric shearlet tight frame for $L_2(\mathbb{R}^2)$. Moreover, for any two different integers $i_1, i_2 \in \{1, ..., Q\}$, $\{D_A^k D_B^m T_n \psi_{\ell;i_1,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L; j = 1, ..., P\}$ and $\{D_A^k D_B^m T_n \psi_{\ell;i_2,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, : \ell = 1, ..., L; j = 1, ..., P\}$ are orthogonal to each other.

Note that if two orthogonal shearlet systems $\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ...L\}$ and $\{D_A^k D_B^m T_n \widetilde{\psi}_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ...L\}$ are shearlet tight frames for $L_2(\mathbb{R}^2)$. Then we

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refer shearlet system $\{D_A^k D_B^m T_n \psi_\ell \bigcup D_A^k D_B^m T_n \widetilde{\psi}_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ...L\}$ as super shearlet frames for $L_2(\mathbb{R}^2) \bigoplus L_2(\mathbb{R}^2)$.

Note that it is easy to see that this construction has a drawback that the directional parameter m runs over the non-compact set $\mathbb{R}$. And then the distribution of directions becomes infinitely dense as m grows. In order to overcome this drawback, we can construct shearlet tight frames on the cone. First, we partition the frequency plane into the following four cones $C_1 - C_4$:

$$C_\kappa = \begin{cases} \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 1, |\xi_2/\xi_1| \leq 1\} : & \kappa = 1, \\ \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 1, |\xi_1/\xi_2| \leq 1\} : & \kappa = 2, \\ \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \leq -1, |\xi_2/\xi_1| \leq 1\} : & \kappa = 3, \\ \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \leq -1, |\xi_1/\xi_2| \leq 1\} : & \kappa = 4, \end{cases} \tag{3.1}$$

and a centered rectangle

$$\mathcal{R} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \|(\xi_1, \xi_2)\|_\infty < 1\} \tag{3.2}$$

Through adapting the construction of Corollary 3.2, we can obtain symmetric shearlet tight frames on the cone.

Proposition 3.3. *Let shearlet system*

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

be a shearlet tight frame for $L_2(\mathbb{R}^2)$*. Then the system*

$$\{D_A^k D_B^m T_n \psi_\ell : k > 0, -2^k \leq m \leq 2^k, n \in \mathbb{Z}^2, \ell = 1, ..., L\}$$

is a shearlet tight frame for $L_2(C_1 \bigcup C_3)^\vee$*. Moreover,*

$$\{D_A^k D_B^m T_n \psi_\ell^\natural : k > 0, -2^k \leq m \leq 2^k, n \in \mathbb{Z}^2, \ell = 1, ..., L\}$$

is also a shearlet tight frame for $L_2(C_2 \bigcup C_4)^\vee$*, where* $\psi_\ell^\natural(\xi_1, \xi_2) = \psi_\ell(\xi_2, \xi_1)$*,* $C_1 - C_4$ *are defined by (3.1).*

Based on the shearlet tight frames on the cone, we give the following theorem for constructing symmetric orthogonal shearlet tight frames on the same cone, which can be proved easily.

Theorem 3.4. *Suppose that shearlet system*

$$\{D_A^k D_B^m T_n \psi_\ell : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2, \ell = 1, 2, ..., L\}$$

is a symmetric shearlet tight frame for $L_2(C_1 \bigcup C_3)^\vee$*. Let* $M(e^{-i\xi})$ *be a* $Q \times P$ *paraunitary symmetric matrix with* $\frac{\pi}{4}$*-periodic entries* $M_{i,j}(e^{-i\xi})$ *and satisfy* $M(e^{-i\xi}) \times M^*(e^{-i\xi}) = I_Q$*, where* $\xi \in C_1 \bigcup C_3$*. Construct* LQP *functions* $\psi_{\ell;i,j}(x)$ *through*

$$\widehat{\psi_{\ell;i,j}}(\xi) = M_{i,j}(e^{-i\xi})\widehat{\psi_\ell}(\xi), \ell = 1, ..., L; i = 1, ..., Q; j = 1, ..., P.$$

Then, for any integer $i \in \{1, ..., Q\}$*,* $\{D_A^k D_B^m T_n \psi_{\ell;i,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L\}$ *is also a symmetric shearlet tight frame for* $L_2(C_1 \bigcup C_3)^\vee$*. Moreover, for any two different integers* $i_1, i_2 \in \{1, ..., Q\}$*,* $\{D_A^k D_B^m T_n \psi_{\ell;i_1,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, \ell = 1, ..., L; j = 1, ..., P\}$ *and* $\{D_A^k D_B^m T_n \psi_{\ell;i_2,j} : k, m \in \mathbb{Z}, j \in \mathbb{Z}^2, : \ell = 1, ..., L; j = 1, ..., P\}$ *are orthogonal to each other.*

In the same way, we can construct symmetric shearlet tight frames for $L_2(C_2 \bigcup C_4)^\vee$. Comparing to Corollary 3.2, the benefit of this construction is that the shear parameter k ranges over a finite set for each j. This is a obvious advantage for the numerical implementation.

4 Examples

It is well known that the shearlet system $\{D_A^k D_B^m T_n \psi : k, m \in \mathbb{Z}, n \in \mathbb{Z}^2\}$ is a shearlet tight frame for $L_2(\mathbb{R}^2)$(see [1]), where function $\psi \in L_2(\mathbb{R}^2)$ satisfies

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1}), \tag{4.1}$$

which is called classical shearlet, $\psi_1 \in L_2(\mathbb{R})$ satisfies the discrete Calderón condition given by $\sum_{j\in\mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$, with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and *supp* $\hat{\psi}_1 \subseteq [-\frac{1}{2}, -\frac{1}{16}] \bigcup [\frac{1}{16}, \frac{1}{2}]$, and $\psi_2 \in L_2(\mathbb{R})$ is a 'bump' function, i.e., for a.e. $\xi \in [-1, 1]$, $\sum_{k=-1}^{1} |\hat{\psi}_2(\xi + k)|^2 = 1$, satisfying $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ and *supp* $\hat{\psi}_2 \subseteq [-1, 1]$. There exist several choices of function ψ_1 and ψ_2 satisfying those conditions. One possible choice is to set ψ_1 to be a Lemariè-Meyer wavelet and ψ_2 to be a spline, for more details see [16] and references therein. Note that this shearlet tight frame is well-localized waveforms with frequency support increasing elongated at finer scale ($k \to -\infty$) and with the directions depending on m and n.

Example 1. Let functions $\psi^1(x)$ and $\psi^2(x)$ be defined by (2.4) in the case of $\ell = 1$, where ψ is classical shearlet given in (4.1). Assume that

$$M(z) = (M_{i,j}(z)) = \begin{bmatrix} \frac{z^4+z^{-4}}{2} & \frac{z^{-4}-z^4}{2} \\ \frac{z^4-z^{-4}}{2} & \frac{z^4+z^{-4}}{2} \end{bmatrix},$$

where $z = e^{-i(\xi_1/2+\xi_2)}$, Construct functions $\{\psi^\gamma_{\ell;i,j}, \gamma, \ell, i, j = 1, 2\}$ through

$$\begin{cases} \widehat{\psi^1_{1;1,j}}(\xi) = M_{1,j}(z)\widehat{\psi^1}(\xi), \\ \widehat{\psi^1_{2;1,j}}(\xi) = M_{1,j}(z)\widehat{\psi^2}(\xi), \\ \widehat{\psi^2_{1;2,j}}(\xi) = M_{2,j}(z)\widehat{\psi^1}(\xi), \\ \widehat{\psi^2_{2;2,j}}(\xi) = M_{2,j}(z)\widehat{\psi^2}(\xi). \end{cases}$$

Then, for every integer $\gamma \in \{1, 2\}$, $\{D_A^k D_B^m T_n \psi^\gamma_{\ell;i,j} : \gamma, \ell, i, j = 1, 2\}$ is symmetric shearlet tight frame for $L_2(\mathbb{R}^2)$. Moreover, $\{D_A^k D_B^m T_n \psi^1_{\ell;i,j} : \ell, i, j = 1, 2\}$ and $\{D_A^k D_B^m T_n \psi^2_{\ell;i,j} : \ell, i, j = 1, 2\}$ are orthogonal, namely, $\{D_A^k D_B^m T_n \psi^\gamma_{\ell;i,j} : \gamma, \ell, i, j = 1, 2\}$ is a shearlet tight frame of $L_2(\mathbb{R}^2) \bigoplus L_2(\mathbb{R}^2)$.

Example 2. Let ψ is also classical shearlet given by (4.1), then according to Theorem 3 in [17], the system

$$\{D_A^k D_B^m T_n \psi : k > 0, -2^k \le m \le 2^k, n \in \mathbb{Z}^2\}$$

is a shearlet tight frame for $L_2(C_1 \bigcup C_3)^\vee$. In the cone of $(C_1 \bigcup C_3)^\vee$, constructing functions $\phi^1(x)$ and $\phi^2(x)$ by (2.4) with $\ell = 1$, then they are (anti)symmetric. Let 2×2 matrix $N(e^{-i\xi})$ be given by.

$$N(e^{-i\xi}) = (N_{i,j}(e^{-i\xi})) = \begin{bmatrix} \frac{e^{-i\xi}+e^{i\xi}}{2} & \frac{e^{i\xi}-e^{-i\xi}}{2} \\ \frac{e^{-i\xi}-e^{i\xi}}{2} & \frac{e^{-i\xi}+e^{i\xi}}{2} \end{bmatrix},$$

where $\xi \in C_1 \bigcup C_3$, Construct functions $\{\phi^\gamma_{\ell;i,j}, \gamma, \ell, i, j = 1, 2\}$ through

$$\begin{cases} \widehat{\phi^1_{1;1,j}}(\xi) = N_{1,j}(z)\widehat{\phi^1}(\xi), \\ \widehat{\phi^1_{2;1,j}}(\xi) = N_{1,j}(z)\widehat{\phi^2}(\xi), \\ \widehat{\phi^2_{1;2,j}}(\xi) = N_{2,j}(z)\widehat{\phi^1}(\xi), \\ \widehat{\phi^2_{2;2,j}}(\xi) = N_{2,j}(z)\widehat{\phi^2}(\xi). \end{cases}$$

Then, for every integer $\gamma \in \{1, 2\}$, system $\{D_A^k D_B^m T_n \phi^\gamma_{\ell;i,j} : k > 0, -2^k \le m \le 2^k, \gamma, \ell, i, j = 1, 2\}$ is symmetric shearlet tight frame for $L_2(C_1 \bigcup C_3)^\vee$. In the same way, let $\widetilde{\psi}(\xi_1, \xi_2) = \psi(\xi_2, \xi_1)$, where $\psi(\xi_1, \xi_2)$ is classical shearlet, we can obtain a shearlet tight frame for $L_2(C_2 \bigcup C_4)^\vee$.

References

[1] G. Kutyniok D. Labate, Shearlets: multiscale analysis for multivariate data, *Birkhäuser, Basel,* 2012.

[2] K. Guo, D. Labate, Optimally sparse multidimensional representation useing shearlets, *SIAM J. Math. Anal.,* 39 (2007) 298-318.

[3] G. R. Easley, D. Labate, and F. Colonna, Shearlet-based total variation for denoising, *IEEE Trans. Image Processing,* 18(2) (2009) 260-268.

[4] K. Guo and D. Labate, Characterization and analysis of edges using the continuous shearlet transform, *SIAM J. Imaging Sciences,* 2 (2009) 959-986.

[5] G. Kutyniok and W. Lim, Image separation using shearlets, in: Curves and Surfaces (*Avignon, France,* 2010), Lecture Notes in Computer Science 6920, *Springer,* 2012.

[6] W. Q. Lim, The discrete shearlet transform: a new directional transform and compactly supported shearlet frames, *IEEE Trans. Image Proc.,* 19(5) (2010) 1166-1180.

[7] S. Yi, D. Labate, G. R. Easley, and H. Krim, A Shearlet approach to edge analysis and detection, *IEEE Trans. Image Proc.,* 18(5) (2009) 929-941.

[8] E. Weber, Orthogonal frames of translates, *Appl. Comput. Harmon. Anal.,* 17 (2004) 69-90.

[9] G. Bhatt, B. Johnson, E. Weber, Orthogonal wavelet frames and vector-valued wavelet transforms, *Appl. Comput. Harmon. Anal.,* 23 (2) (2007) 215-234.

[10] Y. Li, S. Yang, Explicit construction of symmetric orthogonal wavelet frames in $L_2(R^s)$, *J. Approx. Theory,* 162 (2010) 891-909.

[11] H. Kim, R. Kim, J. Lim, Z. Shen, A pair of orthogonal frames, J. Approx. Theory, 147 (2007) 196-204.

[12] Z. Liu, Y. Ren, G. Wu, Orthogonal frames and their dual frames in $L_2(R^d)$, *Journal of Information Computational Science,* 9(5) (2012) 1329-1336.

[13] G. Wu, X. Tian, D. Li, The construction of symmetric or antisymmetric multiwavelets with composite dilations, *Acta Math. Sinica(Chin. Ser.),*54 (2011) 731-738.

[14] Y. Li, S. Yang, Construction of paraunitary symmetric matrix and parametrization of symmetric and orthogonal multiwavelets filter banks, *Acta Math. Sinica(Chin. Ser.),* 53 (2010) 279-290.

[15] K. Guo, D. Labate, W. Lim, G. Weiss, E. Wilson, Wavelets with composite dilations and their MRA properties, *Appl. Comput. Harmon. Anal.,* 20 (2006) 202-236.

[16] G. Easley, D. Labate, W. Lim, Sparse directional image representations using the discrete shearlet transform, *Appl. Comput. Harmon. Anal.,* 25 (2008) 25-46.

[17] K. Guo, G. Kutyniok, D. Labate, Sparse multidimensional representations using anisotropic dilation and shear operators, in Wavelets and Splines *(Athens, GA, 2005), Nashboro Press, Nashville, TN,* 2006, 189-201.

[18] P. Grohs, Continuous shearlet tight frames, *J. Fourier Anal. Appl.,* 17 (2011) 506-518.

[19] P. Kittipoom, G. Kutyniok W. Lim, Construction of compactly supported shearlet frames, *Constr. Approx.,* 35(2012) 21-72. DOI: 10.1007/s00365-011-9142-y.

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Exact orders in simultaneous approximation by complex q-Durrmeyer type operators

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Abstract. In this paper we study the simultaneous approximation properties of the complex q-Durrmeyer type operators which were introduced by Agarwal and Gupta [3]. We obtain the exact orders in approximation by these operators and their derivatives on compact disks.

Keywords: complex q-Durrmeyer type operators; simultaneous approximation; exact orders; q-calculus

Mathematical subject classification: 30E10, 41A25

1. Introduction

Let $q > 0$, for each nonnegative integer k, the $q-$integer $[k]_q$ and the $q-$factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases}$$

and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0 \end{cases}$$

respectively.

Then for $q > 0$ and integers n, k, $n \geq k \geq 0$, we have

$$[k+1]_q = 1 + q[k]_q \quad and \quad [k]_q + q^k[n-k]_q = [n]_q.$$

For the integers $n, k, n \geq k \geq 0$, the $q-$binomial coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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Let $q>0,\ q\neq 1$, we can define the derivative $D_q f$ of functions f in the $q-$calculus by

$$D_q f(x)=\begin{cases}\frac{f(qx)-f(x)}{(q-1)x}, & x\neq 0,\\ f'(0), & x=0.\end{cases}$$

Let $a>0$, the $q-$Jackson integrals in the interval $[0,a]$ is defined as

$$\int_0^a f(t)d_qt=(1-q)a\sum_{j=0}^{\infty}f(aq^j)q^j,\ 0<q<1.$$

The $q-$analogue of Beta function is defined as

$$B_q(m,n)=\int_0^1 t^{m-1}(1-qt)_q^{n-1}d_qt,\quad m,n>0,$$

where

$$(a-b)_q^n=\prod_{j=0}^{n-1}(a-q^jb).$$

Also, it is known that

$$B_q(m,n)=\frac{[m-1]_q![n-1]_q!}{[m+n-1]_q!}.$$

All of the previous concepts can be found in [1, 4, 11].

In 1986, the approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [12]. Recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. (for instance, see [2, 5-10, 13-16]). In 2012, Agarwal and Gupta [3] introduced and studied the complex q-Durrmeyer type operators as follows:

$$\begin{aligned}M_{n,q}(f;z)=[n+1]_q\sum_{k=1}^{n}q^{1-k}p_{n,k}(q;z)\int_0^1 p_{n,k-1}(q;qt)f(t)d_qt\\ +f(0)p_{n,0}(q;z),\end{aligned}\tag{1.1}$$

where $z\in\mathbb{C}, n=1,2,\ldots,0<q<1$ and

$$p_{n,k}(q;z):=\left[\begin{array}{c}n\\k\end{array}\right]_q z^k\prod_{s=0}^{n-k-1}(1-q^sz)=\left[\begin{array}{c}n\\k\end{array}\right]_q z^k(1-z)_q^{n-k}.$$

In [3], the upper bound for the operators (1.1) was obtained as follows: **Theorem 1.** Let $f(z)=\sum\limits_{m=0}^{\infty}c_mz^m$ for all $|z|<R$ and let $1\le r\le R$, then for all $|z|\le r$, $0<q<1$ and $n\in\mathbb{N}$, we have

$$|M_{n,q}(f;z)-f(z)|\le\frac{K_r(f)}{[n+2]_q},$$

where $K_r(f) = (1+r)\sum\limits_{m=1}^{\infty} |c_m| m(m+1)r^{m-1} < \infty$.

The aim of the present article is to obtain the simultaneous approximation results for the complex q-Durrmeyer type operators (1.1) in the case $0 < q < 1$.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1.(see [3]) Let $0 < q < 1$. Then, for all $e_m(t) = t^m$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have the following recurrence relation:

$$M_{n,q}(e_{m+1}; z) = \frac{q^m z(1-z)}{[m+n+2]_q} D_q M_{n,q}(e_m; z) + \frac{[m]_q + zq^m [n]_q}{[m+n+2]_q} M_{n,q}(e_m; z). \quad (2.1)$$

Lemma 2. If $P_m(z)$ is a polynomial of degree m, for all $|z| \le r$, we have

$$|D_q P_m(z)| \le \|P_m'\|_r \le \frac{m}{r} \|P_m\|_r, \quad (2.2)$$

where $\|P_m\|_r = max\{|P_m(z)|; |z| \le r\}$.

Proof. The proof is easy by using the Bernstein inequality and the complex mean value theorem, the proof is omitted in this.

The following Voronovskaja-type result with a quantitative estimate holds.

Lemma 3. Let $0 < q < 1$, $R > 1$, $D_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R, i.e. $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. Then for any fixed $r \in [1, R]$ and for all $n \in \mathbb{N}$, $|z| \le r$, we have

$$\left| M_{n,q}(f; z) - f(z) + \frac{[2]_q z f'(z)}{[n+2]_q} - \frac{z(1-z) f''(z)}{[n+2]_q} \right| \le \frac{M_r(f)}{[n+2]_q^2} + 2(1-q)\sum_{k=1}^{\infty} |c_k| k r^k,$$

where $M_r(f) = \sum\limits_{k=1}^{\infty} |c_k| k F_{k,r} r^k < \infty$ with $F_{k,r} = (k-1)(k-2)(2k-3) + 4(k+1)(k-1)^2 + 2(k-1)(k+1)^2 + 4(k-1)^2 k(1+r)$.

Proof. Denoting $e_k(z) = z^k$, $k = 0, 1, 2, \ldots$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$, we can write $M_{n,q}(f; z) = \sum\limits_{k=0}^{\infty} c_k M_{n,q}(e_k; z)$, thus, for all $z \in D_R$ and $n \in \mathbb{N}$, we have

$$\left| M_{n,q}(f; z) - f(z) + \frac{[2]_q z f'(z)}{[n+2]_q} - \frac{z(1-z) f''(z)}{[n+2]_q} \right|$$

$$\leq \sum_{k=1}^{\infty} |c_k| \left| M_{n,q}(e_k; z) - e_k(z) + \frac{k[2]_q z^k}{[n+2]_q} - \frac{k(k-1)(1-z)z^{k-1}}{[n+2]_q} \right|.$$

Denoting

$$E_{k,n}(q; z) = M_{n,q}(e_k; z) - e_k(z) + \frac{k[2]_q z^k}{[n+2]_q} - \frac{k(k-1)(1-z)z^{k-1}}{[n+2]_q},$$

it is obvious that $E_{k,n}(q; z)$ is a polynomial of degree less than or equal to k and that $E_{1,n}(q; z) = \frac{(1-q^2)[n]_q z}{[n+2]_q}$. For $k \geq 2$, by simple computation and the use of Lemma 1, we can get

$$\begin{aligned} E_{k,n}(q; z) = {} & \frac{q^{k-1} z(1-z)}{[n+k+1]_q} D_q E_{k-1,n}(q; z) \\ & + \frac{q^{k-1}[n]_q z + [k-1]_q}{[n+k+1]_q} E_{k-1,n}(q; z) + G_{k,n}(q; z), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} G_{k,n}(q; z) = {} & \frac{z^{k-2}}{[n+2]_q [n+k+1]_q} \left\{ z^2 \left[k[n+k+1]_q([2]_q + k - 1) \right.\right. \\ & -[n+2]_q[n+k+1]_q - q^{k-1}(k-1)([2]_q + k - 2)[n]_q \\ & +q^{k-1}[n]_q[n+2]_q + q^{k-1}(k-1)[k-1]_q([2]_q + k - 2) \\ & \left. -q^{k-1}[k-1]_q[n+2]_q \right] + z \left[q^{k-1}[k-1]_q[n+2]_q \right. \\ & -q^{k-1}(k-1)[k-1]_q([2]_q + k - 2) - q^{k-1}(k-1)(k-2)[k-2]_q \\ & +[k-1]_q[n+2]_q - (k-1)[k-1]_q([2]_q + k - 2) \\ & \left. +q^{k-1}(k-1)(k-2)[n]_q - k(k-1)[n+k+1]_q \right] \\ & \left. + \left[q^{k-1}(k-1)(k-2)[k-2]_q + (k-1)(k-2)[k-1]_q \right] \right\} \\ =: {} & \frac{z^{k-2}}{[n+2]_q [n+k+1]_q} (z^2 A_{k,n}(q) + z B_{k,n}(q) + C_{k,n}(q)). \end{aligned}$$

For all $k \geq 2$, we easily obtain $|C_{k,n}(q)| \leq (k-1)(k-2)(2k-3)$.

Since $[n+k+1]_q = [k-1]_q + q^{k-1}[n+2]_q$ and $[n]_q = [n+2]_q - q^n - q^{n+1}$, so, for all $k \geq 2$, we can get

$$\begin{aligned} B_{k,n}(q) = {} & [n+2]_q [q^{k-1}[k-1]_q + [k-1]_q - 2q^{k-1}(k-1)] \\ & - q^{k-1}(k-1)[k-1]_q([2]_q + k - 2) - q^{k-1}(k-1)(k-2)[k-2]_q \\ & - q^{n+k-1}(q+1)(k-1)(k-2) - (k-1)[k-1]_q([2]_q + k - 2) \\ & - k(k-1)[k-1]_q. \end{aligned}$$

In view of $[k-1]_q - (k-1) = (q-1) \sum_{j=0}^{k-2} [j]_q$, $[k-1]_q - q^{k-1}(k-1) =$

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$(1-q)\sum\limits_{j=1}^{k-1}[j]_q q^{k-1-j}$ and $[n+2]_q = \frac{q^{n+2}-1}{q-1}$, we have

$$\begin{aligned}&[n+2]_q[q^{k-1}[k-1]_q + [k-1]_q - 2q^{k-1}(k-1)]\\&= q^{k-1}(q^{n+2}-1)\sum_{j=0}^{k-2}[j]_q + (1-q^{n+2})\sum_{j=1}^{k-1}[j]_q q^{k-1-j},\end{aligned}$$

thus, by simple calculation, for all $k \geq 2$, we get $|B_{k,n}(q)| \leq 4(k+1)(k-1)^2$.

Now we estimate $A_{k,n}(q)$. Similar to the calculation of the $B_{k,n}(q)$, for all $k \geq 2$, we have

$$\begin{aligned}A_{k,n}(q) = &-q^{k-1}(q^{n+2}-1)\sum_{j=0}^{k-2}[j]_q - (1-q^{n+2})\sum_{j=1}^{k-1}[j]_q q^{k-1-j}\\&+ (1-q^n)(q^k+q^{k-1})[n+2]_q + k[k-1]_q([2]_q + k-1)\\&+ q^{n+k-1}(q+1)(k-1)([2]_q + k-2)\\&+ q^{k-1}(k-1)[k-1]_q([2]_q + k-2),\end{aligned}$$

by simple calculation, for all $k \geq 2$, it follows that $|A_{k,n}(q)| \leq 2(k-1)(k+1)^2 + 2(1-q^n)[n+2]_q$. Thus, for all $n \in \mathbb{N}$, $k \geq 2$ and $|z| \leq r$, we can obtain

$$\begin{aligned}|G_{k,n}(q;z)| \leq &\frac{r^{k-2}}{[n+2]_q^2}[(k-1)(k-2)(2k-3) + 4r(k+1)(k-1)^2\\&+ 2r^2(k-1)(k+1)^2] + 2r^k(1-q).\end{aligned}$$

By formula (2.3), for all $n \in \mathbb{N}$, $k \geq 2$ and $|z| \leq r$, we have

$$\begin{aligned}|E_{k,n}(q;z)| \leq &\frac{r(1+r)}{[n+k+1]_q}|D_q E_{k-1,n}(q;z)|\\&+ \frac{q^{k-1}[n]_q r + [k-1]_q}{[n+k+1]_q}|E_{k-1,n}(q;z)| + |G_{k,n}(q;z)|,\end{aligned}$$

since $q^{k-1}[n]_q r + [k-1]_q \leq [n+k+1]_q r$, it follows

$$|E_{k,n}(q;z)| \leq \frac{r(1+r)}{[n+k+1]_q}|D_q E_{k-1,n}(q;z)| + r|E_{k-1,n}(q;z)| + |G_{k,n}(q;z)|.$$

Using the estimate in the proof of Theorem 1, we get

$$|M_{n,q}(e_k;z) - e_k(z)| \leq \frac{(1+r)k(k+1)r^{k-1}}{[n+2]_q},$$

for all $k, n \in \mathbb{N}$, $|z| \leq r$, $1 \leq r$.

Denote $\|f\|_r = max\{|f(z)|; |z| \le r\}$, by Lemma 2 we have

$$\begin{aligned}|D_q E_{k-1.n}(q;z)| &\le \frac{k-1}{r} \parallel E_{k-1,n} \parallel_r \\ &\le \frac{k-1}{r}\left[\|M_{n,q}(e_{k-1};\cdot) - e_{k-1}\|_r\right. \\ &\quad \left. + \left\| \frac{(k-1)[2]_q e_{k-1}}{[n+2]_q} - \frac{(k-1)(k-2)(1-e_1)e_{k-2}}{[n+2]_q} \right\|_r \right] \\ &\le \frac{k-1}{r}\left[\frac{k(k-1)(1+r)r^{k-2}}{[n+2]_q} + \frac{k(k-1)(1+r)r^{k-2}}{[n+2]_q}\right] \\ &\le \frac{4(k-1)^2 k r^{k-1}}{[n+2]_q},\end{aligned}$$

so, for all $n \in \mathbb{N}$, $k \ge 2$ and $|z| \le r$, we have

$$|E_{k,n}(q;z)| \le \frac{4(k-1)^2 k(1+r)r^k}{[n+2]_q^2} + r|E_{k-1,n}(q;z)| + |G_{k,n}(q;z)|,$$

where $|G_{k,n}(q;z)| \le \frac{r^k}{[n+2]_q^2} D_k + 2r^k(1-q)$, $D_k = (k-1)(k-2)(2k-3) + 4(k+1)(k-1)^2 + 2(k-1)(k+1)^2$.

On the other hand, for all $n \in \mathbb{N}$ and $|z| \le r$, $|E_{1,n}(q;z)| = |\frac{(1-q^2)[n]_q z}{[n+2]_q}| \le 2r(1-q)$, therefore, for all $k, n \in \mathbb{N}$ and $|z| \le r$, we have $|E_{k,n}(q;z)| \le r|E_{k-1,n}(q;z)| + \frac{r^k}{[n+2]_q^2} F_{k,r} + 2r^k(1-q)$, where $F_{k,r}$ is a polynomial of degree 3 in k defined as $F_{k,r} = D_k + 4(k-1)^2 k(1+r)$.

Since $E_{0.n}(q;z) = 0$ for any $z \in \mathbb{C}$, therefore, by writing the last inequality for $k = 1, 2, \ldots$, we easily obtain step by step the following

$$|E_{k,n}(q;z)| \le \frac{r^k}{[n+2]_q^2} \sum_{j=1}^{k} F_{j,r} + 2kr^k(1-q) \le \frac{kr^k}{[n+2]_q^2} F_{k,r} + 2kr^k(1-q).$$

As a conclusion, we have

$$\left| M_{n,q}(f;z) - f(z) + \frac{[2]_q z f'(z)}{[n+2]_q} - \frac{z(1-z)f''(z)}{[n+2]_q} \right| \le \sum_{k=1}^{\infty} |c_k||E_{k,n}(q;z)|$$

$$\le \frac{1}{[n+2]_q^2} \sum_{k=1}^{\infty} |c_k| k F_{k,r} r^k + 2(1-q) \sum_{k=1}^{\infty} |c_k| k r^k.$$

As $f^{(4)}(z) = \sum\limits_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3) z^{k-4}$ and the series is absolutely con-vergent in $|z| \le r$, it easily follows that $\sum\limits_{k=4}^{\infty} |c_k| k(k-1)(k-2)(k-3) r^{k-4} < \infty$, which implies that $\sum\limits_{k=1}^{\infty} |c_k| k F_{k,r} r^k < \infty$, this completes the proof of theorem.

Remark 1. Let $0 < q < 1$ be fixed. Since we have $\frac{1}{[n+2]_q} \to 1 - q$ as $n \to \infty$,

by passing to limit with $n \to \infty$ in the estimates in Lemma 2 we don't obtain convergence. But the situation can be improved by choosing $1 - \frac{1}{(n+2)^2} \leq q_n < 1$ with $q_n \to 1$ as $n \to \infty$. Indeed, since in this case $\frac{1}{[n+2]_{q_n}} \to 0$ as $n \to \infty$ (see Videnskii [17], formula (2.7)) and $1 - q_n \leq \frac{1}{(n+2)^2} \leq \frac{1}{[n+2]_{q_n}^2}$, from Lemma 2 we get

$$\left| M_{n,q_n}(f;z) - f(z) + \frac{[2]_{q_n} z f'(z)}{[n+2]_{q_n}} - \frac{z(1-z)f''(z)}{[n+2]_{q_n}} \right| \leq \frac{M_r(f)}{[n+2]_{q_n}^2} + \frac{2}{[n+2]_{q_n}^2} \sum_{k=1}^{\infty} |c_k| k r^k, \tag{2.4}$$

that is the order of approximation $\frac{1}{[n+2]_{q_n}^2}$.

3. Main results

In the following theorem, we will obtain the simultaneous approximation properties of the operators (1.1).

Theorem 2. Let $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $|z| < R$ and let $1 \leq r \leq R$, $0 < q < 1$. If $r < r_1 < R$ are arbitrary fired, then for all $|z| \leq r$, $n, p \in \mathbb{N}$, we have

$$|M_{n,q}^{(p)}(f;z) - f^{(p)}(z)| \leq \frac{K_{r_1}(f) p! r_1}{[n+2]_q (r_1 - r)^{p+1}},$$

where $K_{r_1}(f) = (1 + r_1) \sum_{m=1}^{\infty} |c_m| m(m+1) r_1^{m-1} < \infty$.

Proof. Denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v - z| \geq r_1 - r$, by the Cauchy's formulas and the Theorem 1, it follows that for all $|z| \leq r$ and $n, p \in \mathbb{N}$, we have

$$\begin{aligned} |M_{n,q}^{(p)}(f;z) - f^{p}(z)| &= \frac{p!}{2\pi} \left| \int_{\Gamma} \frac{M_{n,q}(f;v) - f(v)}{(v-z)^{p+1}} dv \right| \\ &\leq \frac{K_{r_1}(f)}{[n+2]_q} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= \frac{K_{r_1}(f)}{[n+2]_q} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}}, \end{aligned}$$

which proves the theorem.

Theorem 3. Let $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $|z| < R$ and let $1 - \frac{1}{(n+2)^2} \leq q_n < 1$, $n \in \mathbb{N}$. Suppose that $1 \leq r < r_1 < R$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq p - 1$, then we have

$$\|M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)}\|_r \asymp \frac{1}{[n+2]_{q_n}}, \ n \in \mathbb{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend only on f, r, r_1, p and on the sequence $\{q_n\}_{n \in \mathbb{N}}$.

Proof. Taking into account the upper estimate in Theorem 1, it remains to prove the lower estimate only.

Denoting by Γ the circle of radius $r_1 > r$ and center 0 , by the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$M^{(p)}_{n,q_n}(f;z) - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{M_{n,q_n}(f;v) - f(v)}{(v-z)^{p+1}}dv.$$

Let

$$H_{n,q_n}(f;z) = M_{n,q_n}(f;z) - f(z) + \frac{[2]_{q_n} z f'(z)}{[n+2]_{q_n}} - \frac{z(1-z)f''(z)}{[n+2]_{q_n}}.$$

For all $n \in \mathbb{N}$, we have

$$M_{n,q_n}(f;z) - f(z)$$

$$= \frac{1}{[n+2]_{q_n}}\left\{ z(1-z)f''(z) - [2]_{q_n} z f'(z) + \frac{1}{[n+2]_{q_n}}\left[[n+2]^2_{q_n} H_{n,q_n}(f;z) \right] \right\},$$

by using Cauchy's formula, for all $v \in \Gamma$, we get

$$\begin{aligned} M^{(p)}_{n,q_n}(f;z) - f^{(p)}(z) = & \frac{1}{[n+2]_{q_n}}\left\{ [z(1-z)f''(z) - [2]_{q_n} z f'(z)]^{(p)} \right. \\ & \left. + \frac{1}{[n+2]_{q_n}}\frac{p!}{2\pi i}\int_\Gamma \frac{[n+2]^2_{q_n} H_{n,q_n}(f;v)}{(v-z)^{p+1}}dv \right\}, \end{aligned}$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\begin{aligned} \left\| M^{(p)}_{n,q_n}(f;\cdot) - f^{(p)} \right\|_r \geq & \frac{1}{[n+2]_{q_n}} \left[\| [e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)} \|_r \right. \\ & \left. - \frac{1}{[n+2]_{q_n}} \left\| \frac{p!}{2\pi i}\int_\Gamma \frac{[n+2]^2_{q_n} H_{n,q_n}(f;v)}{(v-\cdot)^{p+1}}dv \right\|_r \right]. \end{aligned}$$

By hypothesis on f, we have $\|[e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)}\|_r > 0$. Indeed, let $p = 1$, supposing the contrary, it follows that $z(1-z)f''(z) - [2]_{q_n} z f'(z)$ is a constant. Clearly, this is possible only if f is constant (since contrariwise $z(1-z)f''(z) - [2]_{q_n} z f'(z)$ is a polynomial of degree at least 1, which cannot be equal to a constant), which implies f is a polynomial of degree $\leq p-1$, a contradiction. let $p = 2$, supposing the contrary, it follows that $z(1-z)f''(z) - [2]_{q_n} z f'(z)$ is a polynomial of degree ≤ 1. Clearly, this is possible only if f is a polynomial of degree ≤ 1 (since contrariwise $z(1-z)f''(z) - [2]_{q_n} z f'(z)$ is a polynomial of degree at least 2, which cannot is a polynomial of degree ≤ 1), which implies f is a polynomial of degree $\leq p-1$, a contradiction.

Now let $p \geq 3$, suppose that $\|[e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)}\|_r = 0$, it follows that $z(1-z)f''(z) - [2]_{q_n} z f'(z)$ is a polynomial of degree $\leq p-1$, that is $z(1-z)f''(z) - [2]_{q_n} z f'(z) = Q_{p-1}(z)$, for all |z| $\leq$ r, where $Q_{p-1}(z)$ is an algebraic polynomial of degree $\leq p-1$, with complex coefficients.

From above we also get $Q_{p-1}(0) = 0$, which means that $Q_{p-1}(z)$ is necessarily of the form $Q_{p-1}(z) = \sum_{k=1}^{p-1} A_k z^k$ and that we can simplify with z in the equation. Now, denoting $f' = F$, the above differential equation one reduces to

$$(1-z)F'(z) - [2]_{q_n} F(z) = H_{p-2}(z), \text{for all } |\mathrm{z}| \le \mathrm{r},$$

where $H_{p-2}(z) = \sum_{k=0}^{p-2} A_{k+1} z^k$ is a polynomial of degree $\le p-2$. In what follows, denote $F(x) = F_1(x) + iF_2(x)$, $A_{k+1} = A_{k+1}^{(1)} + iA_{k+1}^{(2)}$, where $A_{k+1}^{(1)}, A_{k+1}^{(2)} \in \mathbb{R}$, and $H_{p-2,j}(x) = \sum_{k=0}^{p-2} A_{k+1}^{(j)} x^k$, $j = 1, 2$. Evidently we have $H_{p-2}(x) = H_{p-2,1}(x) + iH_{p-2,2}(x)$, for all $x \in [-1, 1]$. Note here that $F_1(x)$, $F_2(x)$, $H_{p-2,1}(x)$ and $H_{p-2,2}(x)$ are real functions of real variable. Also, recall that $i^2 = -1$.

Because $r \ge 1$, it follows that taking $z = x \in [-1, 1]$ in the equation in z for F, the functions F_j, $j = 1, 2$, necessarily verify the differential equations in x

$$(1-x)F_j'(x) - [2]_{q_n} F_j(x) = H_{p-2,j}(x), \text{for all } \mathrm{x} \in [-1, 1], \mathrm{j} = 1, 2.$$

The standard theory says that the general solution of a linear different equations of real functions of real variable is obtained by adding to the general solution of the homogenous equation, a particular solution of the inhomogenous equation. But reasoning exactly as in the proof of Lemma 3, the unique solutions of the homogenous equation are $F_j(x) = 0$, for all $x \in [-1, 1]$, $j = 1, 2$.

On the other hand, if we consider the differential equation of the form

$$(1-x)G'(x) - [2]_{q_n} G(x) = \sum_{k=0}^{p-2} d_{k+1} x^k, \text{for all } \mathrm{x} \in [-1, 1],$$

where $G(x)$ is considered real-valued function and $d_{k+1} \in \mathbb{R}$ for all k, looking for a particular solution of it, of the form $G(x) = \sum_{k=0}^{p-2} C_k x^k$, with $C_k \in \mathbb{R}$, simply calculation show that the differential equation one reduces to

$$\sum_{k=0}^{p-3}(k+1)C_{k+1}x^k - [2]_{q_n} \sum_{k=0}^{p-2}(k+1)C_k x^k = \sum_{k=0}^{p-2} d_{k+1} x^k, \text{for all } \mathrm{x} \in [-1, 1],$$

this immediately leads to the algebraic system

$$C_{p-2} = -\frac{d_{p-1}}{[2]_{q_n}(p-1)}, C_{k+1} - [2]_{q_n} C_k = \frac{d_{k+1}}{k+1}, k \in \{0, 1, \ldots, p-3\},$$

that evidently has unique solution for the unknowns C_k.

Therefore, these considerations show that we can take F_1 and F_2 as polynomials of degree $\le p-2$, solutions of the corresponding inhomogenous equations in x, which implies that necessarily these are the unique solutions of the above inhomogenous different equations in x. This also implies the uniqueness of $F(x)$

too as polynomial of degree $\leq p-2$ in x, solution of the corresponding differential equation in x. Now, because $F(z)$ is the analytic continuation of $F(x)$, from the identity theorem on analytic function, it follows that $F(z)$ as polynomial of degree $\leq p-2$ in z, necessarily is the unique solution of the corresponding differential equation in z, for $|z| \leq r$. This implies that $f'(z)$ is a polynomial of degree $\leq p-2$, which means that $f(z)$ is a polynomial of degree $\leq p-1$, a contradiction with the hypothesis. In conclusion, $\|[e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)}\|_r > 0$.

Since for any $|z| \leq r$ and $v \in \Gamma$ we have $|v - z| \geq r_1 - r$, so, by the formula (2.4), we get

$$\left\| \frac{p!}{2\pi i} \int_\Gamma \frac{[n+2]_{q_n}^2 H_{n,q_n}(f;v)}{(v-\cdot)^{p+1}} dv \right\|_r \leq \frac{p!}{2\pi} \frac{2\pi r_1 [n+2]_{q_n}^2 \|H_{n,q_n}(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}}$$
$$\leq \frac{N_{r_1}(f) p! r_1}{(r_1-r)^{p+1}},$$

where $N_{r_1}(f) = M_{r_1}(f) + 2\sum\limits_{k=1}^{\infty} |c_k| k r_1^k$. Taking into account $\frac{1}{[n+2]_{q_n}} \to 0$ as $n \to \infty$, therefore, there exists an index n_0 depending only on f, r and on sequence $\{q_n\}_{n\in\mathbb{N}}$, such that for all $n \geq n_0$ we have

$$\|e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)}\|_r - \frac{1}{[n+2]_{q_n}} \left\| \frac{p!}{2\pi i} \int_\Gamma \frac{[n+2]_{q_n}^2 H_{n,q_n}(f;v)}{(v-\cdot)^{p+1}} dv \right\|_r$$
$$\geq \frac{1}{2} \left\| [e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p))} \right\|_r,$$

which implies

$$\|M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)}\|_r \geq \frac{1}{2[n+2]_{q_n}} \|[e_1(1-e_1)f'' - [2]_{q_n} e_1 f']^{(p)}\|_r, \forall n \geq n_0.$$

For $n \in \{1, 2, \cdots, n_0 - 1\}$, we have $\|M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)}\|_r \geq \frac{W_{r,n}(f)}{[n+2]_{q_n}}$, where $W_{r,n}(f) = [n+2]_{q_n} \cdot \|M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)}\|_r > 0$.

As a conclusion, we have $\|M_{n,q_n}^{(p)}(f;\cdot) - f^{(p)}\|_r \geq \frac{C_r(f)}{[n+2]_{q_n}}$, for all $n \in \mathbb{N}$, where $C_r(f) = min\{W_{r,1}(f), W_{r,2}(f) \ldots, W_{r,n_0-1}(f), \frac{1}{2}\|[e_1(1-e_1)f'' - [2]_{q_n} e_1 f'\|_r]^{(p)}\}$, this complete the proof.

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References

[1] Andrews G. E., Askey R. and Roy R. *Special Functions.* Cambridge University Press, 1999.

[2] Anastassiou G. A., Gal S. G. *Approximation by complex Bernstien-Durrmeyer polynomials in compact disks.* Mediterr J. Math. 7 (4) (2010), 471-482.

[3] Agarwal R. P., Gupta V. *On q-analogue of a complex summation-integral type operators in compact disks.* J. Inequal. Appl. (2012), doi:10.1186/1029-242X-2012-111.

[4] Gasper G., Rahman M. *Basic Hypergeometric Series, Encyclopedia of Mathematics and its applications.* Cambridge University press, Cambridge, UK, Vol. 35, 1990.

[5] Gal S. G. *Approximation by Complex Bernstein and Convolution-Type Operators.* World Scientfic Publ.Co, Singapore-Hong Kong-London-New Jersey, 2009.

[6] Gal S. G. *Approximation by complex genuine Durrmeyer type polynomials in compact disks.* Appl. Math. Comput. 217 (2010), 1913-1920.

[7] Gal S. G. *Approximation by complex Bernstein-Durrmeyer polynomials with Jacobl weights in conpact disks.*, Math. Balkanica (N.S.) 24 (1-2) (2010), 103-110.

[8] Gal S. G. *Voronovskaja's theorem, shape preserving properties and iterations for complex* $q-$*Bernstein polynomials.* Studia Sci. Math. Hungar. 48 (1) (2011), 23-43.

[9] Gal S. G., Gupta V. *Approximation by a complex Durrmeyer-type operator in compact disks.* Ann. Univ. Ferrara. 57 (2011), 261-274.

[10] Gal S. G., Gupta V. and Mahmudov N. I. *Approximation by a complex q-Durrmeyer type operator.* Ann. Univ. Ferrara. 58 (1) (2012), 65-87.

[11] Kac V. G., Cheung P. *Quantum Calculus, Universitext.* Springer-Verlag, New York, 2002.

[12] Lorentz G. G. *Berstein Polynomials.* Chelsea Publ., Second edition, New York, 1986.

[13] Mahmudov N. I. *Approximation properties of complex q-Szász-Mirakjan operators in compact disks.* Comput. Math. Appl. 60 (2010), 1784-1791.

[14] Mahmudov N. I. *Approximation by Bernstein-Durrmeyer-type operators in compact disks.* Appl. Math. Lett. 24 (7) (2011), 1231-1238.

[15] Mahmudov N. I., Gupta V. *Approximation by genuine Durrmeyer-Stancu polynomials in compact disks.* Math. Comput. Model. 55 (3-4) (2012), 278-285.

[16] Ren M.Y., Zeng X.M. *Approximation by Complex Schurer-Stancu Operators in Compact Disks.* J. Comput. Amal. Appl. 15 (5) (2013), 833-843.

[17] Videnskii V. S. *On q-Bernstein polynomials and related positive linear operators* (in Rusian), in: Problems of Modern Mathematics and Mathematical Education,St.-Pertersburg, 2004, pp.118-126.

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ON POSITIVE SOLUTIONS OF A SYSTEM OF MAX-TYPE DIFFERENCE EQUATIONS

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Abstract. The boundedness character and global stability of positive solutions of the next system of difference equations with maximum

$$x_{n+1}=\max\left\{c,\frac{y_n^p}{z_{n-1}^p}\right\},\ \ y_{n+1}=\max\left\{c,\frac{z_n^p}{x_{n-1}^p}\right\},\ \ z_{n+1}=\max\left\{c,\frac{x_n^p}{y_{n-1}^p}\right\},$$

$n\in\mathbb{N}_0$, where $p,c\in(0,\infty)$, are studied in this paper.

1. Introduction

There has been some recent interest in nonlinear systems of difference equations (see, e.g., [8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 32, 34, 35, 36, 37, 38, 39, 40]), as well as in difference equations containing maximum operator, so called, max-type difference equations (see, e.g., [1, 6, 7, 15, 23, 24, 25, 26, 28, 29, 30, 31, 33, 36, 37, 41, 42, 43]). However, systems of difference equations with the maximum operator are barely touched (see [15, 36, 37]). Another interesting direction in theory of difference equations is investigation of those equations containing non-integer powers of their variables (see, e.g., [2, 3, 4, 5, 16, 19, 22, 23, 24, 25, 26, 27, 28, 29, 31, 43]). Some starting points and motivations for our investigations of difference equations containing non-integer powers of their variables were papers [20], [21] and [22]. These three papers along with some results on difference equations with maximum motivated S. Stević to study in [23] the next difference equation

$$x_{n+1}=\max\left\{c,\frac{x_n^p}{x_{n-1}^p}\right\},\quad n\in\mathbb{N}_0,\tag{1}$$

where initial values x_{-1}, x_0, and parameters c and p are positive numbers.

In view of all above mentioned investigations, it is a natural problem to study systems of max-type difference equations containing non-integer powers of their variables.

One of the first papers in the area was [38] where S. Stević studied solutions of the following max-type system of difference equations

$$x_{n+1}=\max\left\{c,\frac{y_n^p}{x_{n-1}^p}\right\},\quad y_{n+1}=\max\left\{c,\frac{x_n^p}{y_{n-1}^p}\right\},\quad n\in\mathbb{N}_0,$$

with positive initial values x_{-1}, x_0, y_{-1} and y_0 and parameters p and c, which is a natural extension of equation (1).

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Continuing this line of research, here we study long term behavior of positive solutions of the next system of max-type difference equations

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{z_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{c, \frac{z_n^p}{x_{n-1}^p}\right\}, \quad z_{n+1} = \max\left\{c, \frac{x_n^p}{y_{n-1}^p}\right\}, \tag{2}$$

$n \in \mathbb{N}_0$, where parameters p and c are positive. System (2) is a natural three-dimensional extension of scalar difference equation (1). Solution $(x_n, y_n, z_n)_{n\ge -1}$ of system (2) is called positive if $\min\{x_n, y_n, z_n\} > 0$ for every $n \ge -1$.

We say that the system of difference equations

$$\begin{aligned} x_{n+1} &= f_1(x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \\ y_{n+1} &= f_2(x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \\ z_{n+1} &= f_3(x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \end{aligned}$$

is permanent with respect to a class of solutions $\mathcal{F}$, if there are constants m and M such that for every solution $(x_n, y_n, z_n)_{n\ge -1} \in \mathcal{F}$ of the system the following inequalities hold

$$m \le \min\{x_n, y_n, z_n\} \le \max\{x_n, y_n, z_n\} \le M,$$

for sufficiently large n.

Our focus in the study of system (2) will be on the permanence, the existence of unbounded solutions, and on the convergence in the class of positive solutions. Our results are presented in terms of parameters p and c.

2. Permanence and unbounded solutions of system (2)

The permanence and the existence of unbounded positive solutions of system (2) are studied in this section.

Theorem 1. *Assume $p \ge 4$ and $c > 0$. Then there are positive unbounded solutions of system* (2)*.*

Proof. Assume $(x_n, y_n, z_n)_{n\ge -1}$ is a positive solution of system (2). From the equations in (2) we obtain

$$x_{n+1} \ge \left(\frac{y_n}{z_{n-1}}\right)^p, \quad n \in \mathbb{N}_0, \tag{3}$$

$$y_{n+1} \ge \left(\frac{z_n}{x_{n-1}}\right)^p, \quad n \in \mathbb{N}_0, \tag{4}$$

and

$$z_{n+1} \ge \left(\frac{x_n}{y_{n-1}}\right)^p, \quad n \in \mathbb{N}_0. \tag{5}$$

From (3)-(5) we easily get

$$\begin{aligned} \ln x_{n+1} \ge& p\ln y_n - p\ln z_{n-1}, &(6)\\ \ln y_{n+1} \ge& p\ln z_n - p\ln x_{n-1} &(7)\\ \ln z_{n+1} \ge& p\ln x_n - p\ln y_{n-1}, &(8) \end{aligned}$$

for $n \in \mathbb{N}_0$.

Set $v_n = \ln x_n y_n z_n$, $n \geq -1$. Then (6)-(8) imply

$$v_{n+1} - pv_n + pv_{n-1} \geq 0, \quad n \in \mathbb{N}_0. \tag{9}$$

Note that when $p \geq 4$ the polynomial $P(r) = r^2 - pr + p$ has two real roots r_1 and r_2 such that $\min\{r_1, r_2\} > 1$.

From (9) we have

$$v_{n+1} - r_1 v_n - r_2(v_n - r_1 v_{n-1}) \geq 0, \quad n \in \mathbb{N}_0,$$

and consequently

$$\frac{x_{n+1}y_{n+1}z_{n+1}}{(x_n y_n z_n)^{r_1}} \geq \left(\frac{x_n y_n z_n}{(x_{n-1}y_{n-1}z_{n-1})^{r_1}}\right)^{r_2}, \quad n \in \mathbb{N}_0.$$

This implies

$$\frac{x_n y_n z_n}{(x_{n-1}y_{n-1}z_{n-1})^{r_1}} \geq \left(\frac{x_0 y_0 z_0}{(x_{-1}y_{-1}z_{-1})^{r_1}}\right)^{r_2^n}, \quad n \in \mathbb{N}_0. \tag{10}$$

Now choose x_{-1}, y_{-1}, z_{-1}, x_0, y_0 and z_0 such that

$$x_0 y_0 z_0 > \max\left\{1, (x_{-1}y_{-1}z_{-1})^{r_1}\right\}. \tag{11}$$

Then from (10) and (11) we get

$$x_n y_n z_n > (x_{n-1}y_{n-1}z_{n-1})^{r_1}, \quad n \in \mathbb{N}_0,$$

from which along with (11) it follows that

$$x_n y_n z_n > (x_0 y_0 z_0)^{r_1^n} \to +\infty, \quad \text{as } n \to +\infty. \tag{12}$$

The existence of an unbounded solution $(x_n, y_n, z_n)_{n\geq -1}$ of system (2) follows from (12) and by the inequality

$$\sqrt{x_n^2 + y_n^2 + z_n^2} \geq \sqrt{3}\sqrt[3]{x_n y_n z_n}. \quad \square$$

Theorem 2. *Let $p \in (0, 4)$ and $c > 0$. Then system of difference equations (2) is permanent.*

Proof. First note that from equations in system (2) we immediately obtain that for every positive solution $(x_n, y_n, z_n)_{n\geq -1}$ of the system

$$c \leq \min\{x_n, y_n, z_n\}, \tag{13}$$

for every $n \in \mathbb{N}$.

Let $(p_k)_{k\geq 0}$ be defined by

$$p_{k+1} = \frac{p}{p - p_k}, \qquad p_0 = 0. \tag{14}$$

We have

$$\begin{aligned} x_{n+1} &= \max\left\{c, \frac{y_n^p}{z_{n-1}^p}\right\} = \max\left\{c, \left(\frac{y_n}{z_{n-1}}\right)^p\right\} \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \frac{z_{n-1}^{p-p_1}}{x_{n-2}^p}\right\}\right)^p\right\}. \end{aligned} \tag{15}$$

Assume that $p \in (0,1] = (p_0, p_1]$. Then (13) and (15) imply

$$x_{n+1} = \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \frac{1}{z_{n-1}^{p_1-p}x_{n-2}^p}\right\}\right)^p\right\} \le \max\left\{c, 1, \frac{1}{c^p}\right\}, \tag{16}$$

for $n \ge 3$, which along with (13) imply

$$c \le x_n \le \max\left\{c, 1, \frac{1}{c^p}\right\}, \tag{17}$$

for $n \ge 4$.

Assume that $p \in (p_1, p_2]$. By (2) and (15) we get

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\frac{z_{n-1}}{x_{n-2}^{\frac{p}{p-p_1}}}\right)^{p-p_1}\right\}\right)^p\right\} \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \frac{x_{n-2}^{p-p_2}}{y_{n-3}^p}\right\}\right)^{p-p_1}\right\}\right)^p\right\}. \end{aligned} \tag{18}$$

Using (13) in (18), and the facts $(p_2-1)(p-p_1)p = p$ and $p_2(p-p_1)p = p^2$, we get

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \frac{1}{x_{n-2}^{p_2-p}y_{n-3}^p}\right\}\right)^{p-p_1}\right\}\right)^p\right\} \\ &\le \max\left\{c, 1, \frac{1}{c^p}, \frac{1}{c^{p^2}}\right\}, \end{aligned}$$

for $n \ge 4$, which along with (13) implies

$$c \le x_n \le \max\left\{c, 1, \frac{1}{c^p}, \frac{1}{c^{p^2}}\right\}, \tag{19}$$

for $n \ge 5$.

Assume that $p \in (p_2, p_3]$. By (2) and (18) we get

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\frac{x_{n-2}}{y_{n-3}^{p_3}}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^p\right\} \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\max\left\{\frac{c}{y_{n-3}^{p_3}}, \frac{y_{n-3}^{p-p_3}}{z_{n-4}^p}\right\}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^p\right\} \end{aligned} \tag{20}$$

for $n \ge 5$.

Using (13) in (20) we get

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\max\left\{\frac{c}{y_{n-3}^{p_3}}, \frac{1}{y_{n-3}^{p_3-p}z_{n-4}^p}\right\}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^p\right\} \\ &\le \max\left\{c, 1, \frac{1}{c^p}, \frac{1}{c^{(p_3-1)(p-p_2)(p-p_1)p}}, \frac{1}{c^{p_3(p-p_2)(p-p_1)p}}, \right\}, \end{aligned}$$

for $n \ge 5$, which along with (13) implies

$$c \le x_n \le \max\left\{c, 1, \frac{1}{c^p}, \frac{1}{c^{(p_3-1)(p-p_2)(p-p_1)p}}, \frac{1}{c^{p_3(p-p_2)(p-p_1)p}}, \right\}, \tag{21}$$

for $n \ge 6$.

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By induction we get that for each fixed $l \in \mathbb{N}_0$

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\max\left\{\frac{c}{y_{n-3}^{p_3}}, \frac{y_{n-3}^{p-p_3}}{z_{n-4}^{p}}\right\}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^{p}\right\} \\ &= \cdots \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{z_{n-(3l+1)}^{p_{3l+1}}}, \frac{z_{n-(3l+1)}^{p-p_{3l+1}}}{x_{n-(3l+2)}^{p}}\right\}\right)^{p-p_{3l}} \cdots\right)^{p-p_1}\right\}\right)^{p}\right\}, \end{aligned} \tag{22}$$

for $n \geq 3l+1$ and $p \in (p_{3l}, p_{3l+1}]$,

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\max\left\{\frac{c}{y_{n-3}^{p_3}}, \frac{y_{n-3}^{p-p_3}}{z_{n-4}^{p}}\right\}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^{p}\right\} \\ &= \cdots \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{x_{n-(3l+2)}^{p_{3l+2}}}, \frac{x_{n-(3l+2)}^{p-p_{3l+2}}}{y_{n-(3l+3)}^{p}}\right\}\right)^{p-p_{3l+1}} \cdots\right)^{p-p_1}\right\}\right)^{p}\right\}, \end{aligned} \tag{23}$$

for $n \geq 3l+2$ and $p \in (p_{3l+1}, p_{3l+2}]$, and

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\max\left\{\frac{c}{x_{n-2}^{p_2}}, \left(\max\left\{\frac{c}{y_{n-3}^{p_3}}, \frac{y_{n-3}^{p-p_3}}{z_{n-4}^{p}}\right\}\right)^{p-p_2}\right\}\right)^{p-p_1}\right\}\right)^{p}\right\} \\ &= \cdots \\ &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{y_{n-(3l+3)}^{p_{3l+3}}}, \frac{y_{n-(3l+3)}^{p-p_{3l+3}}}{z_{n-(3l+4)}^{p}}\right\}\right)^{p-p_{3l+2}} \cdots\right)^{p-p_1}\right\}\right)^{p}\right\}, \end{aligned} \tag{24}$$

for $n \geq 3l+3$ and $p \in (p_{3l+2}, p_{3l+3}]$.

If $p = p_s$ for some $s \in \mathbb{N}$, difference equation (14) defines p_i for $i = \overline{0, s}$, and the method described above is finished after $s+1$ steps.

The monotonicity of $f(x) = p/(p-x)$ on the interval $(0, p)$, and the fact $0 = p_0 < p_1 = 1$, imply that $p_{k-1} < p_k$ as far as $p_k < p$. The case $p_k \in (0, p)$ for every $k \in \mathbb{N}$, is not possible. Namely, if it were then it would exist $\lim_{k\to\infty} p_k = p^* \in (0, p]$, and it would be $(p^*)^2 - pp^* + p = 0$, which for $p \in (0, 4)$ is not possible. This implies that there is a $k_0 \in \mathbb{N}$ such that $p \in (p_{k_0-1}, p_{k_0}]$.

If $k = k_0 = 3l_0 + 1$, from (22) we get

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{z_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{z_{n-(3l_0+1)}^{p_{3l_0+1}}}, \frac{z_{n-(3l_0+1)}^{p-p_{3l_0+1}}}{x_{n-(3l_0+2)}^{p}}\right\}\right)^{p-p_{3l_0}} \cdots\right)^{p-p_1}\right\}\right)^{p}\right\} \\ &\leq \max\left\{c, \left(\max\left\{1, \left(\cdots, \left(\max\left\{\frac{1}{c^{p_{3l_0+1}-1}}, \frac{1}{c^{p_{3l_0+1}}}\right\}\right)^{p-p_{3l_0}} \cdots\right)^{p-p_1}\right\}\right)^{p}\right\}, \end{aligned} \tag{25}$$

for $n \geq 3l_0 + 3$.

If $k = k_0 = 3l_0 + 2$, from (23) we get

$$\begin{aligned} x_{n+1} &= \max\Bigg\{c, \Bigg(\max\Bigg\{\frac{c}{z_{n-1}}, \Bigg(\cdots, \Bigg(\max\Bigg\{\frac{c}{x^{p_{3l_0+2}}_{n-(3l_0+2)}}, \frac{x^{p-p_{3l_0+2}}_{n-(3l+2)}}{y^{p}_{n-(3l_0+3)}}\Bigg\}\Bigg)^{p-p_{3l_0+1}} \cdots\Bigg)^{p-p_1}\Bigg\}\Bigg)^{p}\Bigg\} \\ &\le \max\Bigg\{c, \Bigg(\max\Bigg\{1, \Bigg(\cdots, \Bigg(\max\Bigg\{\frac{1}{c^{p_{3l_0+2}-1}}, \frac{1}{c^{p_{3l_0+2}}}\Bigg\}\Bigg)^{p-p_{3l_0+1}} \cdots\Bigg)^{p-p_1}\Bigg\}\Bigg)^{p}\Bigg\}, \end{aligned} \tag{26}$$

for $n \ge 3l_0 + 4$.

If $k = k_0 = 3l_0 + 3$, from (24) we get

$$\begin{aligned} x_{n+1} &= \max\Bigg\{c, \Bigg(\max\Bigg\{\frac{c}{z_{n-1}}, \Bigg(\cdots, \Bigg(\max\Bigg\{\frac{c}{y^{p_{3l_0+3}}_{n-(3l_0+3)}}, \frac{y^{p-p_{3l_0+3}}_{n-(3l_0+3)}}{z^{p}_{n-(3l_0+4)}}\Bigg\}\Bigg)^{p-p_{3l_0+2}} \cdots\Bigg)^{p-p_1}\Bigg\}\Bigg)^{p}\Bigg\} \\ &\le \max\Bigg\{c, \Bigg(\max\Bigg\{1, \Bigg(\cdots, \Bigg(\max\Bigg\{\frac{1}{c^{p_{3l_0+3}-1}}, \frac{1}{c^{p_{3l_0+3}}}\Bigg\}\Bigg)^{p-p_{3l_0+2}} \cdots\Bigg)^{p-p_1}\Bigg\}\Bigg)^{p}\Bigg\}, \end{aligned} \tag{27}$$

for $n \ge 3l_0 + 5$.

From (13), (17), (19), (21), (25)-(27) and the method of induction it follows that for every $p \in (0, 4)$ there is a $k_0 \in \mathbb{N}_0$ such that $p \in (p_{k_0-1}, p_{k_0}]$, and

$$c \le x_n \le \max\left\{c, \frac{1}{c^{p_{k_0}\prod_{j=0}^{k_0-1}(p-p_j)}}\right\}, \tag{28}$$

for $n \ge k_0 + 2$.

Bearing in mind that system (2) is invariant with respect to cyclic permutations of variables x_n, y_n and z_n, from (28) it follows that

$$c \le \min\{y_n, z_n\} \le \max\{y_n, z_n\} \le \max\left\{c, \frac{1}{c^{p_{k_0}\prod_{j=0}^{k_0-1}(p-p_j)}}\right\}, \tag{29}$$

for $n \ge k_0 + 2$.

From (28) and (29) the permanence of system (2) follows, as desired. □

The following corollary is a direct consequence of estimates (28) and (29) in Theorem 2.

Corollary 1. *Let $p \in (0, 4)$ and $c \ge 1$. Then every positive solution $(x_n, y_n, z_n)_{n \ge -1}$ of system (2) is eventually equal to (c, c, c).*

3. Convergence of positive solutions

In this section we prove a result on the convergence of positive solutions of system of difference equations (2).

Theorem 3. *If $p \in (0, 1)$ and $c \in (0, 1)$, then every positive solution of system (2) converges to $(1, 1, 1)$.*

Proof. Using (13) and (16) we get

$$c \le x_{n+1} \le \max\left\{c, 1, \frac{1}{c^p}\right\} = \frac{1}{c^p}, \quad \text{for} \quad n \ge 3. \tag{30}$$

On the other hand, the invariance with respect to the cyclic permutations of variables x_n, y_n and z_n of system (2), implies that

$$c \le y_{n+1} \le \max\left\{c, 1, \frac{1}{c^p}\right\} = \frac{1}{c^p}, \quad \text{for} \quad n \ge 3, \tag{31}$$

and

$$c \le z_{n+1} \le \max\left\{c, 1, \frac{1}{c^p}\right\} = \frac{1}{c^p}, \quad \text{for} \quad n \ge 3. \tag{32}$$

Write the equations in (2) as follows

$$\frac{x_{n+1}}{y_n} = \max\left\{\frac{c}{y_n}, \frac{1}{y_n^{1-p} z_{n-1}^p}\right\}, \tag{33}$$

$$\frac{y_{n+1}}{z_n} = \max\left\{\frac{c}{z_n}, \frac{1}{z_n^{1-p} x_{n-1}^p}\right\}, \tag{34}$$

and

$$\frac{z_{n+1}}{x_n} = \max\left\{\frac{c}{x_n}, \frac{1}{x_n^{1-p} y_{n-1}^p}\right\}. \tag{35}$$

Using (31) and (32) in (33), as well as the assumption $p \in (0,1)$, we get

$$c^p \le \frac{x_{n+1}}{y_n} \le \frac{1}{c}, \quad \text{for} \quad n \ge 5; \tag{36}$$

using (30) and (32) in (34), and $p \in (0,1)$, we get

$$c^p \le \frac{y_{n+1}}{z_n} \le \frac{1}{c}, \quad \text{for} \quad n \ge 5; \tag{37}$$

and finally, using (30) and (31) in (35), and $p \in (0,1)$, we get

$$c^p \le \frac{z_{n+1}}{x_n} \le \frac{1}{c}, \quad \text{for} \quad n \ge 5. \tag{38}$$

Using $p, c \in (0,1)$ and (37) in the first equation in (2), we get

$$c^{p^2} \le x_{n+1} \le \frac{1}{c^p}, \quad \text{for} \quad n \ge 6; \tag{39}$$

using $p, c \in (0,1)$ and (38) in the second equation in (2), we get

$$c^{p^2} \le y_{n+1} \le \frac{1}{c^p}, \quad \text{for} \quad n \ge 6; \tag{40}$$

and finally using $p, c \in (0,1)$ and (36) in the third equation in (2), we get

$$c^{p^2} \le z_{n+1} \le \frac{1}{c^p}, \quad \text{for} \quad n \ge 6. \tag{41}$$

From (33)-(35), (39)-(41) it follows that

$$c^p \le \frac{x_{n+1}}{y_n} \le \frac{1}{c^{p^2}}, \quad \text{for} \quad n \ge 8, \tag{42}$$

$$c^p \le \frac{y_{n+1}}{z_n} \le \frac{1}{c^{p^2}}, \quad \text{for} \quad n \ge 8, \tag{43}$$

and

$$c^p \le \frac{z_{n+1}}{x_n} \le \frac{1}{c^{p^2}}, \quad \text{for} \quad n \ge 8. \tag{44}$$

Using (43) in the first equation in (2) it follows that

$$c^{p^2} \le x_{n+1} \le \frac{1}{c^{p^3}}, \quad \text{for} \quad n \ge 9,$$

Using (44) in the second equation in (2) it follows that

$$c^{p^2} \le y_{n+1} \le \frac{1}{c^{p^3}}, \quad \text{for} \quad n \ge 9.$$

Using (42) in the third equation in (2) it follows that

$$c^{p^2} \le z_{n+1} \le \frac{1}{c^{p^3}}, \quad \text{for} \quad n \ge 9.$$

A simple inductive argument shows that

$$c^{p^{2k}} \le \min\{x_{n+1}, y_{n+1}, z_{n+1}\} \le \max\{x_{n+1}, y_{n+1}, z_{n+1}\} \le \frac{1}{c^{p^{2k+1}}}, \tag{45}$$

for $n \ge 6k+3$, and

$$c^{p^{2k+2}} \le \min\{x_{n+1}, y_{n+1}, z_{n+1}\} \le \max\{x_{n+1}, y_{n+1}, z_{n+1}\} \le \frac{1}{c^{p^{2k+1}}}, \tag{46}$$

for $n \ge 6k+6$. From (45), (46) and the assumption $p \in (0,1)$ the result easily follows by letting $k \to \infty$. □

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References

[1] C. Çinar, S. Stević and I. Yalçinkaya, On positive solutions of a reciprocal difference equation with minimum, *J. Appl. Math. Comput.* **17** (1-2) (2005), 307-314.

[2] B. Iričanin, On a higher-order nonlinear difference equation, *Abstr. Appl. Anal.* Vol. 2010, Article ID 418273, (2010), 8 pages.

[3] B. Iričanin, The boundedness character of two Stević-type fourth-order difference equations, *Appl. Math. Comput.* **217** (5) (2010), 1857-1862.

[4] B. Iričanin and S. Stević, On a class of third-order nonlinear difference equations, *Appl. Math. Comput.* **213** (2009), 479-483.

[5] G. Karakostas, Asymptotic behavior of the solutions of the difference equation $x_{n+1} = x_n^2 f(x_{n-1})$, *J. Differ. Equations Appl.* **9** (6) (2003), 599-602.

[6] C. M. Kent, M. Kustesky M, A. Q. Nguyen and B. V. Nguyen, Eventually periodic solutions of $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$ when the parameters are two cycles, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10** (1-3) (2003), 33-49.

[7] C. M. Kent and M. A. Radin, On the boundedness nature of positive solutions of the difference equation $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$, with periodic parameters, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms* **2003**, suppl., 11-15.

[8] G. Papaschinopoulos, M. Radin and C. J. Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, *Appl. Math. Comput.* **218** (2012), 5310-5318.

[9] G. Papaschinopoulos and C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.* **219** (2) (1998), 415-426.

[10] G. Papaschinopoulos and C. J. Schinas, On the behavior of the solutions of a system of two nonlinear difference equations, *Comm. Appl. Nonlinear Anal.* **5** (2) (1998), 47-59.

[11] G. Papaschinopoulos and C. J. Schinas, Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7** (2) (1999), 181-196.

[12] G. Papaschinopoulos and C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal. TMA* **46** (7) (2001), 967-978.

[13] G. Papaschinopoulos and C. J. Schinas, Oscillation and asymptotic stability of two systems of difference equations of rational form, *J. Difference Equat. Appl.* **7** (2001), 601-617.

[14] G. Papaschinopoulos and C. J. Schinas, On the dynamics of two exponential type systems of difference equations, *Comput. Math. Appl.* **64** (7) (2012), 2326-2334.

[15] G. Papaschinopoulos, C. Schinas and V. Hatzifilippidis, Global behavior of the solutions of a max-equation and of a system of two max-equations, *J. Comput. Anal. Appl.* **5** (2) (2003), 237-254.

[16] G. Papaschinopoulos, C. J. Schinas and G. Stefanidou, On the nonautonomous difference equation $x_{n+1} = A_n + (x_{n-1}^p / x_n^q)$, *Appl. Math. Comput.* **217** (2011), 5573-5580.

[17] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, On a system of two exponential type difference equations, *Commun. Appl. Nonlinear Anal.* **17** (2) (2010), 1-13.

[18] S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.* **33** (1) (2002), 45-53.

[19] S. Stević, A note on the difference equation $x_{n+1} = \sum_{i=0}^{k} (\alpha_i / x_{n-i}^{p_i})$, *J. Differ. Equations Appl.* **8** (7) (2002), 641-647.

[20] S. Stević, On the recursive sequence $x_{n+1} = A / \prod_{i=0}^{k} x_{n-i} + 1 / \prod_{j=k+2}^{2(k+1)} x_{n-j}$, *Taiwanese J. Math.* **7** (2) (2003), 249-259.

[21] S. Stević, On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II, *Dynam. Contin. Discrete Impuls. Systems* **10a** (6) (2003), 911-917.

[22] S. Stević, On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}^p / x_n^p)$, *J. Appl. Math. & Computing* **18** (1-2) (2005) 229-234.

[23] S. Stević, On the recursive sequence $x_{n+1} = \max\{c, x_n^p / x_{n-1}^p\}$, *Appl. Math. Lett.* **21** (8) (2008), 791-796.

[24] S. Stević, Boundedness character of a class of difference equations, *Nonlinear Anal. TMA* **70** (2009), 839-848.

[25] S. Stević, Boundedness character of two classes of third-order difference equations, *J. Differ. Equations Appl.* **15** (11-12) (2009), 1193-1209.

[26] S. Stević, Global stability of a difference equation with maximum, *Appl. Math. Comput.* **210** (2009), 525-529.

[27] S. Stević, On a class of higher-order difference equations, *Chaos Solitons Fractals* **42** (2009), 138-145.

[28] S. Stević, Global stability of a max-type equation, *Appl. Math. Comput.* **216** (2010), 354-356.

[29] S. Stević, On a generalized max-type difference equation from automatic control theory, *Nonlinear Anal. TMA* **72** (2010), 1841-1849.

[30] S. Stević, Periodicity of max difference equations, *Util. Math.* **83** (2010), 69-71.

[31] S. Stević, On a nonlinear generalized max-type difference equation, *J. Math. Anal. Appl.* **376** (2011), 317-328.

[32] S. Stević, On a system of difference equations, *Appl. Math. Comput.* **218** (2011), 3372-3378.

[33] S. Stević, Periodicity of a class of nonautonomous max-type difference equations, *Appl. Math. Comput.* **217** (2011), 9562-9566.

[34] S. Stević, On a third-order system of difference equations, *Appl. Math. Comput.* **218** (2012), 7649-7654.

[35] S. Stević, On some solvable systems of difference equations, *Appl. Math. Comput.* **218** (2012), 5010-5018.

[36] S. Stević, On some periodic systems of max-type difference equations, *Appl. Math. Comput.* **218** (2012), 11483-11487.

[37] S. Stević, Solutions of a max-type system of difference equations, *Appl. Math. Comput.* **218** (2012), 9825-9830.

[38] S. Stević, On a symmetric system of max-type difference equations, *Appl. Math. Comput.* **219** (2013) 8407-8412.

[39] S. Stević, On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, *Appl. Math. Comput.* **219** (2013), 4755-4764.

[40] S. Stević, On the system $x_{n+1} = y_n x_{n-k}/(y_{n-k+1}(a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k}/(x_{n-k+1}(c_n + d_n x_n y_{n-k}))$, *Appl. Math. Comput.* **219** (2013), 4526-4534.
[41] H. D. Voulov, Periodic solutions to a difference equation with maximum, *Proc. Amer. Math. Soc.* **131** (7) (2003) 2155-2160.
[42] H. D. Voulov, On the periodic nature of the solutions of the reciprocal difference equation with maximum, *J. Math. Anal. Appl.* **296** (1) (2004) 32-43.
[43] X. Yang and X. Liao, On a difference equation with maximum, *Appl. Math. Comput.* **181** (2006) 1-5.

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SOLUTION AND STABILITY OF A MULTI-VARIABLE FUNCTIONAL EQUATION

JAE-HYEONG BAE AND WON-GIL PARK*

ABSTRACT. We obtain some combinatorial identities and investigate the monomial functional equation

$$\sum_{k=1}^{h}(-1)^{k-1}[\binom{n}{h-k}-d\binom{n}{h-k-1}][f(kx+y)+f(kx-y)]$$
$$+n!(-1)^h(1+d)f(x)-\binom{n}{h}(1-d)f(y)=0,$$

where $h:=\begin{cases}\frac{n}{2}, & n:\text{ even}\\ \frac{n+1}{2}, & n:\text{ odd}\end{cases}$ and $d:=\begin{cases}0, & n:\text{ even}\\ 1, & n:\text{ odd}\end{cases}$.

1. Introduction

Throughout this paper, let X and Y be vector spaces and n a positive integer. For an integer r, $\binom{n}{r}$ is the binomial coefficient. Here $\binom{n}{r}:=0$ for $r<0$ or $r>n$. For a mapping $f:X\to Y$, consider the monomial functional equation:

$$\sum_{k=1}^{h}(-1)^{k-1}\left[\binom{n}{h-k}-d\binom{n}{h-k-1}\right]\left[f(kx+y)+f(kx-y)\right] \tag{1.1}$$
$$+n!\,(-1)^h(1+d)f(x)-\binom{n}{h}(1-d)f(y)=0,$$

where $h=h_n:=\begin{cases}\frac{n}{2}, & n:\text{ even}\\ \frac{n+1}{2}, & n:\text{ odd}\end{cases}$ and $d=d_n:=\begin{cases}0, & n:\text{ even}\\ 1, & n:\text{ odd.}\end{cases}$

For $X=Y=\mathbb{R}$, the monomial $f(x)=cx^n$ is a solution of (1.1) for each $n\ge 2$. If n is even, then the monomial functional equation (1.1) can be rewritten as

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\left[f(kx+y)+f(kx-y)\right]+n!(-1)^{\frac{n}{2}}f(x)-\binom{n}{\frac{n}{2}}f(y)=0. \tag{1.2}$$

The quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is the functional equation (1.2) for $n=2$. If $n\ge 3$ is odd, then the monomial functional equation (1.1) can be rewritten as

$$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\left[f(kx+y)+f(kx-y)\right]+n!(-1)^{\frac{n+1}{2}}2f(x)=0. \tag{1.3}$$

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In 2002, K.-W. Jun and H.-M. Kim [6] solved the cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

which is the functional equation (1.3) for $n=3$. Note that a mapping $f : X \to Y$ satisfies the functional equation (1.3) for $n=1$ if and only if it is a Jensen mapping.

The authors [2, 9] investigated some functional equations in order to induce the monomial functional equation (1.1). Some books [5, 7, 8, 10] provide useful information on functional equations associated with monomials. In this paper, we obtain some combinatorial identities and investigate the monomial functional equation (1.1).

2. Combinatorial Identities

In this section, we prove some combinatorial identities needed to investigate the monomial functional equation (1.1).

Lemma 2.1. *Assume that $n \geq 2$ is an even integer. Then*

(a) *For $n \geq 4$,* $\displaystyle\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}=0$ *for all* $j=1,2,\cdots,\frac{n}{2}-1.$

(b) $\displaystyle 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n}=n!\,(-1)^{\frac{n}{2}-1}.$

Proof. (a) Let $n \geq 4$ be even and $j \in \left\{1,2,\cdots,\frac{n}{2}-1\right\}$. Note that

$$\begin{aligned}(2.1)\qquad &\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}\\ &=\binom{n}{\frac{n}{2}-1}1^{n-2j}-\binom{n}{\frac{n}{2}-2}2^{n-2j}+\binom{n}{\frac{n}{2}-3}3^{n-2j}-\cdots+(-1)^{\frac{n}{2}-1}\binom{n}{0}\left(\frac{n}{2}\right)^{n-2j}\\ &=\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-1-k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n-2j}.\end{aligned}$$

Since $\binom{n}{k}=\binom{n}{n-k}$ for all $k=1,2,\cdots,\frac{n}{2}$, by shifting of indices, we gain

$$(2.2)\qquad \sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}=\sum_{k=\frac{n}{2}+1}^{n}(-1)^{k-\frac{n}{2}-1}\binom{n}{k}\left(k-\frac{n}{2}\right)^{n-2j}.$$

By equalities (2.1) and (2.2), we get

$$(2.3)\qquad \sum_{k=0}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-1-k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n-2j}=\sum_{k=\frac{n}{2}+1}^{n}(-1)^{k-\frac{n}{2}-1}\binom{n}{k}\left(k-\frac{n}{2}\right)^{n-2j}.$$

Since n is even, we have

$$(-1)^{\frac{n}{2}-1-k}=(-1)^{\frac{n}{2}-1-k}(-1)^{-n+2k}=(-1)^{k-\frac{n}{2}-1}$$

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and

$$\left(\frac{n}{2}-k\right)^{n-2j}=\left(k-\frac{n}{2}\right)^{n-2j}$$

for all $k=0,1,\cdots,n$. By equality (2.3) and the above equalities, we obtain

$$\sum_{k=0}^{n}(-1)^{\frac{n}{2}-1-k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n-2j}=2\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-1-k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n-2j}.$$

By equality (2.1) and the above equality, we see that

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}=\frac{1}{2}\sum_{k=0}^{n}(-1)^{\frac{n}{2}-1-k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n-2j}.$$

Since n is even and $(-1)^{-k}=(-1)^{-k}(-1)^{2k}=(-1)^k$ for all $k=0,1,\cdots,n$, the above equality implies that

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}=\frac{1}{2}(-1)^{\frac{n}{2}-1}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(k-\frac{n}{2}\right)^{n-2j}.$$

Note that

$$\sum_{k=0}^{N}(-1)^{k}\binom{N}{k}(\alpha+k)^{m-1}=0 \tag{2.4}$$

for $N\geq m\geq 1,\ N,\ m\in\mathbb{N}$ (see [4]). Replacing N, α and m by n, $-\frac{n}{2}$ and $n-2j+1$ in the combinatorial identity (2.4), respectively, we get $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(k-\frac{n}{2}\right)^{n-2j}=0$ for all $j=1,2,\cdots,\frac{n}{2}-1$. Hence we obtain the desired combinatorial identity.

(b) Since n is even, we gain $\left(-\frac{n}{2}+k\right)^n=\left(\frac{n}{2}-k\right)^n$ for all $k=0,1,\cdots,n$. Shifting of indices, we get

$$\sum_{k=0}^{\frac{n}{2}-1}(-1)^{k}\binom{n}{k}\left(\frac{n}{2}-k\right)^{n}=\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{k-1}\left(\frac{n}{2}-k+1\right)^{n}$$

and

$$\sum_{k=\frac{n}{2}+1}^{n}(-1)^{k}\binom{n}{k}\left(-\frac{n}{2}+k\right)^{n}=\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}+k}\binom{n}{\frac{n}{2}+k}k^{n}.$$

By letting $k=\frac{n}{2}-j+1$ for $k=1,\cdots,\frac{n}{2}$, we have

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{k-1}\left(\frac{n}{2}-k+1\right)^{n}=\sum_{j=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}-j}\binom{n}{\frac{n}{2}-j}j^{n}.$$

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By the above equality and shifting of indices, we obtain

$$\begin{aligned}\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}-k}\binom{n}{\frac{n}{2}-k}k^n &= \sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{k-1}\left(\frac{n}{2}-k+1\right)^n \\ &= \sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\left(\frac{n}{2}-k\right)^n.\end{aligned}$$

Since n is even, the above equality implies that

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}-k}\binom{n}{\frac{n}{2}-k}k^n = \sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\left(-\frac{n}{2}+k\right)^n. \tag{2.5}$$

Shifting of indices, we obtain

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}+k}\binom{n}{\frac{n}{2}+k}k^n = \sum_{k=\frac{n}{2}+1}^{n}(-1)^k\binom{n}{k}\left(-\frac{n}{2}+k\right)^n. \tag{2.6}$$

Since $(-1)^{\frac{n}{2}-k} = (-1)^{\frac{n}{2}+k}$ and $\binom{n}{\frac{n}{2}-k} = \binom{n}{\frac{n}{2}+k}$ for all $k = 1, \cdots, \frac{n}{2}$, by the equalities (2.5) and (2.6), we see that

$$\begin{aligned}2\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}+k}\binom{n}{\frac{n}{2}-k}k^n &= \sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}-k}\binom{n}{\frac{n}{2}-k}k^n + \sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}+k}\binom{n}{\frac{n}{2}+k}k^n \\ &= \sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\left(\frac{n}{2}-k\right)^n + \sum_{k=\frac{n}{2}+1}^{n}(-1)^k\binom{n}{k}\left(-\frac{n}{2}+k\right)^n \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(-\frac{n}{2}+k\right)^n.\end{aligned} \tag{2.7}$$

Note that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^n = n!\,(-1)^n \tag{2.8}$$

for $n \geq 0$ (see [4]). Replacing α by $-\frac{n}{2}$ in the combinatorial identity (2.8), we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(-\frac{n}{2}+k\right)^n = n!\,(-1)^n.$$

By (2.7) and the above equality, the desired combinatorial identity holds. □

Lemma 2.2. *Assume that $n \geq 3$ is an odd integer. Then*

(a) $$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]=\binom{n-1}{\frac{n-1}{2}}-\binom{n-1}{\frac{n-3}{2}}.$$

(b) $$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n-2j}=0 \text{ for all } j=1,2,\cdots,\frac{n-1}{2}.$$

(c) $$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n}=(-1)^{\frac{n-1}{2}}n!.$$

Proof. (a) Let m be a nonnegative integer. Since $(-1)^{k-1}=(-1)^{2k-2}(-1)^{-k+1}=(-1)^{-k+1}$, we gain

$$\begin{aligned}(-1)^m\sum_{k=1}^{m+1}(-1)^{k-1}\binom{n}{m-k+1}&=\sum_{k=1}^{m+1}(-1)^{m-k+1}\binom{n}{m-k+1}\\ =\binom{n}{0}-\binom{n}{1}+-\cdots+(-1)^m\binom{n}{m}&=\sum_{k=0}^{m}(-1)^k\binom{n}{k}.\end{aligned}$$

Since $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$ (see [4]), using the above equality, we get

$$\sum_{k=1}^{m+1}(-1)^{k-1}\binom{n}{m-k+1}=\binom{n-1}{m}. \tag{2.9}$$

Replacing m by $\frac{n-1}{2}$ in equality (2.9), we have

$$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\binom{n}{\frac{n+1}{2}-k}=\binom{n-1}{\frac{n-1}{2}}. \tag{2.10}$$

And replacing m by $\frac{n-3}{2}$ in equality (2.9), we obtain $\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\binom{n}{\frac{n-1}{2}-k}=\binom{n-1}{\frac{n-3}{2}}$. Since $\binom{n}{-1}=0$, we have $\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\binom{n}{\frac{n-1}{2}-k}=\binom{n-1}{\frac{n-3}{2}}$. By equality (2.10) and the above equality, the desired combinatorial identity holds.

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(b) Let n is odd and $j \in \left\{1, 2, \cdots, \frac{n-1}{2}\right\}$. Consider equality

$$\begin{aligned}
&\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n-2j} \\
=&\left[\binom{n}{\frac{n+1}{2}-1}-\binom{n}{\frac{n-1}{2}-1}\right]1^{n-2j}-\left[\binom{n}{\frac{n+1}{2}-2}-\binom{n}{\frac{n-1}{2}-2}\right]2^{n-2j} \\
&+\left[\binom{n}{\frac{n+1}{2}-3}-\binom{n}{\frac{n-1}{2}-3}\right]3^{n-2j}-\cdots+(-1)^{\frac{n-1}{2}}\left[\binom{n}{0}-0\right]\left(\frac{n+1}{2}\right)^{n-2j} \\
=&\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j}.
\end{aligned} \tag{2.11}$$

Since $\binom{n}{k}=\binom{n}{n-k}$ and $\binom{n}{k+1}=\binom{n}{n-k-1}$ for all $k=1,2,\cdots,\frac{n+1}{2}$, by shifting of indices, we gain

$$\begin{aligned}
&\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n-2j} \\
=&\sum_{k=\frac{n+1}{2}}^{n}(-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k+1}\right]\left(k-\frac{n-1}{2}\right)^{n-2j}.
\end{aligned} \tag{2.12}$$

By the equalities (2.11) and (2.14), we get

$$\begin{aligned}
&\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j} \\
=&\sum_{k=\frac{n+1}{2}}^{n}(-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k+1}\right]\left(k-\frac{n-1}{2}\right)^{n-2j}.
\end{aligned}$$

By shifting of indices and the above equality, we have

$$\begin{aligned}
&\sum_{k=\frac{n+3}{2}}^{n+1}(-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
=&\sum_{k=\frac{n+1}{2}}^{n}(-1)^{k-\frac{n-1}{2}}\left[\binom{n}{k+1}-\binom{n}{k}\right]\left(k-\frac{n-1}{2}\right)^{n-2j} \\
=&\sum_{k=\frac{n+1}{2}}^{n}(-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k+1}\right]\left(k-\frac{n-1}{2}\right)^{n-2j} \\
=&\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j}.
\end{aligned}$$

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So we obtain that

$$
\begin{aligned}
(2.13)\qquad & \sum_{k=\frac{n+3}{2}}^{n} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&= \sum_{k=\frac{n+3}{2}}^{n+1} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} + (-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j} \\
&= \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j} + (-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j}.
\end{aligned}
$$

Since n is odd, we have

$$(-1)^{\frac{n-1}{2}-k} = -(-1)^{\frac{n-1}{2}-k}(-1)^{-n+2k} = -(-1)^{k-\frac{n+1}{2}}$$

and

$$\left(\frac{n+1}{2}-k\right)^{n-2j} = -\left(k-\frac{n+1}{2}\right)^{n-2j}$$

for all $k = 0, 1, \cdots, n$. By the above equalities and using equality (2.15), we have

$$
\begin{aligned}
& \sum_{k=0}^{n} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&= \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&\quad + \sum_{k=\frac{n+3}{2}}^{n} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&= \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&\quad + \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j} + (-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j} \\
&= 2\sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}-k}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(\frac{n+1}{2}-k\right)^{n-2j} + (-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j}.
\end{aligned}
$$

By equality (2.11) and the above equality, we gain

$$
\begin{aligned}
(2.14)\qquad & \sum_{k=0}^{n} (-1)^{k-\frac{n+1}{2}}\left[\binom{n}{k}-\binom{n}{k-1}\right]\left(k-\frac{n+1}{2}\right)^{n-2j} \\
&= 2\sum_{k=1}^{\frac{n+1}{2}} (-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n-2j} + (-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j}.
\end{aligned}
$$

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Replacing N, α and m by n, $-\frac{n+1}{2}$ and $n-2j+1$ in the combinatorial identity (2.4), respectively, we get $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(k-\frac{n+1}{2}\right)^{n-2j}=0$. By the above equality, we obtain

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{k-\frac{n+1}{2}}\binom{n}{k}\left(k-\frac{n+1}{2}\right)^{n-2j}\\ =&(-1)^{-\frac{n+1}{2}}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(k-\frac{n+1}{2}\right)^{n-2j}=0.\end{aligned}\tag{2.15}$$

And replacing N, α and m by n, $1-\frac{n+1}{2}$ and $n-2j+1$ in the combinatorial identity (2.4), respectively, we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(k+1-\frac{n+1}{2}\right)^{n-2j}=0$ and so we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}\left(k+1-\frac{n+1}{2}\right)^{n-2j}=-(-1)^n\binom{n}{n}\left(n+1-\frac{n+1}{2}\right)^{n-2j}\\ =&(-1)^{n+1}\left(\frac{n+1}{2}\right)^{n-2j}.\end{aligned}$$

By shifting of indices, the fact that $\binom{n}{-1}=0$ and the above equality, we see that

$$\begin{aligned}&-\sum_{k=0}^{n}(-1)^{k-\frac{n+1}{2}}\binom{n}{k-1}\left(k-\frac{n+1}{2}\right)^{n-2j}\\ =&-(-1)^{-\frac{n+1}{2}}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k-1}\left(k-\frac{n+1}{2}\right)^{n-2j}\\ =&-(-1)^{-\frac{n+1}{2}}\sum_{k=1}^{n}(-1)^{k}\binom{n}{k-1}\left(k-\frac{n+1}{2}\right)^{n-2j}\\ =&-(-1)^{-\frac{n+1}{2}}\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n}{k}\left(k+1-\frac{n+1}{2}\right)^{n-2j}\\ =&(-1)^{-\frac{n+1}{2}}\sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}\left(k+1-\frac{n+1}{2}\right)^{n-2j}\\ =&(-1)^{\frac{n+1}{2}}\left(\frac{n+1}{2}\right)^{n-2j}.\end{aligned}$$

By the equalities (2.14) and (2.15) and the above equality, we obtain the desired combinatorial identity.

(c) Shifting of indices, we gain

$$\sum_{k=\frac{n+1}{2}}^{n}(-1)^k\binom{n}{k}\left(-\frac{n-1}{2}+k\right)^n=\sum_{k=1}^{\frac{n+1}{2}}(-1)^{\frac{n-1}{2}+k}\binom{n}{\frac{n-1}{2}+k}k^n\tag{2.16}$$

and

$$\sum_{k=0}^{\frac{n-3}{2}}(-1)^k\binom{n}{k}\Big(-\frac{n-1}{2}+k\Big)^n=\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\binom{n}{k-1}\Big(-\frac{n+1}{2}+k\Big)^n. \tag{2.17}$$

Setting $j=\frac{n+1}{2}-k$ for $k=1,\cdots,\frac{n-1}{2}$ in the right hand side of equality (2.17), we get

$$\begin{aligned}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\binom{n}{k-1}\Big(-\frac{n+1}{2}+k\Big)^n=&\sum_{j=1}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}-j}\binom{n}{\frac{n-1}{2}-j}(-j)^n\\ =&(-1)^{n-1}\sum_{j=1}^{\frac{n-1}{2}}(-1)^{\frac{n+1}{2}-j}\binom{n}{\frac{n-1}{2}-j}j^n.\end{aligned}$$

Since n is odd, we have

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\binom{n}{k-1}\Big(-\frac{n+1}{2}+k\Big)^n=\sum_{j=1}^{\frac{n-1}{2}}(-1)^{\frac{n+1}{2}-j}\binom{n}{\frac{n-1}{2}-j}j^n \tag{2.18}$$

By the equalities (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned}&\sum_{j=1}^{\frac{n-1}{2}}(-1)^{\frac{n+1}{2}-j}\binom{n}{\frac{n-1}{2}-j}j^n+\sum_{k=1}^{\frac{n+1}{2}}(-1)^{\frac{n-1}{2}+k}\binom{n}{\frac{n-1}{2}+k}k^n\\ =&\sum_{k=0}^{\frac{n-3}{2}}(-1)^k\binom{n}{k}\Big(-\frac{n-1}{2}+k\Big)^n+\sum_{k=\frac{n+1}{2}}^{n}(-1)^k\binom{n}{k}\Big(-\frac{n-1}{2}+k\Big)^n\\ =&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\Big(-\frac{n-1}{2}+k\Big)^n.\end{aligned}$$

Replacing α by $-\frac{n-1}{2}$ in the combinatorial identity (2.8), we see that

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{\frac{n+1}{2}-k}\binom{n}{\frac{n-1}{2}-k}k^n+\sum_{k=1}^{\frac{n+1}{2}}(-1)^{\frac{n-1}{2}+k}\binom{n}{\frac{n-1}{2}+k}k^n=(-1)^n n!.$$

Since $(-1)^{\frac{n+1}{2}-k}=(-1)^{\frac{n+1}{2}+k}$ and $\binom{n}{\frac{n-1}{2}+k}=\binom{n}{\frac{n+1}{2}-k}$ for all $k=1,\cdots,\frac{n+1}{2}$, we have

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{\frac{n+1}{2}+k}\binom{n}{\frac{n-1}{2}-k}k^n+\sum_{k=1}^{\frac{n+1}{2}}(-1)^{\frac{n-1}{2}+k}\binom{n}{\frac{n+1}{2}-k}k^n=(-1)^n n!.$$

Multiplying $(-1)^{\frac{-n+1}{2}}$ by the both hand sides in the above equality, we obtain

$$\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k+1}\binom{n}{\frac{n-1}{2}-k}k^n+\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k}\binom{n}{\frac{n+1}{2}-k}k^n=(-1)^{\frac{n+1}{2}}n!.$$

Dividing -1 by the both hand sides in the above equality and using $(-1)^{k+1} = (-1)^{k-1}$ for all integers k and $\binom{n}{-1} = 0$, we see that

$$\sum_{k=1}^{\frac{n+1}{2}} (-1)^{k-1} \left[\binom{n}{\frac{n+1}{2} - k} - \binom{n}{\frac{n-1}{2} - k} \right] k^n = (-1)^{\frac{n-1}{2}} n!.$$

□

3. Solution of Monomial Functional Equations

The following lemma is needed to investigate the monomial functional equations (1.2) and (1.3).

Lemma 3.1. *Let $n \geq 2$ and let $f : X \to Y$ be a mapping satisfying the monomial functional equation (1.1) for all $x, y \in X$. If n is even, then so is f. And, if n is odd, then so is f.*

Proof. Assume that n is even. Then (1.1) becomes (1.2). Thus, putting in (1.2) $x = y = 0$, we obtain

$$2 \sum_{k=1}^{\frac{n}{2}} (-1)^{k-1} \binom{n}{\frac{n}{2} - k} f(0) + n!(-1)^{\frac{n}{2}} f(0) - \binom{n}{\frac{n}{2}} f(0) = 0.$$

Since $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ and $\binom{n}{k} = \binom{n}{n-k}$ for $k = 0, \cdots, n$, we get

$$2 \sum_{k=1}^{\frac{n}{2}} (-1)^{k-1} \binom{n}{\frac{n}{2} - k} = \binom{n}{\frac{n}{2}}. \tag{3.1}$$

Thus we have $n!(-1)^{\frac{n}{2}} f(0) = 0$. Hence we obtain $f(0) = 0$. Putting in (1.2) $x = 0$, we get

$$\sum_{k=1}^{\frac{n}{2}} (-1)^{k-1} \binom{n}{\frac{n}{2} - k} [f(y) + f(-y)] - \binom{n}{\frac{n}{2}} f(y) = 0$$

for all $y \in X$. By equality (3.1) and the above equation, we have $\binom{n}{\frac{n}{2}} \big[f(-y) - f(y)\big] = 0$ for all $y \in X$. Therefore f is even.

Suppose that n is odd. Setting $x = y = 0$ in the functional equation (1.3), we gain

$$\sum_{k=1}^{\frac{n+1}{2}} (-1)^{k-1} \left[\binom{n}{\frac{n+1}{2} - k} - \binom{n}{\frac{n-1}{2} - k} \right] f(0) + n!(-1)^{\frac{n+1}{2}} f(0) = 0.$$

By Lemma 2.2 (a), we get

$$\left[\binom{n-1}{\frac{n-1}{2}} - \binom{n-1}{\frac{n-3}{2}} + n!(-1)^{\frac{n+1}{2}} \right] f(0) = 0.$$

Thus we have $f(0) = 0$. Taking $x = 0$ in the functional equation (1.3), we obtain that

$$\sum_{k=1}^{\frac{n+1}{2}} (-1)^{k-1} \left[\binom{n}{\frac{n+1}{2} - k} - \binom{n}{\frac{n-1}{2} - k} \right] [f(y) + f(-y)] = 0$$

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for all $y \in X$. By Lemma 2.2 (a), we see that

$$\left[\binom{n-1}{\frac{n-1}{2}} - \binom{n-1}{\frac{n-3}{2}}\right][f(y) + f(-y)] = 0$$

for all $y \in X$. Therefore f is odd. □

In the following theorem, we investigate the solution of the monomial functional equation (1.2).

Theorem 3.2. *Let $n \geq 2$ be even. If a mapping $f : X \to Y$ satisfies the functional equation* (1.2) *for all $x, y \in X$, then there is a symmetric mapping $S_n : X^n \to Y$ satisfying $S_n(x, \cdots, x) = \frac{1}{2^n} f(2x)$ for all $x \in X$. On the contrary, if $S_n : X^n \to Y$ is a symmetric multi-additive mapping and a mapping $f : X \to Y$ satisfies $f(x) = S_n(x, \cdots, x)$ for all $x \in X$, then f satisfies the functional equation* (1.2) *for all $x, y \in X$.*

Proof. Suppose that a mapping $f : X \to Y$ satisfies equation (1.2) for all $x, y \in X$. By Lemma 3.1, f is even. Define the mapping $S_n : X^n \to Y$ by

$$S_n(x_1, \cdots, x_n) := \frac{1}{n!\, 2^{n-1}} \sum_{\sigma_2, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_n f(x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n) \tag{3.2}$$

for all $x_1, \cdots, x_n \in X$. For $2 \leq j \leq n$, we gain

(3.3)

$$\begin{aligned}
&S_n(x_j, x_2, \cdots, x_{j-1}, x_1, x_{j+1}, \cdots, x_n)\\
=&\frac{1}{n!2^{n-1}} \sum_{\sigma_2, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_n f\big(x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n + (\sigma_j - 1)(x_1 - x_j)\big)\\
=&\frac{1}{n!2^{n-1}} \Bigg[\sum_{\sigma_2, \cdots, \sigma_{j-1}, \sigma_{j+1}, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_{j-1} (1) \sigma_{j+1} \cdots \sigma_n\\
&\quad f\big(x_1 + \sigma_2 x_2 + \cdots + \sigma_{j-1} x_{j-1} + x_j + \sigma_{j+1} x_{j+1} + \cdots + \sigma_n x_n\big)\\
&\quad + \sum_{\sigma_2, \cdots, \sigma_{j-1}, \sigma_{j+1}, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_{j-1} (-1) \sigma_{j+1} \cdots \sigma_n\\
&\quad f\big(x_1 + \sigma_2 x_2 + \cdots + \sigma_{j-1} x_{j-1} - x_j + \sigma_{j+1} x_{j+1} + \cdots \sigma_n x_n - 2x_1 + 2x_j\big)\Bigg]\\
=&\frac{1}{n!2^{n-1}} \Bigg[\sum_{\sigma_2, \cdots, \sigma_{j-1}, \sigma_{j+1}, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_{j-1} \sigma_{j+1} \cdots \sigma_n f\big(x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n + (1 - \sigma_j) x_j\big)\\
&\quad - \sum_{\sigma_2, \cdots, \sigma_{j-1}, \sigma_{j+1}, \cdots, \sigma_n \in \{1,-1\}} \sigma_2 \cdots \sigma_{j-1} \sigma_{j+1} \cdots \sigma_n f\big(- x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n + (1 - \sigma_j)\, x_j\big)\Bigg]
\end{aligned}$$

for all $x_1, \cdots x_n \in X$. Since f is even, we get

(3.4)
$$\begin{aligned}
&\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_{j-1}\sigma_{j+1}\cdots\sigma_n f\big(-x_1+\sigma_2x_2+\cdots+\sigma_nx_n+(1-\sigma_j)\,x_j\big)\\
=&\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_{j-1}\sigma_{j+1}\cdots\sigma_n f\big(x_1-\sigma_2x_2-\cdots-\sigma_nx_n-(1-\sigma_j)x_j\big)\\
=&\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}} (-\sigma_2)\cdots(-\sigma_{j-1})(-\sigma_{j+1})\cdots(-\sigma_n) f\big(x_1-\sigma_2x_2-\cdots-\sigma_nx_n-(1-\sigma_j)x_j\big)\\
=&\sum_{\tau_2,\cdots,\tau_{j-1},\tau_{j+1},\cdots,\tau_n\in\{1,-1\}} \tau_2\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n f\big(x_1+\tau_2x_2+\cdots+\tau_nx_n-(1+\tau_j)x_j\big)
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. By the equalities (3.3) and (3.4), we have

(3.5)
$$\begin{aligned}
&S_n(x_j, x_2\cdots, x_{j-1}, x_1, x_{j+1}, \cdots, x_n)\\
&=\frac{1}{n!2^{n-1}}\Bigg[\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_{j-1}\sigma_{j+1}\cdots\sigma_n f\big(x_1+\sigma_2x_2+\cdots+\sigma_nx_n+(1-\sigma_j)x_j\big)\\
&\qquad-\sum_{\tau_2,\cdots,\tau_{j-1},\tau_{j+1},\cdots,\tau_n\in\{1,-1\}} \tau_2\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n f\big(x_1+\tau_2x_2+\cdots+\tau_nx_n-(1+\tau_j)x_j\big)\Bigg]\\
&=\frac{1}{n!\,2^{n-1}}\sum_{\sigma_2,\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_n f(x_1+\sigma_2x_2+\cdots+\sigma_nx_n)\\
&=S_n(x_1,\cdots,x_n)
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. For $2 \le j < k \le n$, putting $\epsilon_i := \sigma_i$ for $i \in \{2, \cdots, n\} \setminus \{j, k\}$, $\epsilon_j := \sigma_k$ and $\epsilon_k := \sigma_j$, we obtain

(3.6)
$$\begin{aligned}
&S_n(x_1,\cdots,x_{j-1},x_k,x_{j+1},\cdots,x_{k-1},x_j,x_{k+1},\cdots,x_n)\\
=&\frac{1}{n!\,2^{n-1}}\sum_{\sigma_2,\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_n f\big(x_1+\sigma_2x_2+\cdots+\sigma_nx_n+(\sigma_k-\sigma_j)x_j+(\sigma_j-\sigma_k)x_k\big)\\
=&\frac{1}{n!\,2^{n-1}}\sum_{\epsilon_2,\cdots,\epsilon_n\in\{1,-1\}} \epsilon_2\cdots\epsilon_n f\big(x_1+\epsilon_2x_2+\cdots+\epsilon_{j-1}x_{j-1}+\epsilon_kx_j+\epsilon_{j+1}x_{j+1}+\cdots+\\
&\qquad\epsilon_{k-1}x_{k-1}+\epsilon_jx_k+\epsilon_{k+1}x_{k+1}+\cdots+\epsilon_nx_n+(\epsilon_j-\epsilon_k)x_j+(\epsilon_k-\epsilon_j)x_k\big)\\
=&\frac{1}{n!\,2^{n-1}}\sum_{\epsilon_2,\cdots,\epsilon_n\in\{1,-1\}} \epsilon_2\cdots\epsilon_n\, f(x_1+\epsilon_2x_2+\cdots+\epsilon_nx_n)\\
=&S_n(x_1,\cdots,x_n)
\end{aligned}$$

for all $x_1, \cdots x_n \in X$. Hence S_n is symmetric.

By the proof of Lemma 3.1, we get $f(0) = 0$. Letting $y = 0$ in (1.2), we have

$$\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}f(kx)=\frac{n!}{2}(-1)^{\frac{n}{2}+1}f(x)$$

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for all $x \in X$. By the above equality, we have

$$\begin{aligned}
&S_n(x,\cdots,x)\\
&= \frac{1}{n!\,2^{n-1}} \sum_{\sigma_2,\cdots,\sigma_n\in\{1,-1\}} \sigma_2\cdots\sigma_n f((1+\sigma_2+\cdots+\sigma_n)x) = \frac{1}{n!\,2^{n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}f((n-2k)x)\\
&= \frac{f(nx)}{n!\,2^{n-1}} + \frac{1}{n!\,2^{n-1}}\sum_{k=1}^{\frac{n}{2}-1}(-1)^k\binom{n-1}{k}f((n-2k)x) + \frac{1}{n!\,2^{n-1}}\sum_{k=\frac{n}{2}+1}^{n-1}(-1)^k\binom{n-1}{n-k-1}f((2k-n)x)\\
&= \frac{f(nx)}{n!\,2^{n-1}} + \frac{1}{n!\,2^{n-1}}\sum_{k=1}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-k}\binom{n-1}{\frac{n}{2}-k}f(2kx) + \frac{1}{n!\,2^{n-1}}\sum_{k=1}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-k}\binom{n-1}{\frac{n}{2}-k-1}f(2kx)\\
&= \frac{1}{n!\,2^{n-1}}\sum_{k=1}^{\frac{n}{2}}(-1)^{\frac{n}{2}-k}\binom{n}{\frac{n}{2}-k}f(2kx) = \frac{(-1)^{\frac{n}{2}+1}}{n!\,2^{n-1}}\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}f(2kx)\\
&= \frac{(-1)^{\frac{n}{2}+1}}{n!\,2^{n-1}}\frac{n!}{2}(-1)^{\frac{n}{2}+1}f(2x) = \frac{1}{2^n}f(2x)
\end{aligned}$$

On the contrary, suppose that there exists a symmetric multi-additive mapping $S_n : X^n \to Y$ such that $f(x) = S_n(x,\cdots,x)$ for all $x \in X$. By Section 11.1 in [1], it suffices to show for $n \geq 4$. By equality (3.1) and Lemma 2.1 (b), we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\big[f(kx+y)+f(kx-y)\big] + n!(-1)^{\frac{n}{2}}f(x) - \binom{n}{\frac{n}{2}}f(y)\\
&= \sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\big[S_n(kx+y,\cdots,kx+y)+S_n(kx-y,\cdots,kx-y)\big]\\
&\qquad + n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x) - \binom{n}{\frac{n}{2}}S_n(y,\cdots,y)\\
&= 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\sum_{j=0}^{\frac{n}{2}}\binom{n}{2j}S_n(\underbrace{kx,\cdots,kx}_{n-2j},\underbrace{y,\cdots,y}_{2j}) + n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x)\\
&\qquad - \binom{n}{\frac{n}{2}}S_n(y,\cdots,y)\\
&= 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{2j}S_n(\underbrace{kx,\cdots,kx}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\qquad + 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}S_n(y,\cdots,y) + n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x) - \binom{n}{\frac{n}{2}}S_n(y,\cdots,y)
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\quad+\left[2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}-\binom{n}{\frac{n}{2}}\right]S_n(y,\cdots,y)+n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x)\\
&= 2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})+n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x)\\
&= 2\sum_{k=1}^{\frac{n}{2}}\sum_{j=1}^{\frac{n}{2}-1}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\quad+2\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^nS_n(x,\cdots,x)+n!(-1)^{\frac{n}{2}}S_n(x,\cdots,x)\\
&= 2\sum_{k=1}^{\frac{n}{2}}\sum_{j=1}^{\frac{n}{2}-1}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&= 2\sum_{j=1}^{\frac{n}{2}-1}\binom{n}{2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}k^{n-2j}
\end{aligned}
$$

By Lemma 2.1 (a), we see that

$$
\sum_{k=1}^{\frac{n}{2}}(-1)^{k-1}\binom{n}{\frac{n}{2}-k}\big[f(kx+y)+f(kx-y)\big]+n!(-1)^{\frac{n}{2}}f(x)-\binom{n}{\frac{n}{2}}f(y)=0.
$$

□

Theorem 3.3. *Let $n\geq 3$ be odd. If a mapping $f:X\to Y$ satisfies the functional equation* (1.3) *for all $x,y\in X$, then there is a symmetric mapping $S_n:X^n\to Y$ satisfying $S_n(x,\cdots,x)=\frac{1}{2^n}f(2x)$ for all $x\in X$. On the contrary, if $S_n:X^n\to Y$ is a symmetric multi-additive mapping and a mapping $f:X\to Y$ satisfies $f(x)=S_n(x,\cdots,x)$ for all $x\in X$, then f satisfies the functional equation* (1.3) *for all $x,y\in X$.*

Proof. Suppose that a mapping $f:X\to Y$ satisfies equation (1.3) for all $x,y\in X$. By Lemma 3.1, f is odd. Define the mapping $S_n:X^n\to Y$ by equality (3.2) for all $x_1,\cdots,x_n\in X$. For $2\leq j\leq n$, we gain equality (3.3) for all $x_1,\cdots x_n\in X$. Since f is odd, we get

$$
\begin{aligned}
&\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}}\sigma_2\cdots\sigma_{j-1}\sigma_{j+1}\cdots\sigma_n f\big(-x_1+\sigma_2x_2+\cdots+\sigma_nx_n+(1-\sigma_j)\,x_j\big)\\
&=-\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}}\sigma_2\cdots\sigma_{j-1}\sigma_{j+1}\cdots\sigma_n f\big(x_1-\sigma_2x_2-\cdots-\sigma_nx_n-(1-\sigma_j)x_j\big)\\
&=\sum_{\sigma_2,\cdots,\sigma_{j-1},\sigma_{j+1},\cdots,\sigma_n\in\{1,-1\}}(-\sigma_2)\cdots(-\sigma_{j-1})(-\sigma_{j+1})\cdots(-\sigma_n)f\big(x_1-\sigma_2x_2-\cdots-\sigma_nx_n-(1-\sigma_j)x_j\big)
\end{aligned}
$$

$$= \sum_{\tau_2,\cdots,\tau_{j-1},\tau_{j+1},\cdots,\tau_n\in\{1,-1\}} \tau_2\cdots\tau_{j-1}\tau_{j+1}\cdots\tau_n f\big(x_1+\tau_2x_2+\cdots+\tau_nx_n-(1+\tau_j)x_j\big)$$

for all $x_1,\cdots,x_n \in X$. By equality (3.3) and the above equality, we have equality (3.5) for all $x_1,\cdots,x_n\in X$. For $2\le j<k\le n$, putting $\epsilon_i:=\sigma_i$ for $i\in\{2,\cdots,n\}\setminus\{j,k\}$, $\epsilon_j:=\sigma_k$ and $\epsilon_k:=\sigma_j$, we obtain equality (3.6) for all $x_1,\cdots x_n\in X$. Hence S_n is symmetric. By a similar method to the proof of Theorem 3.2, we obtain $S_n(x,\cdots,x)=\frac{1}{2^n}f(2x)$ for all $x\in X$.

On the contrary, suppose that there exists a symmetric multi-additive mapping $S_n: X^n\to Y$ such that $f(x)=S_n(x,\cdots,x)$ for all $x\in X$. By Lemma 2.2 (c), we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\big[f(kx+y)+f(kx-y)\big]+n!(-1)^{\frac{n+1}{2}}2f(x)\\
&=\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\big[S_n(kx+y,\cdots,kx+y)\\
&\quad+S_n(kx-y,\cdots,kx-y)\big]+n!(-1)^{\frac{n+1}{2}}2S_n(x,\cdots,x)\\
&=2\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{2j}S_n(\underbrace{kx,\cdots,kx}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\quad+n!(-1)^{\frac{n+1}{2}}2S_n(x,\cdots,x)\\
&=2\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\quad+n!(-1)^{\frac{n+1}{2}}2S_n(x,\cdots,x)\\
&=2\sum_{k=1}^{\frac{n+1}{2}}\sum_{j=1}^{\frac{n-1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&\quad+2\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^nS_n(x,\cdots,x)\\
&\quad+n!(-1)^{\frac{n+1}{2}}2S_n(x,\cdots,x)
\end{aligned}$$

$$\begin{aligned}
&=2\sum_{k=1}^{\frac{n+1}{2}}\sum_{j=1}^{\frac{n-1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\binom{n}{2j}k^{n-2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\\
&=2\sum_{j=1}^{\frac{n-1}{2}}\binom{n}{2j}S_n(\underbrace{x,\cdots,x}_{n-2j},\underbrace{y,\cdots,y}_{2j})\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]k^{n-2j}
\end{aligned}$$

By Lemma 2.2 (b), we see that

$$\sum_{k=1}^{\frac{n+1}{2}}(-1)^{k-1}\left[\binom{n}{\frac{n+1}{2}-k}-\binom{n}{\frac{n-1}{2}-k}\right]\left[f(kx+y)+f(kx-y)\right]+n!(-1)^{\frac{n+1}{2}}2f(x)=0.$$

□

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References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.
[2] J.-H. Bae and W.-G. Park, *On stability of a functional equation with n variables*, Nonlinear Anal. **64** (2006), 856–868.
[3] J.K. Chung and P.K. Sahoo, *On the general solution of a quartic functional equation*, Bull. Korean Math. Soc. **40** (2003), 565–576.
[4] I.S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier Inc., Amsterdam, 2007.
[5] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Boston, 1998.
[6] K.-W. Jun and H.-M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 867–878.
[7] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.
[8] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, New York, 2009.
[9] W.-G. Park and J.-H. Bae, *On a bi-quadratic functional equation and its stability*, Nonlinear Anal. **62** (2005), 643–654.
[10] P.K. Sahoo and Pl. Kannappan, *Introduction to Functional Equations*, Chapman & Hall / CRC Press, Boca Raton, Florida, 2011.
[11] L. Székelyhidi, *Convolution Type Functional Equation on Topological Abelian Groups*, World Scientific, Singapore, 1991.

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On the Behaviour of the Solutions of Difference Equation Systems

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ABSTRACT

In this paper, we investigate the behaviour of the solutions of difference equations systems

$$x_{n+1} = \frac{y_{n-5}}{\pm 1 + y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1} = \frac{x_{n-5}}{\pm 1 + x_{n-1}y_{n-3}x_{n-5}},$$

where the initial values are arbitrary real numbers such that the denominator is always nonzero.

Keywords: System of difference equation, Explicit solutions, Periodicity.

AMS Classification: 39A10, 39A12.

1 Introduction

Our aim in this study is to investigate the periodic character of all solutions of the following difference equations systems

$$x_{n+1} = \frac{y_{n-5}}{\pm 1 + y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1} = \frac{x_{n-5}}{\pm 1 + x_{n-1}y_{n-3}x_{n-5}},\ n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary real numbers. Throughout this paper, we will assume that our solutions are well-defined, that is, the denominator is always nonzero. Also, we take $\overline{1,n}$ instead of $1, 2, \ldots, n$.

Nonlinear difference equations have long interested mathematics as well as other sciences. They play a key concept in many applications such as the natural model of a discrete process. There have been many recent investigations and interest in the field of nonlinear difference equations by several authors [1–22]. For instance, in [16], Stevic obtained behaviour of the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}.$$

Karatas et al., in [12], gave that the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

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In [6], Elsayed dealed with the dynamics and found the solution of the following rational recursive sequences

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 + x_{n-1}x_{n-3}x_{n-5}}.$$

Grove et al., in [11], have studied existence and behavior of solutions of the rational system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n},\ y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}.$$

Ozban [15], has investigated the positive solutions of the rational difference system

$$x_{n+1} = \frac{1}{y_{n-k}},\ y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}.$$

In [13], Kurbanli et al. have studied the positive solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

In [17], Touafek and Elsayed dealed with the periodic nature and the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}.$$

Similar nonlinear system of rational difference equations were investigated; see [4,5,8,18-22].

2 On System $x_{n+1} = \frac{y_{n-5}}{1+y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1} = \frac{x_{n-5}}{1+x_{n-1}y_{n-3}x_{n-5}}$

In this section we study the solutions of the difference equation system

$$x_{n+1} = \frac{y_{n-5}}{1 + y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}y_{n-3}x_{n-5}},\ n \in \mathbb{N}_0. \tag{2.1}$$

Theorem 2.1. *Suppose that initial conditions are any positive real numbers. Let* $\{x_n, y_n\}_{n=1}^{\infty}$ *be a solution of system* (2.1)*. For* $k = \left\lceil \frac{n}{6} \right\rceil$ *, all solutions of system* (2.1) ; *are given by:*

i) *If k is odd and p is equal to 2 or 3,*

$$x_n = y_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\alpha}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\alpha},\ y_n = x_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\beta}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\beta}.$$

ii) *If k is odd and p is equal to otherwise,*

$$x_n = y_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\beta}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\beta},\ y_n = x_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\alpha}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\alpha}.$$

iii) *If k is even and p is equal to 2 or 3*

$$x_n = x_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\beta}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\beta},\ y_n = y_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil - 1\right)\beta}{1 + \left(3j + \left\lceil \frac{i}{2} \right\rceil\right)\beta}.$$

iv) *If k is even and p is equal to otherwise*

$$x_n = x_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1+\left(3j+\left\lceil \frac{i}{2} \right\rceil - 1\right)\alpha}{1+\left(3j+\left\lceil \frac{i}{2} \right\rceil\right)\alpha},\ y_n = y_{-p} \prod_{j=0}^{\left\lfloor \frac{n-i}{6} \right\rfloor} \frac{1+\left(3j+\left\lceil \frac{i}{2} \right\rceil - 1\right)\alpha}{1+\left(3j+\left\lceil \frac{i}{2} \right\rceil\right)\alpha},$$

where for $\left(i=\overline{1,6}\right)$, $n-i \equiv 0 \,(\text{mod } 6)$ *and for* $\left(p=\overline{0,5}\right)$, $n+p \equiv 0 \ (\text{mod } 6)$, $r+n \equiv 0 \ (\text{mod } 2)$, $s=r+2,\ t=r+4,\ \alpha = x_{-r}y_{-s}x_{-t},\ \beta = y_{-r}x_{-s}y_{-t}$.

Proof. We will prove this theorem by mathematical induction on n. For $n=1$, we obtain as

$$k=1,\ i=1,\ p=5,\ r=1,\ s=3,\ t=5.$$

Then, we get

$$x_1 = \frac{y_{-5}}{1+x_{-1}y_{-3}x_{-5}}, y_1 = \frac{x_{-5}}{1+x_{-1}y_{-3}x_{-5}}.$$

Now, suppose that our assumption holds as follows:

$$\begin{aligned}
x_{12n-5} &= x_{-5} \prod_{j=0}^{2n-1} \frac{1+3j\alpha}{1+(3j+1)\alpha},\ y_{12n-5} = y_{-5} \prod_{j=0}^{2n-1} \frac{1+3j\beta}{1+(3j+1)\beta}, \\
x_{12n-4} &= x_{-4} \prod_{j=0}^{2n-1} \frac{1+3j\alpha}{1+(3j+1)\alpha},\ y_{12n-4} = y_{-4} \prod_{j=0}^{2n-1} \frac{1+3j\beta}{1+(3j+1)\beta}, \\
x_{12n-3} &= x_{-3} \prod_{j=0}^{2n-1} \frac{1+(3j+1)\beta}{1+(3j+2)\beta},\ y_{12n-3} = y_{-3} \prod_{j=0}^{2n-1} \frac{1+(3j+1)\alpha}{1+(3j+2)\alpha}, \\
x_{12n-2} &= x_{-2} \prod_{j=0}^{2n-1} \frac{1+(3j+1)\beta}{1+(3j+2)\beta},\ y_{12n-2} = y_{-2} \prod_{j=0}^{2n-1} \frac{1+(3j+1)\alpha}{1+(3j+2)\alpha}, \\
x_{12n-1} &= x_{-1} \prod_{j=0}^{2n-1} \frac{1+(3j+2)\alpha}{1+(3j+3)\alpha},\ y_{12n-1} = y_{-1} \prod_{j=0}^{2n-1} \frac{1+(3j+2)\beta}{1+(3j+3)\beta}, \\
x_{12n} &= x_0 \prod_{j=0}^{2n-1} \frac{1+(3j+2)\alpha}{1+(3j+3)\alpha},\ \ y_{12n} = y_0 \prod_{j=0}^{2n-1} \frac{1+(3j+2)\beta}{1+(3j+3)\beta},
\end{aligned}$$

where $k=2n$. To end up the proof, we have to show that the cases in $\{x_m, y_m\}$ hold for $m=\overline{12n+1, 12n+6}$. For x_{12n+1} and y_{12n+1}, we obtain

$$i=1,\ p=5,\ k=\left\lceil \frac{12n+1}{6} \right\rceil = 2n+1,\ \left\lfloor \frac{12n}{6} \right\rfloor = 2n,\ r=1.$$

Firstly, we consider $x_{12n+1} = \frac{y_{12n-5}}{1+y_{12n-1}x_{12n-3}y_{12n-5}}$.Therefore, we can write

$$\begin{aligned}
x_{12n+1} &= \frac{y_{-5} \prod_{j=0}^{2n-1} \frac{1+3j\beta}{1+(3j+1)\beta}}{1+y_{-1} \prod_{j=0}^{2n-1} \frac{1+(3j+2)\beta}{1+(3j+3)\beta} x_{-3} \prod_{j=0}^{2n-1} \frac{1+(3j+1)\beta}{1+(3j+2)\beta} y_{-5} \prod_{j=0}^{2n-1} \frac{1+3j\beta}{1+(3j+1)\beta}} \\
&= \frac{y_{-5} \prod_{j=0}^{2n-1} \frac{1+3j\beta}{1+(3j+1)\beta}}{\frac{1+(6n+1)\beta}{1+6n\beta}} = y_{-5} \prod_{i=0}^{2n} \frac{1+3j\beta}{1+(3j+1)\beta}.
\end{aligned}$$

Secondly, we consider $y_{12n+1} = \frac{x_{12n-5}}{1+x_{12n-1}y_{12n-3}x_{12n-5}}$. Then we can write

$$\begin{aligned} y_{12n+1} &= \frac{x_{-5}\prod\limits_{j=0}^{2n-1}\frac{1+3j\alpha}{1+(3j+1)\alpha}}{1+x_{-1}\prod\limits_{j=0}^{2n-1}\frac{1+(3j+2)\alpha}{1+(3j+3)\alpha}y_{-3}\prod\limits_{j=0}^{2n-1}\frac{1+(3j+1)\alpha}{1+(3j+2)\alpha}x_{-5}\prod\limits_{j=0}^{2n-1}\frac{1+3j\alpha}{1+(3j+1)\alpha}} \\ &= \frac{y_{-5}\prod\limits_{j=0}^{2n-1}\frac{1+3j\alpha}{1+(3j+1)\alpha}}{\frac{1+(6n+1)\alpha}{1+6n\alpha}} = y_{-5}\prod_{i=0}^{2n}\frac{1+3j\alpha}{1+(3j+1)\,\alpha}. \end{aligned}$$

Similarly one can prove the other relations. The proof is complete. □

Remark 2.1. If $\alpha = x_{-r}y_{-s}x_{-t} \neq -1/n$ or $\beta = y_{-r}x_{-s}y_{-t} \neq -1/n$ for all $n \in \mathbb{Z}^+$, then Theorem 2.1 also represents solutions of system (2.1) in the case where initial conditions are real numbers.

Theorem 2.2. *System* (2.1) *has one equilibrium point which is* $(0,0)$.

Proof. For the equilibrium points of system (2.1), we can write

$$\overline{x} = \frac{\overline{y}}{1+\overline{y}^2\overline{x}} \text{ and } \overline{y} = \frac{\overline{x}}{1+\overline{x}^2\overline{y}}.$$

Then we have

$$\overline{x} + \overline{y}^2\overline{x}^2 = \overline{y} \text{ and } \overline{y} + \overline{x}^2\overline{y}^2 = \overline{x},$$

or,

$$\overline{y} - \overline{x} = \overline{x} - \overline{y}.$$

Hence, we obtain $\overline{x} = \overline{y} = 0$, which is desired. □

Theorem 2.3. *For all* $n \in \mathbb{Z}^+,\ \alpha = 0$ *and* $\beta = 0$ *iff the system (2.1) has periodic solutions of period* 12.

Proof. First, let $\alpha = 0$ and $\beta = 0$. By considering Theorem 2.1, the solutions of system (2.1) is reduced as

$$\begin{aligned} x_n &= y_{-p},\ y_n = x_{-p};\ k \text{ is odd,} \\ x_n &= x_{-p},\ y_n = y_{-p};\ k \text{ is even,} \end{aligned}$$

where for $\left(p = \overline{0,5}\right)$, $n + p \equiv 0 \pmod 6$. Therefore, for $n = 1, 2, \ldots$, we get

$$\begin{aligned} x_{12n-11} &= y_{-5},\ y_{12n-11} = x_{-5};\ x_{12n-10} = y_{-4},\ y_{12n-10} = x_{-4}, \\ x_{12n-9} &= y_{-3},\ y_{12n-9} = x_{-3};\ x_{12n-8} = y_{-2},\ y_{12n-8} = x_{-2}, \\ x_{12n-7} &= y_{-1},\ y_{12n-7} = x_{-1};\ x_{12n-6} = y_{0},\ y_{12n-6} = x_{0}, \\ x_{12n-5} &= x_{-5},\ y_{12n-5} = y_{-5};\ x_{12n-4} = x_{-4},\ y_{12n-4} = y_{-4}, \\ x_{12n-3} &= x_{-3},\ y_{12n-3} = y_{-3};\ x_{12n-2} = x_{-2},\ y_{12n-2} = y_{-2}, \\ x_{12n-1} &= x_{-1},\ y_{12n-1} = y_{-1};\ x_{12n} = x_{0},\ y_{12n} = y_{0}, \end{aligned}$$

which is desired. Second assume that the system (2.1) has periodic solutions of period 12. Then, we have

$$\begin{aligned} x_{12n-11} &= y_{-5}\prod_{j=0}^{2n-2}\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\alpha}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha} \\ &= y_{-5}\prod_{j=0}^{2n}\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\alpha}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha} = x_{12n+1}, \end{aligned}$$

and

$$\begin{aligned} y_{12n-11} &= x_{-5}\prod_{j=0}^{2n-2}\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\beta}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\beta} \\ &= x_{-5}\prod_{j=0}^{2n}\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\beta}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\beta} = y_{12n+1}. \end{aligned}$$

In here, in order to ensure above equalities iff $\alpha, \beta = 0$. Similarly one can prove the other conditions. The proof is complete. □

Theorem 2.4. *Assume that* $\alpha, \beta \neq 0$. *Then every solution of system (2.1) converges to* $(0,0)$.

Proof. In here, there are 16 different states. We will present only the case $\alpha < 0$ for x_n in Theorem 2.1-(i). By considering Theorem 2.1, we obtain

$$\begin{aligned} x_n &= y_{-p}\prod_{j=0}^{\left\lfloor\frac{n-i}{6}\right\rfloor}\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\alpha}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha} \\ &= y_{-p}\exp\prod_{j=0}^{\left\lfloor\frac{n-i}{6}\right\rfloor}\ln\frac{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil-1\right)\alpha}{1+\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha} \\ &= y_{-p}\exp\left(-\sum_{j=0}^{\left\lfloor\frac{n-i}{6}\right\rfloor}\ln\left(1+\frac{\alpha}{\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha+1}\right)\right) \\ &= y_{-p}c(n_0)\exp\left(-\alpha\sum_{j=n_0}^{\left\lfloor\frac{n-i}{6}\right\rfloor}\left(\frac{1}{\left(3j+\left\lceil\frac{i}{2}\right\rceil\right)\alpha+1}+O\left(\frac{1}{i^2}\right)\right)\right) \to 0, \quad n\to\infty. \end{aligned}$$

Here, $c(n_0)$ is a positive constant depending on $n_0 \in \mathbb{N}$. Similarly one can prove the other relations. The proof is complete. □

3 On System $x_{n+1} = \frac{y_{n-5}}{-1+y_{n-1}x_{n-3}y_{n-5}}$, $y_{n+1} = \frac{x_{n-5}}{-1+x_{n-1}y_{n-3}x_{n-5}}$

In here, we investigate on the solutions of the difference equation system

$$x_{n+1} = \frac{y_{n-5}}{-1+y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1} = \frac{x_{n-5}}{-1+x_{n-1}y_{n-3}x_{n-5}},\ n \in \mathbb{N}_0. \tag{3.1}$$

Theorem 3.1. *Let* $\{x_n, y_n\}$ *be a solution of system* (3.1). *For each* $n \in \mathbb{N}$, *assume that* $\alpha, \beta \neq 1$. *For* $k = \left\lceil\frac{n}{6}\right\rceil$, *then all solutions of system* (3.1)

i) *If k is even,*

$$x_n = x_{-p},\ y_n = y_{-p}.$$

ii) *If k is odd and p is equal to 2 or 3*

$$x_n = y_{-p}\left(-1+\alpha\right),\ y_n = x_{-p}\left(-1+\alpha\right).$$

iii) *If k is odd and p is equal to otherwise,*

$$x_n = \frac{y_{-p}}{-1+\beta},\ y_n = \frac{x_{-p}}{-1+\alpha},$$

where for $(p = 0,1,2,3,4,5)$, $n+p \equiv 0 \pmod 6$, $r+n \equiv 0 \pmod 2$, $s = r+2,\ t = r+4,\ \alpha = x_{-r}y_{-s}x_{-t},\ \beta = y_{-r}x_{-s}y_{-t}$.

Proof. By induction. For $n = 1$, we obtain as $k = 1,\ p = 5,\ r = 1$. Then we can write

$$x_1 = x_n = \frac{y_{-5}}{-1+\beta},\ y_n = \frac{x_{-5}}{-1+\alpha},$$

where $\alpha = x_{-1}y_{-3}x_{-5},\ \beta = y_{-1}x_{-3}y_{-5}$, and the relation holds. Suppose that our assumption holds as follows:

$$\begin{aligned}
x_{12n-5} &= x_{-5},\ y_{12n-5} = y_{-5};\ x_{12n-4} = x_{-4},\ y_{12n-4} = y_{-4},\\
x_{12n-3} &= x_{-3},\ y_{12n-3} = y_{-3};\ x_{12n-2} = x_{-2},\ y_{12n-2} = y_{-2},\\
x_{12n-1} &= x_{-1},\ y_{12n-1} = y_{-1};\ x_{12n} = x_0,\ \ y_{12n} = y_0.
\end{aligned}$$

To end up the proof, we have to show that the cases in $\{x_m, y_m\}$ hold for $m = \overline{12n+1, 12n+6}$. For x_{12n+1} and y_{12n+1}, we obtain

$$k = \left\lceil \frac{12n+1}{6} \right\rceil = 2n+1,\ \ p = 5,\ r = 1.$$

Firstly, we consider $x_{12n+1} = \frac{y_{12n-5}}{-1+y_{12n-1}x_{12n-3}y_{12n-5}}$.Therefore, we can write that

$$\begin{aligned}
x_{12n+1} &= \frac{x_{-5}}{-1+y_{-1}x_{-3}y_{-5}}\\
&= \frac{x_{-5}}{-1+\beta}.
\end{aligned}$$

Secondly, we consider $y_{12n+1} = \frac{x_{12n-5}}{-1+x_{12n-1}y_{12n-3}x_{12n-5}}$. Then we obtain that

$$\begin{aligned}
y_{12n+1} &= \frac{y_{-5}}{-1+x_{-1}y_{-3}x_{-5}}\\
&= \frac{x_{-5}}{-1+\alpha}.
\end{aligned}$$

Similarly one can prove the other relations. The proof is complete. $\square$

Corollary 3.2. *Let $\{x_n, y_n\}$ be a solution of system (3.1). Then all solutions of system (3.1) are periodic with period 12.*

4 On System $x_{n+1}=\frac{y_{n-5}}{\pm 1+y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1}=\frac{x_{n-5}}{\mp 1+x_{n-1}y_{n-3}x_{n-5}}$

In this section, firstly, we consider the solution of the following system

$$x_{n+1}=\frac{y_{n-5}}{1+y_{n-1}x_{n-3}y_{n-5}},\ y_{n+1}=\frac{x_{n-5}}{-1+x_{n-1}y_{n-3}x_{n-5}} \tag{4.1}$$

where the initial values are arbitrary real numbers such that the denominator is always nonzero. If we interchange x_n and y_n in the following theorem, then we obtain the solution of the other system. So, we can omit the solution of other system.

Theorem 4.1. *Suppose that $\{x_n, y_n\}$ are solutions of system (4.1). Then every solutions of system (4) are periodic with period 24 and given by the following formula for $n=0,1,2,...,$*

$$\begin{aligned}
x_{24n-5} &= x_{-5},\ x_{24n-4}=x_{-4},\ x_{24n-3}=x_{-3},\ x_{24n-2}=x_{-2},\ x_{24n-1}=x_{-1},\\
x_{24n} &= x_0,\ x_{24n+1}=\frac{y_{-5}}{1+y_{-5}x_{-3}y_{-1}},\ x_{24n+2}=\frac{y_{-4}}{1+y_{-4}x_{-2}y_0},\\
x_{24n+3} &= \frac{y_{-3}(-1+x_{-5}y_{-3}x_{-1})}{(-1+2x_{-5}y_{-3}x_{-1})},\ x_{24n+4}=\frac{y_{-2}(-1+x_{-4}y_{-2}x_0)}{(-1+2x_{-4}y_{-2}x_0)},\\
x_{24n+5} &= \frac{y_{-1}}{1-y_{-5}x_{-3}y_{-1}},\ x_{24n+6}=\frac{y_0}{1-y_{-4}x_{-2}y_0},
\end{aligned}$$

$$\begin{aligned}
x_{24n+7} &= -x_{-5},\ x_{24n+8}=-x_{-4},\ x_{24n+9}=-x_{-3},\ x_{24n+10}=-x_{-2},\\
x_{24n+11} &= -x_{-1},\ x_{24n+12}=-x_0,\ \ x_{24n+13}=\frac{-y_{-5}}{1+y_{-5}x_{-3}y_{-1}},\\
x_{24n+14} &= \frac{-y_{-4}}{1+y_{-4}x_{-2}y_0},\\
x_{24n+15} &= \frac{-y_{-3}(-1+x_{-5}y_{-3}x_{-1})}{(-1+2x_{-5}y_{-3}x_{-1})},\ \ x_{24n+16}=\frac{-y_{-2}(-1+x_{-4}y_{-2}x_0)}{(-1+2x_{-4}y_{-2}x_0)},\\
x_{24n+17} &= \frac{-y_{-1}}{1-y_{-5}x_{-3}y_{-1}},\ x_{24n+18}=\frac{-y_0}{1-y_{-4}x_{-2}y_0},
\end{aligned}$$

$$\begin{aligned}
y_{24n-5} &= y_{-5},\ y_{24n-4}=y_{-4},\ y_{24n-3}=y_{-3},\ y_{24n-2}=y_{-2},\ y_{24n-1}=y_{-1},\\
y_{24n} &= y_0,\ y_{24n+1}=\frac{x_{-5}}{-1+x_{-5}y_{-3}x_{-1}},\ \ y_{24n+2}=\frac{x_{-4}}{-1+x_{-4}y_{-2}x_0},\\
y_{24n+3} &= -x_{-3}(1+y_{-5}x_{-3}y_{-1}),\ y_{24n+4}=-x_{-2}(1+y_{-4}x_{-2}y_0),\\
y_{24n+5} &= \frac{x_{-1}(1-2x_{-5}y_{-3}x_{-1})}{(-1+x_{-5}y_{-3}x_{-1})},\ y_{24n+6}=\frac{x_0(1-2x_{-4}y_{-2}x_0)}{(-1+x_{-4}y_{-2}x_0)},
\end{aligned}$$

$$\begin{aligned}
y_{24n+7} &= \frac{y_{-5}(-1+y_{-5}x_{-3}y_{-1})}{(1+y_{-5}x_{-3}y_{-1})},\ y_{24n+8}=\frac{y_{-4}(-1+y_{-4}x_{-2}y_0)}{(1+y_{-4}x_{-2}y_0)},\\
y_{24n+9} &= \frac{y_{-3}}{(-1+2x_{-5}y_{-3}x_{-1})},\\
y_{24n+10} &= \frac{y_{-2}}{(-1+2x_{-4}y_{-2}x_0)},\ y_{24n+11}=\frac{y_{-1}(1+y_{-5}x_{-3}y_{-1})}{(-1+y_{-5}x_{-3}y_{-1})},\\
y_{24n+12} &= \frac{y_0(1+y_{-4}x_{-2}y_0)}{(-1+y_{-4}x_{-2}y_0)},\ y_{24n+13}=\frac{x_{-5}(-1+2x_{-5}y_{-3}x_{-1})}{(-1+x_{-5}y_{-3}x_{-1})},\\
y_{24n+14} &= \frac{x_{-4}(-1+2x_{-4}y_{-2}x_0)}{(-1+x_{-4}y_{-2}x_0)},\\
y_{24n+15} &= x_{-3}(1-y_{-5}x_{-3}y_{-1}),\ y_{24n+16}=x_{-2}(1-y_{-4}x_{-2}y_0),\\
y_{24n+17} &= \frac{x_{-1}}{1-x_{-5}y_{-3}x_{-1}},\ y_{24n+18}=\frac{x_0}{1-x_{-4}y_{-2}x_0}.
\end{aligned}$$

5 Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations systems.

Example 1. Consider the difference system equation (2.1) with the initial conditions $x_{-5} = .5,\ x_{-4} = .2,\ x_{-3} = .11,\ x_{-2} = .7,\ x_{-1} = .4,\ x_0 = -3,\ y_{-5} = .7,\ y_{-4} = .12,\ y_{-3} = -1.2,$ $y_{-2} = .2, y_{-1} = -.11,\ and\ y_0 = .13.$ (See Fig. 1).

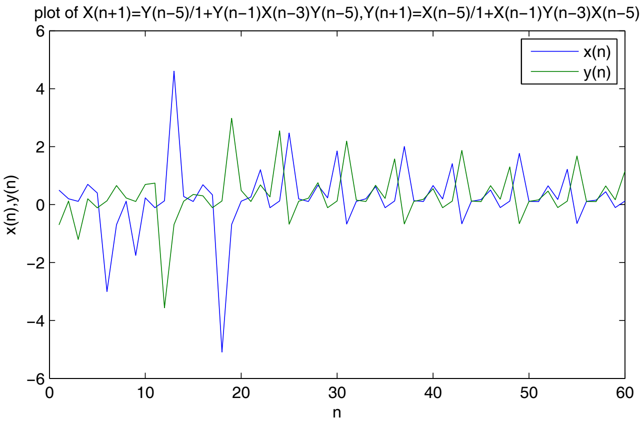

Figure 1.

Example 2. For the initial conditions $x_{-5} = 9,\ x_{-4} = 11,\ x_{-3} = 7,\ x_{-2} = 5,\ x_{-1} = 4,\ x_0 = 3,\ y_{-5} = 8,\ y_{-4} = 3,\ y_{-3} = 1.2, y_{-2} = 3.4, y_{-1} = 1.9,\ and\ y_0 = 13,$ when we take the system (2.1). (See Fig. 2).

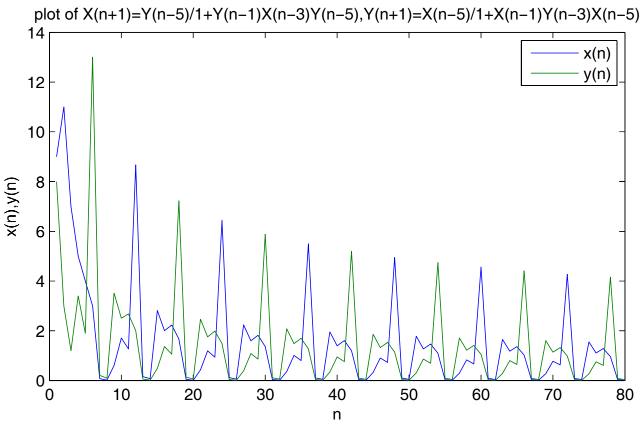

Figure 2.

Example 3. If we consider the difference equation system (3.1) with the initial conditions $x_{-5} = .7,\ x_{-4} = -.16,\ x_{-3} = 1.5,\ x_{-2} = -.3,\ x_{-1} = .24,\ x_0 = -.2,\ y_{-5} = .8,\ y_{-4} = 1.1,\ y_{-3} = -1.2,$ $y_{-2} = .4, y_{-1} = 1.9,\ and\ y_0 = -.13.$ (See Fig. 3).

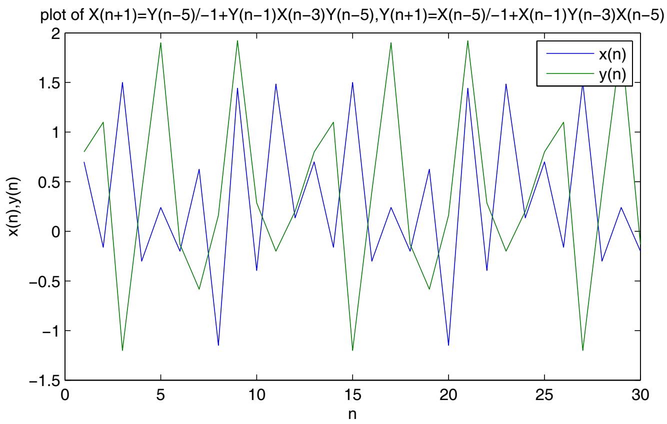

Figure 3.

Example 4. See Figure 4, since we take the difference system equation (4.1) with the initial conditions $x_{-5} = -.7,\ x_{-4} = .16,\ x_{-3} = -1.5,\ x_{-2} = .3,\ x_{-1} = -.24,\ x_0 = .2,\ y_{-5} = -.8,\ y_{-4} = 1.1,\ y_{-3} = -1.2, y_{-2} = 1.4, y_{-1} = -1.9,\ and\ y_0 = .13.$

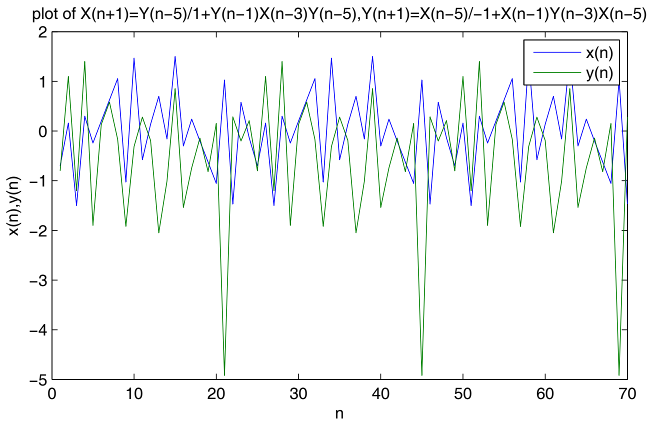

Figure 4.

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References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (1992).

[2] R.P. Agarwal, E.M. Elsayed, On the solution of fourth-order rational recursive sequence, *Adv. Stud. Contem. Math.,* **20(4)** (2010), 525–545.

[3] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = x_{n-1}/(-1 + ax_n x_{n-1})$, *Appl. Math. Comp.,* **158(3)** (2004), 793-797.

[4] C. Cinar, On the positive solutions of the difference equation system $x_{n+1} = 1/y_n$, $y_{n+1} = y_n/\left(x_{n-1}y_{n-1}\right)$, *Appl. Math. Comp.*, **158(2)** (2004), 303–305.

[5] D. Clark and M. Kulenovic, A coupled system of rational difference equations, *Comput. Math. Appl.,* **43** (2002), 849–867.

[6] E. M. Elsayed, Dynamics of a rational recursive sequences, *Int. J. Differ. Equ.,* **4(2)** (2009), 185–200.

[7] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, *J. Comp. Anal. Appl.*, **15** (**1**) (2013), 73-81.

[8] E. M. Elsayed, Solutions of rational difference systems of order two, *Math. Comput. Mod.,* **55(3-4)** (2012) 378–384.

[9] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Disc. Dyn. Nat. Soc., Volume 2011, Article ID 982309, 17 pages.

[10] E. M. Elsayed and H. A. El-Metwally, On the solutions of some nonlinear systems of difference equations, *Adv. Differ. Equ.*, 2013, 2013:16, doi:10.1186/1687-1847-2013-161.

[11] E.A. Grove, G. Ladas, L.C McGrath, C.T. Teixeira, Existence and behavior of solutions of a rational system, *Commun. Appl. Nonlinear Anal.* **8** (2001), 1–25.

[12] R. Karatas, C. Cinar, D. Simsek, On positive solutions of the difference equation $x_{n+1} = x_{n-5}/(1 + x_{n-2}x_{n-5})$, *Int. J. Contemp. Math. Sci.,* **1(10)** (2006) 495 500.

[13] A. S. Kurbanli, C. Cinar, I. Yalcinkaya, On the behavior of positive solutions of the system of rational difference equations, *Math. Comput. Mod.,* **53** (2011), 1261–1267.

[14] R. E. Mickens, *Difference equations, Theory and applications*, Van Nostrand Reinhold Co., New York (1990).

[15] A.Y. Ozban, On the positive solutions of the system of rational difference equations, $x_{n+1} = 1/y_{n-k}, y_{n+1} = y_n/x_{n-m}y_{n-m-k}$, *J. Math. Anal. Appl.,* **323** (2006), 26–32.

[16] S. Stevic, More on a rational recurrence relation, *Appl. Math. E-notes* **4** (2004), 80-84.

[17] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, *Math. Comput. Mod.,* **55** (2012) 1987-1997.

[18] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, *Bull. Math. Soc. Sci. Math. Roumanie,* ***2** (2012), 217-224.*

[19] I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, *Disc. Dyn. Nat. Soc.,* (2008), Article ID 860152.

[20] I. Yalcinkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, *ARS Combinatoria,* **95** (2010) 151–159.

[21] I. Yalcinkaya, C. Cinar, M. Atalay, On the solutions of systems of difference equations, *Adv. Differ. Equ.,* (2008) **9**. Article ID 143943.

[22] X. Yang, On the system of rational difference equations $x_n = A + y_{n-1}/x_{n-p}y_{n-q}$, $y_n = A + x_{n-1}/x_{n-r}y_{n-s}$, *J. Math. Anal. Appl.,* **307** (2005) 305–311.

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The shared set of meromorphic functions and differential polynomials *

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Abstract
The purpose of our paper is to deal with some uniqueness problem of meromorphic functions whose differential polynomials with more general form then the formers sharing a set with finite weight. These results in this paper complement some results given by Lin, Yi.
Key words: Meromorphic function, small function, weighted sharing, differential polynomial.
Mathematical Subject Classification (2010): 30D 30, 30D 35.

1 Introduction and Main Results

In this paper the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. We shall use the following standard notations of the value distribution theory(see Hayman [7], Yi and Yang [17]). Let f be a nonconstant meromorphic function and $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and S be a subset of $\overline{\mathbb{C}}$. Define

$$E(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \ \textit{counting multiplicity}\},$$

$$\overline{E}(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \ \textit{ignoring multiplicity}\}.$$

If $E(S, f) = E(S, g)$ we say that f and g share the set S CM; if $\overline{E}(S, f) = \overline{E}(S, g)$, we say that f and g share the set S IM. Especially, let $S = \{a\}$, we say that f and g share the value a CM if $E(S, f) = E(S, g)$; and we say that f and g share the value a IM if $\overline{E}(S, f) = \overline{E}(S, g)$(see [6]).

Let m be a nonnegative integer, we denote by $E_m(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_m(a; f)$ the set of distinct a-points of f with multiplicities not greater than m. If $E_\infty(a; f) = E_\infty(a; g)$ for some $a \in \overline{\mathbb{C}}$, we say that f, g share the value a CM. For any positive integer m, we define

$$E_m(S, f) = \bigcup_{a\in S} E_m(a; f), \quad \text{and} \quad \overline{E}_m(S, f) = \bigcup_{a\in S} \overline{E}_m(a; f).$$

In 1997, Yang and Hua [16] proved the following result.

Theorem 1.1 (see [16]). *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n + 1)$th root of unity d or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$ where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

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In 2001, an idea of gradation of sharing of values was introduced in [8, 10] which measures how close a shared value is to being shared CM or to being shared IM. This notion is known as weighted sharing. The author studied some problem on the uniqueness of meromorphic function sharing some values and sets with finite weight (see [15, 14, 13])

In 2002, Fang and Fang [5] employed the idea of weighted sharing of values and obtained the following results:

Theorem 1.2 (see [5]). *Let f and g be two nonconstant entire functions, n be a positive integer. If $E_k(1, f^n(f-1)f') = E_k(1, g^n(g-1)g')$ and one of the following conditions is satisfied: (a) $k \geq 3$ and $n \geq 8$, (b) $k = 2$ and $n \geq 9$, (c) $k = 1$ and $n \geq 14$, then $f \equiv g$.*

The following example shows that Theorem 1.2 is not valid when f and g are two meromorphic functions.

Example 1.1 (see [11]). *Let*

$$f = \frac{(n+2)(h-h^{n+2})}{(n+1)(1-h^{n+2})}, \qquad g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where $h = e^z$. Then $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, but $g \not\equiv f$.

For meromorphic functions, Fang and Fang [5], Lin and Yi [11] obtained some unicity theorems corresponding to the above theorems.

Theorem 1.3 (see [5]). *Let f and g be two nonconstant meromorphic functions, n be a positive integer. If $E_k(1, f^n(f-1)^2f') = E_k(1, g^n(g-1)^2g')$ and one of the following conditions is satisfied: (a) $k \geq 3$ and $n \geq 13$, (b) $k = 2$ and $n \geq 15$, (c) $k = 1$ and $n \geq 23$, then $f \equiv g$.*

Theorem 1.4 (see [11]). *Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n+1}$, $n \geq 12$. If $[f^n(f-1)]f'$ and $[g^n(g-1)]g'$ share 1 CM, then $f \equiv g$.*

In the mean time, Lahiri and Sarkar [9] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from the forms previously mentioned, and proved the following result.

Theorem 1.5 (see [9]). *Let f and g be two nonconstant meromerophic functions, $n(\geq 13)$ is an integer. If $E_2(1, f^n(f^2-1)f') = E_2(1, g^n(g^2-1)g')$, then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

In this paper, we will investigate the uniqueness of meromorphic functions when two nonlinear differential polynomials of more general form namely $f^n(f-a)(f-b)f'$ and $g^n(g-a)(g-b)g'$ where $a \neq b$ and $a, b \neq 0$, share a set $S_m = \{1, \omega, \omega^2, \cdots, \omega^{m-1}\}$ where $\omega = e^{\frac{2\pi}{m}i}$, m is a integer.

Now, we state our main results of the paper as follows.

Theorem 1.6 *Let f and g be two nonconstant meromorphic functions, n and $m(\geq 2)$ be two positive integers. Let $E_k(S_m, f^n(f-a)(f-b)f') = E_k(S_m, g^n(g-a)(g-b)g')$, and f or g be meromorphic function only having multiple poles, and let the two functions $\frac{a+b}{n+2}g\sum_{s=0}^{n+1}(\frac{f}{g})^s - \frac{ab}{n+1}\sum_{s=0}^{n}(\frac{f}{g})^s$ and $\sum_{s=0}^{n+2}(\frac{f}{g})^s$ have no common simple zeros. If one of the following conditions is satisfied:*

(i) *$k \geq 3$: $n > 4 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;*

(ii) *$k = 2$: $n > 4 + \frac{11}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;*

(iii) *$k = 1$: $n > 4 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$.*

Then $f \equiv g$.

Remark 1.1 *A. Banerjee [2] obtained some theorems when* $m = 1$, *that is.* $S_m = \{1\}$.

Theorem 1.7 *Let* f *and* g *be two nonconstant meromorphic functions,* n *and* $m(\geq 2)$ *be two positive integers. If* $E_k(S_m, f^n(f-a)^2f') = E_k(S_m, g^n(g-a)^2g')$, *and one of the following conditions is satisfied:*

(i) $k \geq 3$ *and* $n > 4 + \frac{8}{m}$;

(ii) $k = 2$ *and* $n > \max\left\{4 + \frac{4}{m}, 2 + \frac{10}{m}\right\}$;

(iii) $k = 1$: $n > 4 + \frac{20}{m}$ *when* $2 \leq m \leq 3$ *and* $n > 4 + \frac{4}{m}$ *when* $m \geq 4$.

Then $f \equiv g$.

Next, some definitions and notations used in the paper are explained as follows.

For $a \in \overline{\mathbb{C}}$ and a positive integer k, we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f, and denote by $N(r, a; f| \leq k)$ $(N(r, a; f| \geq k))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than k where each a-point is counted according to its multiplicity(see [7]). $\overline{N}(r, a; f| \leq k)(\overline{N}(r, a; f| \geq k))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Set $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2) + \cdots + \overline{N}(r, a; f| \geq k)$.

Definition 1.1 [1, 17] *When* f *and* g *share* 1 IM, *We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of the* 1*-points of* f *whose multiplicities are greater than* 1*-points of* g, *where each zero is counted only once; Similarly, we have* $\overline{N}_L(r, 1; g)$. *We also denote by* $N_E^{1)}(r, 1; f)$ *the counting function of common simple* 1*-points of* f *and* g; $\overline{N}_E^{(2}(r, 1; f)$ *denotes the counting function of those multiplicity 1-points of* f *and* g,*each point in these counting functions is counted only once. In the same way ,one can define* $N_E^{1)}(r, 1; g), \overline{N}_E^{(2}(r, 1; g)$.

2 Some Lemmas

To prove our results, we need the following Lemmas.

Lemma 2.1 (see [12]). *Let* f *be a nonconstant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{p} a_k f^k}{\sum_{j=0}^{q} b_j f^j}$$

be an irreducible rational function in f *with constant coefficient* $\{a_k\}$ *and* $\{b_j\}$, *where* $a_p \neq 0$ *and* $b_q \neq 0$. *Then*

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{p, q\}$.

Lemma 2.2 (see [17]). *Let* f *be a nonconstant meromorphic function, then*

$$T\left(r, f^{(k)}\right) \leq T(r, f) + k\overline{N}(r, \infty; f) + S(r, f),$$

and

$$N\left(r, 0; f^{(k)}\right) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.3 *Let f and g be two nonconstant meromorphic functions and n, m be two positive integers such that $n > 1 + \frac{4}{m}$. If f or g is meromorphic function having only multiple poles and the two expressions $\frac{a+b}{n+2} g \sum_{s=0}^{n+1} (\frac{f}{g})^s - \frac{ab}{n+1} \sum_{s=0}^{n} (\frac{f}{g})^s$ and $\sum_{s=0}^{n+2} (\frac{f}{g})^s$ have no common simple zeros, and*

$$\left(\frac{f^{n+3}}{n+3} - \frac{(a+b)f^{n+2}}{n+2} + \frac{abf^{n+1}}{n+1}\right)^m \equiv \left(\frac{g^{n+3}}{n+3} - \frac{(a+b)g^{n+2}}{n+2} + \frac{abg^{n+1}}{n+1}\right)^m,$$

where $a, b \neq \mathbb{C} - \{0\}$, then $f \equiv g$.

Proof: We propose to follow the idea in the proof of [3, Lemma 2.11]. From the assumption of Lemma 2.3, we have

$$\frac{f^{n+3}}{n+3} - \frac{(a+b)f^{n+2}}{n+2} + \frac{abf^{n+1}}{n+1} \equiv t\left(\frac{g^{n+3}}{n+3} - \frac{(a+b)g^{n+2}}{n+2} + \frac{abg^{n+1}}{n+1}\right), \tag{1}$$

where $t^m = 1$. From (1), we get that f and g share ∞ CM. Without loss of generality, from the assumption of Lemma 2.3, we may assume that g has some multiple poles. Let $h = \frac{f}{g}$. From (1), we have

$$Ag^2(h^{n+3} - t) + Bg(h^{n+2} - t) + C(h^{n+1} - t) \equiv 0,$$

i.e.,

$$Ag^2 = -Bg\frac{h^{n+2} - t}{h^{n+3} - t} - C\frac{h^{n+1} - t}{h^{n+3} - t}, \tag{2}$$

where $A = \frac{1}{n+3}$, $B = -\frac{a+b}{n+2}$ and $C = \frac{ab}{n+1}$.

Let z_0 be a pole of g with multiplicity $p_1 (\geq 2)$, which is not a root of $h - u_k = 0$, where $u_k^{n+3} = t$. From (2), we have $2p_1 = p_1$ *i.e.*, $p_1 = 0$. Thus, we get a contradiction.

Therefore, we can see that the poles of g are precisely the roots of $h - u_k = 0$.

Let z_1 be a zero of $h - u_k$ of multiplicity p_2 which is a pole of g with multiplicity q_2, then from (2) we have $2q_2 = p_1 + q_2$ *i.e.*, $p_2 = q_2$.

Since g has no simple pole, it follows that such points are multiple zeros of $h - u_k$.

From (2), we have

$$Ag^2 = -\frac{Bg\sum_{j=0}^{n+1} h^j + C\sum_{j=0}^{n} h^j}{\sum_{j=0}^{n+2} h^j}. \tag{3}$$

Suppose z_2 be a simple zero of $h - u_k$ where $k = 1, 2, \ldots, n+2$, which is a zero of multiplicity $q_1 (\geq 2)$ of numerator of (3). Then from (3), z_2 would be a zero of order $q_1 - 1$ of g^2. So it follows that z_2 would be a zero of $\sum_{j=0}^{n} h^j$. Since $\sum_{j=0}^{n} h^j$ and $\sum_{j=0}^{n+2} h^j$ may have at most one common factor and a meromorphic function can not have more than two Picard exceptional values, we see that $h - u_k$ has multiple zeros for at least $n - 1$ values of $k \in \{1, 2, \ldots, n+2\}$. Therefore, we have $\Theta(u_k; h) \geq \frac{1}{2}$ for at least $n - 1$ values of k, which implies a contradiction as $n > 1 + \frac{4}{m}$.

Thus, we complete the proof of Lemma 2.3. □

Lemma 2.4 *Let f and g be two nonconstant meromorphic functions, $n(> 3), m$ be two positive integers. Then we have*

$$(f^n(f-a)(f-b)f')^m (g^n(g-a)(g-b)g')^m \not\equiv 1,$$

where $a, b \in \mathbb{C} - \{0\}$.

Proof: Suppose $(f^n(f-a)(f-b)f')^m (g^n(g-a)(g-b)g')^m \not\equiv 1$, then we have

$$f^n(f-a)(f-b)f'g^n(g-a)(g-b)g' \equiv t, \tag{4}$$

where $t^m = 1$. Without loss of generality , we suppose that there exists a set I with infinite measure such that $T(r, g) \leq T(r, f)$, $r \in I$. Next, we consider the two following cases.

Case 1: $a = b$. From (4), we have

$$f^n(f-a)^2 f' g^n (g-a)^2 g' \equiv t.$$

Using the similar method of [5, P.611], we can get that the equality is impossible.

Case 2: $a \neq b$.

Let z_0 be a zero of f with multiplicity p_1, then from (4) we can see that z_0 is a pole of g (say with multiplicity q_1). Thus, we have $np_1+p_1-1 = nq_1+2q_1+q_1+1$ *i.e.*, $2q_1+2 = (n+1)(p_1-q_1) \geq n+1$, that is, $q_1 \geq \frac{n-1}{2}$. Hence, from this we can deduce that

$$(n+1)p_1 \geq \frac{(n+3)(n-1)+4}{2} \qquad i.e., \qquad p_1 \geq \frac{n+1}{2}. \tag{5}$$

Let z_1 be a zero of $f-a$ with multiplicity p_2, then from (4) we can see that z_1 is a pole of g (say with multiplicity q_2). Thus, we have

$$2p_2 - 1 = (n+3)q_2 + 1 \qquad i.e., \qquad p_2 = \frac{(n+3)q_2+2}{2} \geq \frac{n+5}{2}. \tag{6}$$

Let z_2 be a zero of $f-b$ with multiplicity p_3, then from (4) we can see that z_2 is a pole of g (say with multiplicity q_3). Similarly, we have

$$p_3 \geq \frac{n+5}{2}. \tag{7}$$

Let z_3 be a zero of f' with multiplicity p_4 which is not a zero of $f(f-a)(f-b)$, then from (4) we get that z_3 is a pole of g (say with multiplicity q_4). Therefore, we have

$$p_4 = (n+3)q_4 + 1 \geq n+4. \tag{8}$$

Similarly, we have the same results for the zeros of $g(g-a)(g-b)g'$. Thus, we have

$$\begin{aligned}
\overline{N}(r,\infty;f) &= \overline{N}(r,\infty;f(f-a)(f-b)f') \\
&\leq \overline{N}(r,0;g) + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N}_0(r,0;g') \\
&\leq \frac{2}{n+1}N(r,0;g) + \frac{2}{n+5}N(r,a;g) + \frac{2}{n+5}N(r,b;g) + \frac{1}{n+4}N_0(r,0;g') \\
&\leq \left(\frac{2}{n+1} + \frac{4}{n+5} + \frac{1}{n+4}\right) T(r,g) + S(r,g).
\end{aligned}$$

By the second main theorem and from the above inequality, we have

$$\begin{aligned}
2T(r,f) \leq & \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + S(r,f) \\
\leq & \frac{2}{n+1}N(r,0;f) + \frac{2}{n+5}N(r,a;f) + \frac{2}{n+5}N(r,b;f) \\
& + \left(\frac{2}{n+1} + \frac{4}{n+5} + \frac{1}{n+4}\right) T(r,g) + S(r,g) + S(r,f) \\
\leq & \left(\frac{4}{n+1} + \frac{8}{n+5} + \frac{2}{n+4}\right) T(r,f) + S(r,f).
\end{aligned}$$

Since $n > 3$, we can get a contradiction.

Thus, we complete the proof of Lemma 2.4. $\square$

Lemma 2.5 (see [5]). *Let f and g be two meromorphic functions, and let k be a positive integer. If $E_k(1,f) = E_k(1,g)$, then one of the following cases must occur:*

(i)

$$\begin{aligned}T(r,f)+T(r,g)\leq &N_2(r,\infty;f)+N_2(r,0;f)+N_2(r,\infty;g+N_2(r,0;g))\\&+\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_E^{1)}(r,1;f)\\&+\overline{N}(r,1;f|\geq k+1)+\overline{N}(r,1;g|\geq k+1)+S(r,f)+S(r,g);\end{aligned}$$

(ii) $f=\frac{(B+1)g+(A-B-1)}{Bg+(A-B)}$, *where* $A(\neq 0), B$ *are two constants.*

Lemma 2.6 *Let* f *and* g *be two nonconstant meromorphic functions and* n, m *be two positive integers such that* $n>4+\frac{4}{m}$. *Let* $F=f^n(f-a)(f-b)f'$ *and* $G=g^n(g-a)(g-b)g'$, *where* $a\neq b$ *and* $a, b\in\mathbb{C}-\{0\}$. *If one of* f *and* g *is meromorphic function having and only having multiple poles and the two expressions* $\frac{a+b}{n+2}g\sum_{s=0}^{n+1}(\frac{f}{g})^s-\frac{ab}{n+1}\sum_{s=0}^{n}(\frac{f}{g})^s$ *and* $\sum_{s=0}^{n+2}(\frac{f}{g})^s$ *have no common simple zeros, and*

$$F^m=\frac{(B+1)G^m+A-B-1}{BG^m+A-B}, \tag{9}$$

where $A(\neq 0)$ *and* B *are constants, then* $f\equiv g$.

Proof: Let

$$P(z)=\frac{1}{n+3}z^{n+3}-\frac{a+b}{n+2}z^{n+2}+\frac{ab}{n+1}z^{n+1}. \tag{10}$$

Then we have

$$F=(P(f))'=f^n(f-a)(f-b)f',\quad G=(P(g))'=g^n(g-a)(g-b)g'. \tag{11}$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}T(r,F)&\leq T(r,f^n(f-a)(f-b))+T(r,f')\\&\leq (n+2)T(r,f)+2T(r,f)+S(r,f)\\&=(n+4)T(r,f)+S(r,f).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(n+2)T(r,f)=&T(r,f^n(f-a)(f-b))\\\leq&T(r,f^n(f-a)(f-b)f')+T(r,f')+O(1)\\\leq&T(r,F)+2T(r,f)+S(r,f).\end{aligned}$$

Therefore, we have

$$nT(r,f)+S(r,f)\leq T(r,F)\leq (n+4)T(r,f)+S(r,f). \tag{12}$$

Thus, we get $S(r,F)=S(r,f)$. Similarly, we can get $S(r,G)=S(r,g)$.
By Lemma 2.1, we have

$$\begin{aligned}(n+3)T(r,f)&=T(r,P(f))\\&\leq T(r,(P(f))')+N(r,0;P(f))-N(r,0;(P(f))')+S(r,f)\\&=T(r,F)+N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)\\&\quad-N(r,a;f)-N(r,b;f)-N(r,0;f')+S(r,f),\end{aligned} \tag{13}$$

where γ_1,γ_2 are the two roots of the equation $\frac{1}{n+3}z^2-\frac{a+b}{n+2}z+\frac{ab}{n+1}=0$.

Similarly, we can get

$$\begin{aligned}(n+3)T(r,g) &= T(r,P(g)) \\ &\leq T(r,G)+nN(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g) \\ &\quad -N(r,a;g)-N(r,b;g)-N(r,0;g')+S(r,g).\end{aligned} \tag{14}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,g)\leq T(r,f)$, $r\in I$. Next we consider three cases as follows.

Case 1. Suppose $B\neq 0,-1$. From (9), we have $\overline{N}(r,\frac{B+1}{B};F^m)=\overline{N}(r,\infty;G^m)$. By the second main theorem and $S(r,F^m)=S(r,f)$, we get

$$\begin{aligned}mT(r,F) &= T(r,F^m) \\ &\leq \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+\overline{N}(r,\frac{B+1}{B};F^m)+S(r,f) \\ &= \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+\overline{N}(r,\infty;G^m)+S(r,f) \\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+N(r,a;f)+N(r,b;f) \\ &\quad +N(r,0;f')+\overline{N}(r,\infty;g)+S(r,f).\end{aligned} \tag{15}$$

From (13) and (15), we can get

$$\begin{aligned}(n+3)T(r,f) =&T(r,P(f)) \\ \leq&\frac{1}{m}\overline{N}(r,\infty;f)+(1+\frac{1}{m})N(r,0;f)+N(r,\gamma_1;f) \\ &+N(r,\gamma_2;f)+\frac{1}{m}\overline{N}(r,\infty;g)+S(r,f) \\ \leq&(3+\frac{2}{m})T(r,f)+\frac{1}{m}T(r,g)+S(r,f) \\ \leq&(3+\frac{3}{m})T(r,f)+S(r,f),\end{aligned}$$

i.e.,

$$(n-\frac{3}{m})T(r,f)\leq S(r,f).$$

Since $n>4+\frac{4}{m}$, we get a contradiction.

Case 2. Suppose $B=0$. From (9), we have $\overline{N}(r,\frac{A-1}{A};F^m)=\overline{N}(r,0;G^m)$. We consider two subcases as follows.

Subcase 2.1. $A\neq 1$. By the second main theorem and $S(r,F^m)=S(r,f)$, we get

$$\begin{aligned}mT(r,F) &= T(r,F^m) \\ &\leq \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+\overline{N}\left(r,\frac{A-1}{A};F^m\right)+S(r,f) \\ &= \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+\overline{N}(r,0;G^m)+S(r,f) \\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+N(r,a;f)+N(r,b;f)+N(r,0;f') \\ &\quad +\overline{N}(r,0;g)+N(r,a;g)+N(r,b;g)+\overline{N}(r,0;g')+S(r,f).\end{aligned} \tag{16}$$

From (13) and (16), we can get

$$\begin{aligned}(n+3)T(r,f) =&T(r,P(f)) \\ \leq&(3+\frac{2}{m})T(r,f)+\frac{5}{m}T(r,g)+S(r,f) \\ \leq&(3+\frac{7}{m})T(r,f)+S(r,f),\end{aligned}$$

Thus, we get $(n-\frac{7}{m})T(r,f)\le S(r,f)$. Since $n>4+\frac{4}{m}$, we can deduce a contradiction.

Subcase 2.2. $A=1$. Then we have $F^m=G^m$, that is $F=tG$, where $t^m=1$. By integration we have $P(f)=tP(g)+s$, where s is a constant. If $s\neq 0$, by the second main theorem we have

$$\begin{aligned}(n+3)T(r,f)=&T(r,P(f))\\ \le&\overline{N}(r,\infty;P(f))+\overline{N}(r,0;P(f))+\overline{N}(r,s;P(f))+S(r,f)\\ =&\overline{N}(r,\infty;P(f))+\overline{N}(r,0;P(f))+\overline{N}(r,0;tP(g))+S(r,f)\\ \le&\overline{N}(r,\infty;f)+N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)\\ &+N(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g)+S(r,f)\\ \le&4T(r,f)+3T(r,g)+S(r,f)\\ \le&7T(r,f)+S(r,f)\end{aligned}$$

Since $n>4+\frac{4}{m}$, we get a contradiction. Hence $s=0$, that is. $P(f)\equiv tP(g)$. By Lemma 2.3 we get $f\equiv g$.

Case 3. $B+1=0$. Proceeding as in the proof of Case 2, we can get $F^mG^m\equiv 1$, that is. $f^n(f-a)(f-b)f'g^n(g-a)(g-b)g'\equiv t$, where $t^m=1$. By Lemma 2.4, we have $f\equiv g$.

Therefore, we complete the proof of Lemma 2.6. □

Lemma 2.7 (see [4]). *Let*

$$Q(\omega)=(n-1)^2(\omega^n-1)(\omega^{n-2}-1)-n(n-2)(\omega^{n-1}-1)^2,$$

then

$$Q(\omega)=(\omega-1)^4(\omega-\beta_1)(\omega-\beta_2)\cdots(\omega-\beta_{2n-6}),$$

where $\beta_j\in\mathbb{C}-\{0,1\}$ $(j=1,2,\ldots,2n-6)$*, which are distinct respectively.*

3 Proofs of Theorems 1.7 and 1.8

3.1 Proof of Theorem 1.7

Proof: Let F and G be given by (11), and $P(z)$ by (10). From the assumptions of Theorem 1.7, we have $E_k(S_m,F)=E_k(S_m,G)$ that is. $E_k(1,F^m)=E_k(1,G^m)$ and

$$\begin{aligned}N_2(r,0;F^m)+N_2(r,\infty;F^m)\le&2N(r,0;f)+2\overline{N}(r,a;f)+2\overline{N}(r,b;f)+2\overline{N}(r,0;f')\\ &+2\overline{N}(r,\infty;f)+S(r,f),\end{aligned}\tag{17}$$

$$\begin{aligned}N_2(r,0;G^m)+N_2(r,\infty;G^m)\le&2N(r,0;g)+2\overline{N}(r,a;g)+2\overline{N}(r,b;g)+2\overline{N}(r,0;g')\\ &+2\overline{N}(r,\infty;g)+S(r,g),\end{aligned}\tag{18}$$

(i) $k\ge 3$. Since

$$\begin{aligned}&\overline{N}(r,1;F^m)+\overline{N}(r,1;G^m)+\overline{N}(r,1;F^m|\ge k+1)\\ &+\overline{N}(r,1;G^m|\ge k+1)-N_E^{1)}(r,1;F^m)\\ \le&\frac{1}{2}N(r,1;F^m)+\frac{1}{2}N(r,1;G^m)+S(r,F^m)+S(r,G^m)\\ \le&\frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{19}$$

Suppose that F^m, G^m satisfy (i) of Lemma 2.5, then from Lemma 2.1, we can get

$$\begin{aligned} mT(r,F)+mT(r,G) =& T(r,F^m)+T(r,G^m) \\ \leq& \frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+N_2(r,0;F^m)+N_2(r,\infty;F^m) \\ &+N_2(r,0;G^m)+N_2(r,\infty;G^m)+S(r,F)+S(r,G), \end{aligned}$$

i.e.,

$$\begin{aligned} T(r,F)+T(r,G) \leq& \frac{2}{m}N_2(r,0;F^m)+\frac{2}{m}N_2(r,\infty;F^m)+\frac{2}{m}N_2(r,0;G^m) \\ &+\frac{2}{m}N_2(r,\infty;G^m)+S(r,F)+S(r,G). \end{aligned} \tag{20}$$

(i_1) $2 \leq m \leq 3$. From (13),(14),(17),(18),(20) and the definitions of F, G, we have

$$\begin{aligned} &(n+3)T(r,f)+(n+3)T(r,g) \\ \leq&(1+\frac{4}{m})N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,a;f) \\ &+N(r,b;f)+\frac{4}{m}\overline{N}(r,\infty;f)+N(r,0;f')+(1+\frac{4}{m})N(r,0;g) \\ &+N(r,\gamma_1;g)+N(r,\gamma_2;g)+N(r,a;g)+N(r,b;g) \\ &+\frac{4}{m}\overline{N}(r,\infty;g)+N(r,0;g')+S(r,f)+S(r,g) \\ \leq&(7+\frac{8}{m})T(r,f)+(7+\frac{8}{m})T(r,g)+S(r,f)+S(r,g), \end{aligned}$$

i.e.,

$$\left(n-4-\frac{8}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g). \tag{21}$$

Since $n > 4+\frac{8}{m}$, we can get a contradiction.

By Lemma 2.5, we have

$$F^m = \frac{(B+1)G^m+A-B-1}{BG^m+A-B},$$

where $A(\neq 0)$ and B are constants.

From Lemma 2.6 and $n > 4+\frac{8}{m}$, we can get $f \equiv g$.

(i_2) If $m \geq 4$, then $\frac{4}{m}-1 \leq 0$. Thus, from (13),(14),(17),(18) and (20), we can get

$$\begin{aligned} &(n+3)T(r,f)+(n+3)T(r,g) \\ \leq&(1+\frac{4}{m})N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+\frac{4}{m}\overline{N}(r,\infty;f)+(1+\frac{4}{m})N(r,0;g) \\ &+N(r,\gamma_1;g)+N(r,\gamma_2;g)+\frac{4}{m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\ \leq&(3+\frac{8}{m})T(r,f)+(3+\frac{8}{m})T(r,g)+S(r,f)+S(r,g), \end{aligned}$$

i.e.,

$$\left(n-\frac{8}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

Since $n > 4+\frac{4}{m}$, we can get a contradiction.

By Lemma 2.5, we have

$$F^m = \frac{(B+1)G^m+A-B-1}{BG^m+A-B},$$

where $A(\neq 0)$ and B are constants.

From Lemma 2.6 and $n > 4 + \frac{4}{m}$, we can get $f \equiv g$.

(ii) $k = 2$. Since

$$\begin{aligned}&\overline{N}(r,1;F^m)+\overline{N}(r,1;G^m)+\frac{1}{2}\overline{N}(r,1;F^m|\geq 3)\\&+\frac{1}{2}\overline{N}(r,1;G^m|\geq 3)-\overline{N}_E^{1)}(r,1;F^m)\\\leq&\frac{1}{2}N(r,1;F^m)+\frac{1}{2}N(r,1;G^m)+S(r,F)+S(r,G)\\\leq&\frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+S(r,F)+S(r,G).\end{aligned}\tag{22}$$

Suppose that F^m, G^m satisfy (i) of Lemma 2.5, then from Lemma 2.1 and (22), we can get

$$\begin{aligned}\frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)\leq&N_2(r,0;F^m)+N_2(r,\infty;F^m)+N_2(r,0;G^m)+N_2(r,\infty;G^m)\\&+\frac{1}{2}\overline{N}(r,1;F^m|\geq 3)+\frac{1}{2}\overline{N}(r,1;G^m|\geq 3)+S(r,F)+S(r,G).\end{aligned}\tag{23}$$

Since

$$\begin{aligned}\overline{N}(r,1;F^m|\geq 3)\leq&\frac{1}{2}N\left(r,\infty;\frac{F^m}{(F^m)'}\right)=\frac{1}{2}N\left(r,\infty;\frac{(F^m)'}{F^m}\right)+S(r,F)\\\leq&\frac{1}{2}\overline{N}(r,\infty;F^m)+\frac{1}{2}\overline{N}(r,0;F^m)+S(r,F)\\\leq&\frac{1}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,a;f)\\&+\frac{1}{2}\overline{N}(r,b;f)+\frac{1}{2}\overline{N}(r,0;f')+S(r,f),\end{aligned}\tag{24}$$

and

$$\begin{aligned}\overline{N}(r,1;G^m|\geq 3)\leq&\frac{1}{2}\overline{N}(r,\infty;g)+\frac{1}{2}\overline{N}(r,0;g)+\frac{1}{2}\overline{N}(r,a;g)\\&+\frac{1}{2}\overline{N}(r,b;g)+\frac{1}{2}\overline{N}(r,0;g')+S(r,g).\end{aligned}\tag{25}$$

(ii_1) $2 \leq m \leq 3$. From (13),(14), (17),(18) and (23)-(25), we can get

$$\begin{aligned}&(n+3)T(r,f)+(n+3)T(r,g)\\\leq&T(r,F)+T(r,G)\\\leq&\left(1+\frac{9}{2m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+\left(1+\frac{1}{2m}\right)N(r,a;f)\\&+\left(1+\frac{1}{2m}\right)N(r,b;f)+\left(1+\frac{1}{2m}\right)N(r,0;f')+\frac{9}{2m}\overline{N}(r,\infty;f)\\&+\left(1+\frac{9}{2m}\right)N(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g)+\left(1+\frac{1}{2m}\right)N(r,a;g)\\&+\left(1+\frac{1}{2m}\right)N(r,b;g)+\left(1+\frac{1}{2m}\right)N(r,0;g')+\frac{9}{2m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\\leq&\left(7+\frac{11}{m}\right)T(r,f)+\left(7+\frac{11}{m}\right)T(r,g)+S(r,f)+S(r,g),\end{aligned}$$

that is,

$$\left(n-4-\frac{11}{m}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g).\tag{26}$$

Since $n > 4 + \frac{11}{m}$, we can get a contradiction.

Thus, from Lemma 2.5, we can get that F^m, G^m satisfy the equality

$$F^m = \frac{(B+1)G^m + A - B - 1}{BG^m + A - B},$$

where $A(\neq 0)$ and B are constants.

From Lemma 2.6 and $n > 4 + \frac{11}{m}$, we can get $f \equiv g$.

(ii_2) $m \geq 4$. Like (i_2), we can get

$$\begin{aligned}
&(n+3)T(r,f) + (n+3)T(r,g) \\
\leq &T(r,F) + T(r,G) \\
\leq &\left(1 + \frac{9}{2m}\right) N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + \frac{1}{2m}N(r,a;f) + \frac{1}{2m}N(r,b;f) \\
&+ \frac{1}{2m}N(r,0;f') + \frac{9}{2m}\overline{N}(r,\infty;f) + \left(1 + \frac{9}{2m}\right) N(r,0;g) + N(r,\gamma_1;g) + N(r,\gamma_2;g) \\
&+ \frac{1}{2m}N(r,a;g) + \frac{1}{2m}N(r,b;g) + \frac{1}{2m}N(r,0;g') + \frac{9}{2m}\overline{N}(r,\infty;g) + S(r,f) + S(r,g),
\end{aligned}$$

that is,

$$\left(n - \frac{11}{m}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g).$$

Since $n > 4 + \frac{4}{m}$ and $m \geq 4$, then a contradiction $n - \frac{11}{m} > 4 - \frac{7}{m} > 0$ exists.

Thus, from Lemma 2.5, we can get that F^m, G^m satisfy the equality

$$F^m = \frac{(B+1)G^m + A - B - 1}{BG^m + A - B},$$

where $A(\neq 0)$ and B are constants.

From Lemma 2.6 and $n > 4 + \frac{4}{m}$, we can get $f \equiv g$.

(iii) $k = 1$. Since

$$\begin{aligned}
&\overline{N}(r,1;F^m) + \overline{N}(r,1;G^m) - N_E^{1)}(r,1;F^m) \\
\leq &\frac{1}{2}N(r,1;F^m) + \frac{1}{2}N(r,1;G^m) + S(r,F) + S(r,G) \\
\leq &\frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + S(r,F) + S(r,G),
\end{aligned} \tag{27}$$

$$\begin{aligned}
\overline{N}(r,1;F^m| \geq 2) \leq &N\left(r,\infty;\frac{F^m}{(F^m)'}\right) = N\left(r,\infty;\frac{(F^m)'}{F^m}\right) + S(r,F) \\
\leq &\overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + S(r,F) \\
\leq &\overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) + \overline{N}(r,0;f') + \overline{N}(r,\infty;f) + S(r,f),
\end{aligned} \tag{28}$$

and

$$\overline{N}(r,1;G^m| \geq 2) \leq \overline{N}(r,0;g) + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N}(r,0;g') + \overline{N}(r,\infty;g) + S(r,g). \tag{29}$$

Suppose that F^m, G^m satisfy (i) of Lemma 2.5, then from Lemma 2.1 and (27), we can get

$$\begin{aligned}
\frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) \leq &N_2(r,0;F^m) + N_2(r,\infty;F^m) + N_2(r,0;G^m) + N_2(r,\infty;G^m) \\
&+ \overline{N}(r,1;F^m| \geq 2) + \overline{N}(r,1;G^m| \geq 2) + S(r,F) + S(r,G).
\end{aligned} \tag{30}$$

(iii_1) $2 \leq m \leq 3$. From (13),(14),(17),(18) and (28)-(30), we can get

$$\begin{aligned}
&(n+3)T(r,f) + (n+3)T(r,g)\\
\leq& T(r,F) + T(r,G)\\
\leq& \left(1+\frac{6}{m}\right)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + \left(1+\frac{2}{m}\right)N(r,a;f)\\
&+\left(1+\frac{2}{m}\right)N(r,b;f) + \left(1+\frac{2}{m}\right)N(r,0;f') + \frac{6}{m}\overline{N}(r,\infty;f)\\
&+N(r,\gamma_1;g) + N(r,\gamma_2;g) + \left(1+\frac{2}{m}\right)N(r,a;g) + \left(1+\frac{2}{m}\right)N(r,b;g)\\
&+\left(1+\frac{2}{m}\right)N(r,0;g') + \frac{6}{m}\overline{N}(r,\infty;g) + S(r,f) + S(r,g),
\end{aligned}$$

that is,

$$\left(n-4-\frac{20}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g). \tag{31}$$

Since $n > 4 + \frac{20}{m}$, we get a contradiction.
Thus, from Lemma 2.5, we can get that F^m, G^m satisfy the equality

$$F^m = \frac{(B+1)G^m + A - B - 1}{BG^m + A - B},$$

where $A(\neq 0)$ and B are constants.
From Lemma 2.6 and $n > 4 + \frac{11}{m}$, we can get $f \equiv g$.
(iii_2) $m \geq 4$. Like (i_2) and (iii_1), we can get

$$\begin{aligned}
&(n+3)T(r,f) + (n+3)T(r,g)\\
\leq& T(r,F) + T(r,G)\\
\leq& \left(1+\frac{6}{m}\right)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + \frac{2}{m}N(r,a;f) + \frac{2}{m}N(r,b;f)\\
&+\frac{2}{m}N(r,0;f') + \frac{6}{m}\overline{N}(r,\infty;f) + \left(1+\frac{6}{m}\right)N(r,0;g) + N(r,\gamma_1;g) + N(r,\gamma_2;g)\\
&+\frac{2}{m}N(r,a;g) + \frac{2}{m}N(r,b;g) + \frac{2}{m}N(r,0;g') + \frac{6}{m}\overline{N}(r,\infty;g) + S(r,f) + S(r,g),
\end{aligned}$$

that is,

$$\left(n-\frac{20}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

Since $n > 4 + \frac{4}{m}$ and $m \geq 4$, then $n - \frac{20}{m} > 4 - \frac{16}{m} \geq 0$, a contradiction.
Thus, from Lemma 2.5, we can get that F^m, G^m satisfy the equality

$$F^m = \frac{(B+1)G^m + A - B - 1}{BG^m + A - B},$$

where $A(\neq 0)$ and B are constants.
From Lemma 2.6 and $n > 4 + \frac{4}{m}$, we can get $f \equiv g$.
Thus, the proof of Theorem 1.7 is completed. □

3.2 Proof of Theorem 1.8

Proof: By Lemma 2.7 and using the same argument in Theorem 1.7, we can easily get the conclusions of Theorem 1.8. Here, the process of the proof of Theorem 1.8 is omitted. □

References

[1] A. Banerjee, Weighted sharing of a small function by a meromorphic function and its derivative, Comput. Math. Appl. 53 (2007), 1750-1761.

[2] A. Banerjee, S. Mukherjee, Nonlinear differential polynomials sharing a small function, Arch. Math. (Brno) 44 (2008), 41-56

[3] A. Banerjee, S. Mukherjee, Corrigendum to "Nonlinear differential polynomials sharing a small function" [Arch. Math. (Brno) 44(2008): 41-56], Arch. Math. (Brno) 44 (2008), 335-337.

[4] G. Frank, M. Reinders, A unique range set for meromorphic functions with 11 elements, Complex Var. Elliptic Equ. 37 (1998), no. 9, 185-193.

[5] C.-Y. Fang, M.-L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl. 44 (2002), 607-617.

[6] G.G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc. 20 (1979), 457-466.

[7] W.K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[8] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J. 161 (2001), 193-206.

[9] I. Lahiri, A. Sarkar, Nonlinear differential polynomials sharing 1-points with weight two, Chinese J. Contemp. Math. 25 (2004), no. 3, 325-334.

[10] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Var. Elliptic Equ. 46 (2001), 241-253.

[11] W. C. Lin, H. X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math. 35 (2004), no. 2, 121-132.

[12] A. Z. Mokhonko, The Navanlinna charateristic of some meromorphic functions, Funct. Anal. Appl. 14 (1971), 83-87.

[13] H. Y. Xu, T. B. Cao, Uniqueness of entire ormeromorphic functions sharing one value or a function with finiteweight, J. Inequal. Pure and Appl. Math. 10(3) (2009), Art. 88, 14 pp.

[14] H. Y. Xu, T. B. Cao, S. Liu, Uniquenessresults of meromorphic functions whose nonlinear differentialpolynomials have one nonzero pseudo value, Mat. Vesnik 64 (1) (2012), 1-16.

[15] H. Y. Xu, C. F. Yi, T. B. Cao, Uniqueness ofmeromorphic functions and differential polynomials sharing onevalue with finite weight, Ann. Polon. Math. 95 (2009), 55-66.

[16] C. C. Yang, X. H. Hua, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math. 22 (1997), 395-406.

[17] H. X. Yi, C. C. Yang, Uniqueness theory of meromorphic functions, Kluwer Academic Publishers, Dordrecht, 2003; Chinese original: Science Press, Beijing, 1995.

QUASICONFORMAL HARMONIC MAPPINGS RELATED TO STARLIKE FUNCTIONS

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Abstract

Let $f = h(z) + \overline{g(z)}$ be a univalent sense-preserving harmonic mapping of the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. If f satisfies the condition $|w(z)| = \left|\frac{g'(z)}{h'(z)}\right| < k$, $(0 \leq k < 1)$, then f is called $k-$quasiconformal harmonic mapping in $\mathbb{D}$.
The aim of this paper is to investigate a subclass of $k-$quasiconformal harmonic mappings.

1 Introduction

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.
Next, let S^* denote the family of functions $h(z) = z + c_2 z^2 + ...$ regular in $\mathbb{D}$ such that $h(z)$ is in S^* if and only if

$$z\frac{h'(z)}{h(z)} = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

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for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.
Moreover, let $s_1(z) = z + d_2z^2 + d_3z^3 + ...$ and $s_2(z) = z + e_2z^2 + e_3z^3 + ...$ be analytic functions in the open unit disc $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. Specially if $s_2(z)$ univalent in $\mathbb{D}$, then $s_1(z) \prec s_2(z)$ if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$ and $s_1(0) \subset s_2(0)$ implies $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$ where $\mathbb{D}_r = \{z \in \mathbb{C}||z| < r, 0 < r < 1\}$ (Subordination and Lindelof Principle [3]).
Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_nz^n, g(z) = \sum_{n=0}^{\infty} b_nz^n,$$

Where a_n, $b_n \in \mathbb{C}$, $n = 0, 1, 2, ...$ as usual, we call $h(z)$ the analytic part of f and $g(z)$ is co-analytic part of f. An elegant and complete account of the theory of harmonic mappings is given in Duren's monograph [1].
Lewy [4] proved in 1936 that the harmonic function f is locally univalent in $\mathbb{D}$ if and only if its Jacobien

$$J_f = |h'(z)|^2 - |g'(z)|^2$$

is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc $\mathbb{D}$ are either sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ doesn't vanish in $\mathbb{D}$ and the second dilatation $w(z) = (\frac{g'(z)}{h'(z)})$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by S_H. Thus S_H contains standard class S of univalent functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by S_H^0. Hence it is clear that $S \subset S_H^0 \subset S_H$.
For the aim of this paper we need the following lemma and theorem.

Lemma 1.1. *([6])Let $\phi(z)$ be a non-constant and analytic function in the unit disc $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.*

Theorem 1.2. *([3]) Let $h(z)$ be an element of S^*, then*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2}$$

$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}$$

A univalent harmonic mapping is called $k-$quasiconformal $(0 \leq k < 1)$ if $|w(z)| < k$. For the general definition of quasiconformal mapping see [1], [5]. The main idea of this paper is to investigate the subclass of $k-$quasiconformal harmonic mappings

$$S_{H(kq)}^{(*)} = \left\{ f = h(z) + \overline{g(z)} \in S_H \mid |w(z)| < k, 0 \leq k < 1, h(z) \in S^* \right\}. \quad (1.2)$$

2 Main Results

Theorem 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{H(kq)}^{(*)}$, then*

$$\frac{g(z)}{h(z)} \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z} \quad (2.1)$$

Proof. We consider the linear transformation

$$\left(\frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z}\right).$$

The transfromation maps $|z| < k$ onto itself. On the other hand we have,

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{(b_1 z + b_2 z^2 + ...)'}{(z + a_2 z^2 + ...)'} = \frac{b_1 + 2b_2 z + ...}{1 + 2a_2 z + ...} \Rightarrow w(0) = b_1$$

Therefore the function

$$\phi(z) = \frac{k^2(b_1 - w(z))}{M^2 - \overline{b_1}w(z)}$$

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satisfies the conditions of Schwarz Lemma, then we have

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z} \tag{2.2}$$

and the transformation

$$(\frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z})$$

maps $|z| = r$ onto the disc with the centre

$$C(r) = (\frac{k^2 Reb_1(1 - r^2)}{k^2 - |b_1|^2 r^2}, \frac{k^2 Imb_1(1 - r^2)}{k^2 - |b_1|^2 r^2})$$

and the radius

$$\rho(r) = \frac{k(k^2 - |b_1|^2)r}{k^2 - |b_1|^2 r^2}$$

then we can write

$$W(D_r) = \left\{ z \mid \left| w(z) - \frac{k^2(1 - r^2)b_1}{k^2 - |b_1|^2 r^2} \right| \leq \frac{k(k^2 - |b_1|^2)r}{k^2 - |b_1|^2 r^2} \right\}. \tag{2.3}$$

Now we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} \tag{2.4}$$

Then $\phi(z)$ is analytic and

$$\phi(0) = \frac{0}{|b_1|^2 - k^2} = 0.$$

If we take the derivative of (2.4) and after the brief calculations we get

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} + \frac{k^2(|b_1|^2 + k^2 - 2b_1\phi(z))z\phi(z)}{(k^2 - b_1\phi(z))^2} . \frac{1 - \phi(z)}{1 + \phi(z)} \tag{2.5}$$

Now it is easy to realize that the subordination (2.1) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, we assume the contrary; then there is a $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. So by I. S. Jack's Lemma

$$z_1\phi'(z_1) = m\phi(z_1)$$

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for some $m \geq 1$ and for such z_1 we have

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = \frac{k^2(b_1 - \phi(z_1))}{k^2 - \overline{b_1}\phi(z_1)} + \frac{k^2(|b_1|^2 + k^2 - 2b_1\phi(z_1))z\phi(z_1)}{(k^2 - b_1\phi(z))^2}.\frac{1-\phi(z_1)}{1+\phi(z_1)}$$
$$= w(\phi(z_1)) \in W(\mathbb{D})$$

but this contradicts to (2.1); so our assumption is wrong, i. e, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. □

Corollary 2.2. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^{(*)}_{H(kq)}$, then*

$$rF(k, |b_1|, -r) \leq |g(z)| \leq rF(k, |b_1|, r) \tag{2.6}$$

$$G(k, |b_1|, -r) \leq |g'(z)| \leq G(k, |b_1|, r) \tag{2.7}$$

where

$$F(k, |b_1|, r) = \frac{1}{(1-r^2)}\frac{k(|b_1| + kr)}{k + |b_1|r}$$

$$G(k, |b_1|, r) = \frac{1+r}{(1-r)^3}\frac{k(|b_1| + kr)}{k + |b_1|r}$$

Proof. These inequalities is a simple consequence of Theorem 2.1 and the defination of $S^{(*)}_{H(kq)}$. □

Corollary 2.3. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^{(*)}_{H(kq)}$, then*

$$\frac{(1-r)^2}{(1+r)^6}F_2(k, |b_1|, r) \leq J_f \leq \frac{(1+r)^2}{(1-r)^6}F_1(k, |b_1|, r) \tag{2.8}$$

where

$$F_1(k, |b_1|, r) = \frac{[(k + k|b_1|) - (|b_1| + k^2)r][(k - k|b_1|) - (|b_1| - k^2)r]}{(k - |b_1|r)^2}$$

$$F_2(k, |b_1|, r) = \frac{[(k + k|b_1|) + (|b_1| + k^2)r][(k - k|b_1|) + (|b_1| - k^2)r]}{(k + |b_1|r)^2}$$

Proof. Using (2.3) we obtain

$$F_2(k, |b_1|, r) \leq (1 - |w(z)|^2) \leq F_1(k, |b_1|, r) \tag{2.9}$$

On the other hand we have

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 - |h'(z)|^2|w(z)|^2 = |h'(z)|^2(1 - |w(z)|^2) \tag{2.10}$$

Considering (2.9) and (2.10) with the Theorem 2.1 we obtain (2.8). □

Corollary 2.4. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^{(*)}_{H(kq)}$, then*

$$\int_0^r \frac{1-\rho}{(1+\rho)^3}\frac{k(1-|b_1|)+(|b_1|+k^2)\rho}{k+|b_1|\rho}d\rho \leq |f| \leq \int_0^r \frac{1+\rho}{(1-\rho)^3}\frac{k(1+|b_1|)+(|b_1|+k^2)\rho}{k+|b_1|\rho}d\rho \tag{2.11}$$

Proof. Using (2.3) we get

$$\frac{(k-k|b_1|)+(|b_1|-k^2)r}{(k+|b_1|r)} \leq (1-|w(z)|) \leq \frac{(k-k|b_1|)-(|b_1|-k^2)r}{(k-|b_1|r)} \tag{2.12}$$

$$\frac{(k+k|b_1|)-(|b_1|+k^2)r}{(k-|b_1|r)} \leq (1+|w(z)|) \leq \frac{(k+k|b_1|)+(|b_1|+k^2)r}{(k+|b_1|r)} \tag{2.13}$$

On the other hand

$$(|h'(z)|-|g'(z)|)|dz| \leq |df| \leq (|h'(z)|+|g'(z)|)|dz| \Rightarrow$$

$$|h'(z)|(1-|w(z)|)|dz| \leq |df| \leq |h'(z)|(1+|w(z)|)|dz| \tag{2.14}$$

Using (2.12), (2.13) and Theorem 1.2 in the inequality (2.14) and integrating we obtain (2.11). □

Theorem 2.5. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S^{(*)}_{H(kq)}$, then*

$$\sum_{m=2}^{n} k^4 |b_m - b_1 a_m|^2 \leq (|b_1|^2 - k^2)^2 + \sum_{m=2}^{n} \left|\overline{b_1} b_m - k^2 a_m\right|^2 \tag{2.15}$$

Proof. Using Theorem 2.1, then we can write

$$\frac{g(z)}{h(z)} = \frac{k^2(b_1-\phi(z))}{k^2-\overline{b_1}\phi(z)} \Leftrightarrow (k^2 g(z) - k^2 b_1 h(z)) = (\overline{b_1} g(z) - k^2 h(z))\phi(z)$$

Therefore we have

$$\sum_{m=2}^{n}(k^2 b_m - k^2 b_1 a_m)z^m + \sum_{m=n+1}^{\infty} d_m z^m = ((|b_1|^2-k^2)z + \sum_{m=2}^{\infty}(\overline{b_1}b_m - k^2 a_m)z^m)\phi(z) \tag{2.16}$$

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Where the coefficients d_m have been chosen suitably. The equality (2.16) can be written in the

$$F(z) = G(z)\phi(z), |\phi(z)| < 1,$$

then we have

$$|F(z)|^2 = |G(z)\phi(z)|^2 = |G(z)|^2 |\phi(z)|^2 \Rightarrow |F(z)|^2 \leq |G(z)|^2 \Rightarrow$$

$$\left|\sum_{m=2}^{n}(k^2 b_m - k^2 b_1 a_m)z^m + \sum_{m=n+1}^{\infty} d_m z^m\right|^2 \leq \left|(|b_1|^2 - k^2)z + \sum_{m=2}^{n}(\overline{b_1} b_m - k^2 a_m)z^m\right|^2$$

Assume $z = re^{i\theta}$, $0 < r < 1$, $0 \leq \theta$ and integrate the resulting inequality in the interval $[0, 2\pi]$. Then we find the inequality

$$\sum_{m=2}^{n} k^4 |b_m - b_1 a_m|^2 r^{2k} + \sum_{m=n+1}^{\infty} |d_m|^2 r^{2k} \leq (|b_1|^2 - k^2)^2 r^{2k} + \sum_{m=2}^{n} \left|\overline{b_1} b_m - k^2 a_m\right|^2 r^{2k}.$$

Hence we get

$$\sum_{m=2}^{n} k^4 |b_m - b_1 a_m|^2 r^{2k} \leq (|b_1|^2 - k^2)^2 r^{2k} + \sum_{m=2}^{n} \left|\overline{b_1} b_m - k^2 a_m\right|^2 r^{2k}.$$

passing to the limit as $r \to 1$ we obtain (2.15). The proof of this method was introduced by J. Clunie [2]. □

References

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nastrand Princeton, N. J, (1966).

[2] J. Clunie, *On Meromorphic Schicht Functions*, J. London Math. Soc. 34 (1959), 215-216.

[3] P. Duren, *Univalent Functions*, Springer Verlag, (1983).

[4] P. Duren, *Harmonic Mappings in the Plane*, Vol. 156 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge UK, (2004).

[5] D. Kolaj, *Quasiconformal Harmonic Mappings and Close-to-Convex Domains*, Filomat 24:1, (2010), 63-68.

[6] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2), 3, (1971), no. 2, $469 - 474$.

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On the superstability of ternary Jordan C^*-homomorphisms

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Abstract. In this paper, we prove the superstability of ternary Jordan C^*-homomorphisms associated with the following Cauchy-Jensen functional equation:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2\left[f(x)+f(y)+f(z)\right] \tag{1}$$

and the Hyers-Ulam stability of ternary Jordan C^*-homomorphisms associated with the following generalized Cauchy-Jensen functional equation:

$$\sum_{i=1}^{p} f\left(\frac{1}{p-1}\sum_{\substack{j=1\\ j\neq i}}^{p} x_j + x_i\right) = 2\sum_{i=1}^{p} f(x_i). \tag{2}$$

Keywords: Superstability; Hyers-Ulam stability; Cauchy-Jensen functional equation; Ternary Jordan C^*-homomorphism.

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1. Introduction and preliminaries

It is clear that the functional equation (2) is a generalized form of the functional equation (1). In order to investigate of the functional equation (2), we will suppose that $p \geq 2$, and so the simplest case of (2) is the Cauchy equation with $p = 2$.

Definition 1.1. [25] Let A, B be C^*-ternary algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary Jordan homomorphism* if

$$H([x,x,x]) = [H(x), H(x), H(x)]$$

for all $x \in A$. If, in addition,

$$H(x^*) = H(x)^*$$

for all $x \in A$, then H is called a *ternary Jordan C^*-homomorphism*.

Definition 1.2. [25] Let A be a C^*-ternary algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary Jordan derivation* if

$$\delta([x,x,x]) = [\delta(x), x, x] + [x, \delta(x), x] + [x, x, \delta(x)]$$

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for all $x \in A$. If, in addition,

$$\delta(x^*) = \delta(x)^*$$

for all $x \in A$, then δ is called a *ternary Jordan C^*-derivation.*

In this paper, we will just obtain our results for ternary Jordan C^*-homomorphisms, and the reader can also investigate and get the results for ternary Jordan C^*-derivations similarly.

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). We say that a functional equation is *superstable* if every approximately solution is an exact solution of it [33].

The stability of functional equations originated from Ulam [37] in 1940. Ulam proposed the following question "When does an exact solution of functional equation (ξ), near an approximately solution of that exist?" In 1941, Hyers [18] affirmatively answered to this question of Ulam for Banach spaces. In 1950, Aoki [1] generalized the Hyers' theorem for approximately additive mappings. In 1978, Rassias [32] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. In 1994, a generalization of Rassias' theorem was obtained by Ğavruţa [17]. He proved the following:

Theorem 1.3. [17] *Let G be an abelian group and E a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a mapping such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. Suppose $f : G \to E$ is a mapping satisfying

$$\| f(x + y) - f(x) - f(y) \| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $A : G \to E$ such that

$$\| f(x) - A(x) \| \leq \phi(x, x)$$

for all $x \in G$.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning these problems. A list of references concerning these results can be found in [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 24, 26, 27, 28, 29, 31, 33, 35, 36].

2. Superstability of ternary Jordan C^*-homomorphisms

Throughout this section, we prove the superstability of ternary Jordan C^*-homomorphisms associated with the functional equation (1).

From now on, A and B are C^*-ternary algebras with norm $\| \cdot \|_A$ and $\| \cdot \|_B$ respectively.

We will use the following lemmas in the proof of our theorems.

Lemma 2.1. [22] *Let X and Y be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in X$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. [22] *Let $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}_n$ is a convergent sequence in A.*

Lemma 2.3. *Let $f : A \to B$ be a mapping such that*

$$\left\|2[f(x)+f(y)+f(z)] - f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x+z}{2}+y\right)\right\|_B \le \left\|f\left(\frac{y+z}{2}+x\right)\right\|_B \tag{2.1}$$

for all $x, y, z \in A$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|4f(0)\|_B \le \|f(0)\|_B.$$

So $f(0) = 0$. Letting $y = z = -x$ in (2.1), we get

$$\|2f(x) + 2f(-x)\|_B \le \|f(0)\|_B = 0$$

for all $x \in A$. Hence $f(-x) = -f(x)$ for all $x \in A$. Letting $z = -2x - y$ in (2.1), we obtain

$$2f(x) + 2f(y) - 2f(2x+y) + f\left(\frac{3x+y}{2}\right) - f\left(\frac{y-x}{2}\right) = 0 \tag{2.2}$$

for all $x, y \in A$. Letting $x = \frac{-y-z}{2}$ in (2.1), we get

$$-2f\left(\frac{y+z}{2}\right) + 2f(y) + 2f(z) - f\left(\frac{3z+y}{4}\right) - f\left(\frac{3y+z}{4}\right) = 0$$

and so

$$-2f\left(\frac{x+y}{2}\right) + 2f(x) + 2f(y) - f\left(\frac{3y+x}{4}\right) - f\left(\frac{3x+y}{4}\right) = 0 \tag{2.3}$$

for all $x, y \in A$. Now by (2.2) and (2.3), we get

$$2f(2x+y) + f\left(\frac{y-x}{2}\right) - 2f\left(\frac{x+y}{2}\right) - f\left(\frac{3x+y}{2}\right) - f\left(\frac{3x+y}{4}\right) - f\left(\frac{3y+x}{4}\right) = 0$$

for all $x, y \in A$. Letting $y = 0$ in the top line, we obtain

$$f(x) + 3f(2x) + f(3x) + f(6x) - 2f(8x) = 0 \tag{2.4}$$

for all $x \in A$.

Letting $y = -3x,\ z = x$ in (2.1), and then $y = 0,\ z = -2x$ and then $y = 3x,\ z = -5x$ and then $y = -\frac{3}{2}x,\ z = -\frac{1}{2}x$ and then $y = 2x,\ z = -4x$ and then $y = -\frac{2}{3}x,\ z = -\frac{4}{3}x$, respectively, we get

$$\begin{aligned}
&4f(x) + f(2x) - 2f(3x) = 0,\\
&f(x) + 2f(2x) + f(3x) - 2f(4x) = 0,\\
&f(x) + 3f(3x) - 2f(5x) = 0,\\
&2f(2x) - f(3x) - 2f(4x) - f(5x) + 2f(6x) = 0,\\
&f(x) - 2f(2x) - 2f(4x) - f(5x) + 2f(8x) = 0,\\
&2f(4x) - f(5x) - 2f(6x) - f(7x) + 2f(8x) = 0
\end{aligned}$$

for all $x \in A$. By these equations and by (2.4), we obtain $f(nx) = nf(x)$ with $n = 1, \cdots, 8$. So by (2.2), we have

$$f(2x+y) = f(x) + f(y) - \frac{1}{4}f(y-x) + \frac{1}{4}f(3x+y) \tag{2.5}$$

for all $x, y \in A$. Letting $x = u+v,\ y = u-v,\ z = -3u-v$ in (2.1), we get

$$f(v) + 2f(u+v) - 2f(v-u) + f(2u+v) - 2f(3u+v) = 0$$

and so

$$f(2x+y) = 2f(3x+y) + 2f(y-x) - f(y) - 2f(x+y)$$

for all $x, y \in A$. By this equation and (2.5), we obtain that

$$f(3x+y) = \frac{4}{7}f(x) + \frac{8}{7}f(y) + \frac{8}{7}f(x+y) - \frac{9}{7}f(y-x)$$

and so

$$f(3y+x) = \frac{4}{7}f(y) + \frac{8}{7}f(x) + \frac{8}{7}f(x+y) + \frac{9}{7}f(y-x)$$

for all $x, y \in A$. By (2.3), we can obtain the result. □

Theorem 2.4. *Let $\varphi : A^3 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 2^{3n}\varphi(2^{-n}x, 2^{-n}x, 2^{-n}x) = 0$$

or

$$\lim_{n\to\infty} 2^{-n}\varphi(2^{n}x, 2^{n}x, 2^{n}x) = 0$$

for all $x \in A$. Let $f : A \to B$ be a mapping satisfying

$$\begin{aligned} &\left\|2\mu[f(x)+f(y)+f(z)] - f\left(\mu\frac{x+y}{2} + \mu z\right) - f\left(\mu\frac{x+z}{2} + \mu y\right)\right\|_B \\ &\quad \le \left\|f\left(\mu\frac{y+z}{2} + \mu x\right)\right\|_B, \end{aligned} \tag{2.6}$$

$$\|f([x,x,x]) - [f(x), f(x), f(x)]\|_B \le \varphi(x,x,x), \tag{2.7}$$

$$\|f(x^*) - f(x)^*\|_B \le \varphi(x^*, x^*, x^*) \tag{2.8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : A \to B$ is a ternary Jordan C^*-homomorphism.

Proof. Assume that $\lim_{n\to\infty} 2^{3n}\varphi(2^{-n}x, 2^{-n}x, 2^{-n}x) = 0$.

Let $\mu = 1$ in (2.6). By Lemma 2.3, the mapping $f : A \to B$ is additive. Letting $y = z = -x$ in (2.6), we get

$$\|-2\mu f(x) + 2f(\mu x)\|_B = \|2\mu f(x) + 4\mu f(-x) - 2f(-\mu x)\|_B \le \|f(0)\|_B = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By Lemma 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear. By (2.7) and Lemma 2.2, we have

$$\begin{aligned} &\|f([x,x,x]) - [f(x), f(x), f(x)]\|_B \\ &= \lim_{n\to\infty} 2^{3n}\left\|f\left(\left[\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right]\right\|_B \\ &\le \lim_{n\to\infty} 2^{3n}\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence $f([x,x,x]) = [f(x), f(x), f(x)]$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

It follows from (2.8) that

$$\begin{aligned} \|f(x^*) - f(x)^*\|_B &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x}{2^n}^*\right) - f\left(\frac{x}{2^n}\right)^*\right\|_B \\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x^*}{2^n}, \frac{x^*}{2^n}, \frac{x^*}{2^n}\right) \le \lim_{n\to\infty} 2^{3n}\varphi\left(\frac{x^*}{2^n}, \frac{x^*}{2^n}, \frac{x^*}{2^n}\right) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence $f(x^*) = f(x)^*$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Therefore, the mapping $f : A \to B$ is a ternary Jordan C^*-homomorphism.

Assume that $\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n x, 2^n x) = 0$.

By (2.7), (2.8) and Lemma 2.2, we have

$$\begin{aligned}
&\|f([x,x,x]) - [f(x), f(x), f(x)]\|_B \\
&= \lim_{n\to\infty} \frac{1}{2^{3n}} \|f([2^n x, 2^n x, 2^n x]) - [f(2^n x), f(2^n x), f(2^n x)]\|_B \\
&\le \lim_{n\to\infty} \frac{1}{2^{3n}} \varphi(2^n x, 2^n x, 2^n x) \le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n x, 2^n x) = 0, \\
&\|f(x^*) - f(x)^*\|_B = \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n x^*) - f(2^n x)^*\|_B \\
&\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x^*, 2^n x^*, 2^n x^*) = 0
\end{aligned}$$

for all $x \in A$.

Therefore, the mapping $f : A \to B$ is a ternary Jordan C^*-homomorphism. $\square$

Corollary 2.5. *Let θ be a nonnegative real number and q_1, q_2, q_3 be positive real numbers such that $q_1, q_2, q_3 > 3$ or $q_1, q_2, q_3 < 1$. Let $f : A \to B$ be a mapping satisfying (2.6) and*

$$\begin{aligned}
\|f([x,x,x]) - [f(x), f(x), f(x)]\|_B &\le \theta\left(\|x\|_A^{q_1} + \|x\|_A^{q_2} + \|x\|_A^{q_3}\right), \\
\|f(x^*) - f(x)^*\|_B &\le \theta\left(\|x^*\|_A^{q_1} + \|x^*\|_A^{q_2} + \|x^*\|_A^{q_3}\right)
\end{aligned}$$

for all $x \in A$. Then the mapping $f : A \to B$ is a ternary Jordan C^*-homomorphism.

Proof. Defining $\varphi(x, y, z) = \theta\left(\|x\|_A^{q_1} + \|y\|_A^{q_2} + \|z\|_A^{q_3}\right)$ and applying Theorem 2.4, we get the result. $\square$

Corollary 2.6. *Let θ be a nonnegative real number and q_1, q_2, q_3 be positive real numbers such that $q_1 + q_2 + q_3 > 3$ or $q_1 + q_2 + q_3 < 1$. Let $f : A \to B$ be a mapping satisfying (2.6) and*

$$\begin{aligned}
\|f([x,x,x]) - [f(x), f(x), f(x)]\|_B &\le \theta\|x\|_A^{q_1+q_2+q_3}, \\
\|f(x^*) - f(x)^*\|_B &\le \theta\|x^*\|_A^{q_1+q_2+q_3}
\end{aligned}$$

for all $x \in A$. Then the mapping $f : A \to B$ is a ternary Jordan C^*-homomorphism.

Proof. Defining $\varphi(x, y, z) = \theta\left(\|x\|_A^{q_1}\|y\|_A^{q_2}\|z\|_A^{q_3}\right)$ and applying Theorem 2.4, we get the result. $\square$

We can also put $q_1 = q_2 = q_3 = q$ in Corollaries 2.5 and 2.6, and obtain better and simpler results.

3. Hyers-Ulam stability of ternary Jordan C^*-homomorphisms

Throughout this section, We prove the Hyers-Ulam stability of ternary Jordan C^*-homomorphisms associated with the functional equation (2).

In order to do that, for a given mapping $f : A \to B$, we define

$$\Gamma_\mu f(x_1, \cdots, x_p) := \sum_{i=1}^{p} f\left(\frac{1}{p-1}\sum_{\substack{j=1\\ j\ne i}}^{p} \mu x_j + \mu x_i\right) - 2\sum_{i=1}^{p} \mu f(x_i)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$.

Lemma 3.1. [25] *A mapping $f : A \to B$ is a $\mathbb{C}$-linear mapping if and only if*

$$\Gamma_\mu f(x_1, \cdots, x_p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$.

Theorem 3.2. *Let $\varphi : A^p \to [0, \infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} 2^n \varphi\left(2^{-(n+1)}x_1, \cdots, 2^{-(n+1)}x_p\right) < \infty, \tag{3.1}$$

$$\lim_{n\to\infty} 2^{3n} \varphi\left(2^{-n}x, \cdots, 2^{-n}x\right) = 0, \tag{3.2}$$

or

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi(2^n x_1, \cdots, 2^n x_p) < \infty \tag{3.3}$$

for all $x, x_1, \cdots, x_p \in A$. Suppose that $f : A \to B$ is a mapping satisfying

$$\begin{aligned} \| \Gamma_\mu f(x_1, \cdots, x_p) \|_B &\leq \varphi(x_1, \cdots, x_p), &(3.4)\\ \|f([x, x, x]) - [f(x), f(x), f(x)]\|_B &\leq \varphi(x, \cdots, x), &(3.5)\\ \|f(x^*) - f(x)^*\|_B &\leq \varphi(x^*, \cdots, x^*) &(3.6) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_p \in A$. Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{p}\phi(x, \cdots, x) \tag{3.7}$$

for all $x \in A$.

Proof. Firstly, assume that (3.1) and (3.2) hold.

Putting $\mu = 1$ and $x_1 = \cdots = x_p = x$ in (3.4), we get

$$\begin{aligned} \|f(2x) - 2f(x)\|_B &\leq \frac{1}{p}\varphi(x, \cdots, x)\\ \left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B &\leq \frac{1}{p}\varphi\left(\frac{x}{2}, \cdots, \frac{x}{2}\right) \end{aligned}$$

for all $x \in A$. Using the induction method, we obtain

$$\left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\|_B \leq \frac{1}{p} \sum_{s=0}^{n-1} 2^s \varphi\left(2^{-(s+1)}x, \cdots, 2^{-(s+1)}x\right) \tag{3.8}$$

for each $n \geq 1$ and all $x \in A$. Now assume that m, l are positive integers, with $m > l$.

By (3.8), we have

$$\begin{aligned} \left\|2^m f\left(\frac{x}{2^m}\right) - 2^l f\left(\frac{x}{2^l}\right)\right\|_B &= 2^l \left\|2^{m-l} f\left(\frac{1}{2^{m-l}}\ \frac{x}{2^l}\right) - f\left(\frac{x}{2^l}\right)\right\|_B\\ &\leq \frac{1}{p} \sum_{s=l}^{m-1} 2^s \varphi\left(2^{-(s+1)}x, \cdots, 2^{-(s+1)}x\right)\\ &\leq \frac{1}{p} \sum_{s=l}^{\infty} 2^s \varphi\left(2^{-(s+1)}x, \cdots, 2^{-(s+1)}x\right). \end{aligned}$$

Now, the relation (3.1) shows that the right side converges to 0 when $l \to \infty$, and this clarifies that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence. Since A is a complete space, the sequence $\{2^n f(\frac{x}{2^n})\}$ is a convergent sequence. Therefore, we can define, for all $x \in A$, the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Passing the limit $n \to \infty$ in (3.8) and by (3.1), we obtain (3.7).

It follows from (3.4) and (3.1) that

$$\begin{aligned} \|\Gamma_\mu H(x_1, \cdots, x_p)\|_B &= \lim_{n\to\infty} 2^n \left\|\Gamma_\mu f\left(\frac{x_1}{2^n}, \cdots, \frac{x_p}{2^n}\right)\right\|_B \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x_1}{2^n}, \cdots, \frac{x_p}{2^n}\right) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$. By Lemma 3.1, H is $\mathbb{C}$-linear.

By Lemma 2.2 and replacing x by $\frac{x}{2^n}$ in (3.5) and by (3.2), we obtain

$$\begin{aligned} &\|H([x,x,x]) - [H(x), H(x), H(x)]\|_B \\ &= \lim_{n\to\infty} 2^{3n} \left\| f\left(\left[\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right]\right\|_B \\ &\leq \lim_{n\to\infty} 2^{3n} \varphi\left(\frac{x}{2^n}, \cdots, \frac{x}{2^n}\right) = 0 \end{aligned}$$

for all $x \in A$. Thus $H([x,x,x]) = [H(x), H(x), H(x)]$ for all $x \in A$.

By (3.1) and replacing x by $\frac{x}{2^n}$ in (3.6), we get

$$\begin{aligned} \|H(x^*) - H(x)^*\|_B &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^*\right\|_B \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x^*}{2^n}, \cdots, \frac{x^*}{2^n}\right) = 0 \end{aligned}$$

for all $x \in A$. Thus $H(x^*) = H(x)^*$ for all $x \in A$. Therefore, $H : A \to B$ is a ternary Jordan C^*-homomorphism.

Let $T : A \to B$ be another ternary Jordan C^*-homomorphism that satisfies (3.7). Then we have

$$\begin{aligned} \|H(x) - T(x)\|_B &\leq 2^n \left\|f\left(\frac{x}{2^n}\right) - H\left(\frac{x}{2^n}\right)\right\|_B + 2^n \left\|f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_B \\ &\leq 2^n \left(\frac{2}{p}\phi\left(\frac{x}{2^n}, \cdots, \frac{x}{2^n}\right)\right) \\ &\leq \frac{2^{n+1}}{p} \sum_{s=0}^{\infty} 2^s \varphi\left(2^{-(s+1)}\frac{x}{2^n}, \cdots, 2^{-(s+1)}\frac{x}{2^n}\right) \\ &= \frac{2}{p} \sum_{s=n}^{\infty} 2^s \varphi\left(2^{-(s+1)}x, \cdots, 2^{-(s+1)}x\right) \end{aligned}$$

for all $x \in A$. Now if $n \to \infty$, then (3.1) shows that the right side converges to 0. So $H : A \to B$ is unique.

If we assume that (3.3) holds, then by the same method as in the proof of last part, one can obtain a $\mathbb{C}$-linear mapping $H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ satisfying (3.7), and get the desired result. □

Corollary 3.3. *Let θ be a nonnegative real number and, for every $1 \le j \le p$, q_j be positive real numbers such that all $q_j > 3$ or all $q_j < 1$, and let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned} \|\Gamma_\mu f(x_1, \cdots, x_p)\|_B &\le \theta(\|x_1\|_A^{q_1} + \cdots + \|x_p\|_A^{q_p}), \\ \|f([x, x, x]) - [f(x), f(x), f(x)]\|_B &\le \theta(\|x\|_A^{q_1} + \cdots + \|x\|_A^{q_p}), \\ \|f(x^*) - f(x)^*\|_B &\le \theta(\|x^*\|_A^{q_1} + \cdots + \|x^*\|_A^{q_p}) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_p \in A$. Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \sum_{j=1}^{p} \frac{\theta\|x\|_A^{q_j}}{p\,|2^{q_j} - 2|}$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1, \cdots, x_p) = \theta \sum_{j=1}^{p} \|x_j\|_A^{q_j}$$

and applying Theorem 3.2, we get the result. □

Corollary 3.4. *Let θ be a nonnegative real number and, for every $1 \le j \le p$, q_j be positive real numbers such that $q_1 + \cdots + q_p > 3$ or $q_1 + \cdots + q_p < 1$, and let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned} \|\Gamma_\mu f(x_1, \cdots, x_p)\|_B &\le \theta(\|x_1\|_A^{q_1} \cdots \|x_p\|_A^{q_p}), \\ \|f([x, x, x]) - [f(x), f(x), f(x)]\|_B &\le \theta\|x\|_A^{\sum_{j=1}^{p} q_j}, \\ \|f(x^*) - f(x)^*\|_B &\le \theta\|x^*\|_A^{\sum_{j=1}^{p} q_j} \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_p \in A$. Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{\theta\|x\|_A^{\sum_{j=1}^{p} q_j}}{p\left|2^{\sum_{j=1}^{p} q_j} - 2\right|}$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1, \cdots, x_p) = \theta \prod_{j=1}^{p} \|x_j\|_A^{q_j}$$

and applying Theorem 3.2, we get the result. □

One can also put $q_1 = \cdots = q_p = q$ in these last corollaries, and obtain better and simpler results.

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach space*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] I. Chang, *Stability of higher ring derivations in fuzzy Banach algebras,* J. Computat. Anal. Appl. **14** (2012), 1059–1066.

[3] I. Cho, D. Kang, H. Koh, *Stability problems of cubic mappings with the fixed point alternative,* J. Computat. Anal. Appl. **14** (2012), 132–142.

[4] P. Czerwik, *Functional Equations and Inequalities in Several Variables*, Word Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.

[5] A. Ebadian, N. Ghobadipour, M. Eshaghi Gordji, *A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras,* J. Math. Phys. **51**, No. 10, Art. ID 103508, 10 pages (2010).

[6] A. Ebadian, I. Nikoufar, M. Eshaghi Gordji, *Nearly $(\theta_1, \theta_2, \theta_3, \phi)$-derivations on C^*-modules,* Int. J. Geom. Methods Mod. Phys. **9**, No. 3, Art. ID 1250019, 12 pages (2012).

[7] M. Eshaghi Gordji, M. Bavand Savadkouhi, M. Bidkham, *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **12** (2010), 454–462.

[8] M. Eshaghi Gordji, A. Bodaghi, *On the stability of quadratic double centralizers on Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 724–729.

[9] M. Eshaghi Gordji, R. Farokhzad Rostami, S.A.R. Hosseinioun, *Nearly higher derivations in unital C^*-algebras,* J. Computat. Anal. Appl. **13** (2011), 734–742.

[10] M. Eshaghi Gordji, A. Fazeli, C. Park, *3-Lie multipliers on Banach 3-Lie algebras,* Int. J. Geom. Methods Mod. Phys. (to appear).

[11] M. Eshaghi Gordji, M.B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, A. Ebadian, *On the stability of J^*-derivations,* J. Geom. Phys. **60** (2010), 454–459.

[12] M. Eshaghi Gordji, M.B. Ghaemi, B. Alizadeh, *A fixed point method for perturbation of higher ring derivationsin non-Archimedean Banach algebras,* Int. J. Geom. Methods Mod. Phys. **8** (2011), 1611–1625.

[13] M. Eshaghi Gordji, N. Ghobadipour, *Stability of (α, β, γ)-derivations on Lie C^*-algebras,* Int. J. Geom. Methods Mod. Phys. **7** (2010), 1097–1102.

[14] M. Eshaghi Gordji, S. Kaboli Gharetapeh, T. Karimi, E. Rashidi, M. Aghaei, *Ternary Jordan derivations on C^*-ternary algebras,* J. Computat. Anal. Appl. **12** (2010), 463–470.

[15] M. Eshaghi Gordji, A. Najati, *Approximately J^*-homomorphisms: a fixed point approach,* J. Geom. Phys. **60** (2010), 809–814.

[16] Z. Gajda, *On stability of the additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.

[17] P. Ğavruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive maooings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[18] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[19] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[20] H.A. Kenary, J. Lee, C. Park, *Non-Archimedean stability of an AQQ-functional equation,* J. Computat. Anal. Appl. **14** (2012), 211–227.

[21] M.S. Moslehian, *Almost derivations on C^*-ternary rings*, Bull. Belgian Math. Soc–Simon Stevin **14** (2007), 135–142.

[22] A. Najati, C. Park, J. Lee, *Homomorphisms and derivations in C^*-ternary algebras*, Abs. Appl. Anal. **2009**, Art. ID 612392, 16 pages (2009).

[23] C. Park, *Isomorphisms between C^*-ternary algebras*, J. Math. Phys. **47**, Art. ID 103512, 12 pages (2006).

[24] C. Park, Y. Cho, H.A. Kenary, *Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces,* J. Computat. Anal. Appl. **14** (2012), 526–535.

[25] C. Park, S. Farhadabadi *Ternary Jordan C^*-homomorphisms and ternary Jordan C^*-derivations for a generalized Cauchy-Jensen functional equation*, (preprint).

[26] C. Park, S. Jang, R. Saadati, *Fuzzy approximate of homomorphisms,* J. Computat. Anal. Appl. **14** (2012), 833–841.

[27] C. Park, J. Lee, D. Shin, *Stability of J^*-derivations,* Int. J. Geom. Meth. Mod. Phys. **9**, No. 5, Art. ID 1220009, 10 pages (2012).

[28] C. Park, A. Najati, *Homomorphisms and derivations in C^*-algebras*, Abs. Appl. Anal. **2007**, Art. ID 80630, 12 pages (2007).

[29] C. Park, W. Park, *On the Jensen's equation in Banach modules*, Taiwanese J. Math. **6** (2002), 523–531.

[30] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.

[31] J.M. Rassias, *Solution of the Ulam stability problem for the quartic mapping*, Glasnik Math. **34(54)** (1999), 243–252.

[32] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[33] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[34] Th.M. Rassias, P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[35] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivations on ternary Banach algebras,* J. Computat. Anal. Appl. **13** (2011), 1097–1105.

[36] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Nearly ternary cubic homomorphism in ternary Fréchet algebras*, J. Computat. Anal. Appl. **13** (2011), 1106–1114.

[37] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, USA, 1960.

[38] H. Zettl, *A Chararcterization of ternary rings of operators*, Adv. Math. **48** (1983), 117–143.

Coupled fixed points for generalized weakly contractive mappings in partial metric spaces

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Abstract

In this paper, we establish coupled fixed point results for generalized weakly contractive mappings having the mixed monotone property in ordered partial metric spaces. The results on fixed point theorems are generalizations of the recent results of Alsulami, Hussain and Alotaibi [S. Alsulami, N. Hussain and A. Alotaibi, Coupled Fixed and Coincidence Points for Monotone Operators in Partial Metric Spaces, Fixed Point Theory and Applications 2012, 2012:173].

Keywords: Coupled fixed point; Partial metric space; Generalized weakly contractive mapping; Coupled coincidence point

1. Introduction and Preliminaries

The existence and uniqueness of fixed and common fixed point theorems of operators has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922. Many authors generalized the Banach contraction principle in various spaces such as quasi-metric spaces, generalized metric spaces, cone metric spaces and fuzzy metric spaces. Matthews [2] introduced the notion of partial metric spaces in such a way that each object does not necessarily have to have a zero distance from itself and proved a modified version of the Banach contraction principle. Afterwards, many authors proved many existing fixed point theorems in partial metric spaces (see [5]-[44] for examples).

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. Some of these works are noted in [11, 17, 18, 36]. Bhaskar and Lakshmikantham [18] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. After the publication of this work, several coupled fixed point and coincidence

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point results have appeared in the recent literature. Works noted in [31, 38, 39] are some examples of these works.

We recall below the definition of partial metric space and some of its properties.

Definition 1.1. [2] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}_0^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p1) and (p2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. The function $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$ defines a partial metric on $\mathbb{R}^+$. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If p is a partial metric on X, then the function $d_p : X \times X \longrightarrow \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 1.2. Let (X,p) be a partial metric space. Then

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) **converges** to a point $x \in X$ if and only if $p(x,x) = \lim_{n\longrightarrow\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a **Cauchy sequence** iff $\lim_{n,m\longrightarrow\infty} p(x_n,x_m)$ exists (and is finite).
(iii) A partial metric space (X,p) is said to be **complete** if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\longrightarrow\infty} p(x_n,x_m)$.
(iv) A subset A of a partial metric space (X,p) is **closed** if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 1.3. The limit in a partial metric space is not unique.

Theorem 1.4. *Let (Y,d') be a subspace of metric space (X,d). If (X,d) is a complete metric space and Y is a closed set in X, then (Y,d') is a complete metric space.*

Lemma 1.5. [2, 33] Let (X,p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).

(b) A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Furthermore, $\lim_{n\longrightarrow\infty} d_p(x_n,x)=0$ if and only if

$$p(x,x)=\lim_{n\longrightarrow\infty} p(x_n,x)=\lim_{n,m\longrightarrow\infty} p(x_n,x_m).$$

Let (X,p) be a partial metric. We endow the product space $X\times X$ with the partial metric q defined as follows:

$$\text{for } (x,y),(u,v)\in X\times X,\quad q((x,y),(u,v))=p(x,u)+p(y,v).$$

A mapping $F:X\times X\to X$ is said to be continuous at $(x,y)\in X\times X$ is for each $\varepsilon>0$, there exists $\delta>0$ such that

$$F(B_q((x,y),\delta))\subset B_p((x,y),\varepsilon).$$

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham [18].

Definition 1.6 ([18]). Let $(X,\preceq)$ be a partial ordered set. A mapping $F:X\times X\to X$ is said to be have *mixed monotone property* if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in X$

$$x_1,x_2\in X, x_1\preceq x_2 \quad\Rightarrow\quad F(x_1,y)\preceq F(x_2,y)$$

and

$$y_1,y_2\in X, y_1\preceq y_2 \quad\Rightarrow\quad F(x,y_1)\succeq F(x,y_2).$$

Definition 1.7 ([18]). Let $F:X\times X\to X$. An element $(x,y)\in X\times X$ is said to be a *coupled fixed point* of a mapping F if

$$x=F(x,y) \text{ and } y=F(y,x).$$

Let Φ denote the set of all functions $\phi:[0,\infty)\to[0,\infty)$ which satisfy

($\phi1$) ϕ is continuous and non-decreasing,
($\phi2$) $\phi(t)=0$ if and only if $t=0$,
($\phi3$) $\phi(t+s)\le\phi(t)+\phi(s)$ for all $t,s\in[0,\infty)$,
($\phi4$) $\phi(\alpha t)\le\alpha\phi(t)$ for all $\alpha\in(0,\infty)$.

and let Ψ denote the set of all functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy

$$\lim_{t\to r}\psi(t)>0 \text{ for all } r>0 \text{ and } \lim_{t\to 0^+}\psi(t)=0.$$

Alsulami, Hussain and Alotaibi [4] proved some coupled fixed point results for (ϕ,φ)- weakly contractive mappings in ordered partial metric spaces. More precisely, they obtained the following results.

Theorem 1.8. [4, Theorem 3.1] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(p(F(x,y), F(u,v))) \;\leq\; \frac{1}{2}\phi(p(x,u) + p(y,v)) - \psi\left(\frac{p(x,u) + p(y,v)}{2}\right) \tag{1.1}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

and also suppose either
(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$,then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$,then $y \preceq y_n$ for all n,*
then F has a coupled fixed point in X, that is there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Starting from the results in Alsulami, Hussain and Alotaibi [4], our main aim in this paper is to obtain more general coupled fixed point theorems for mixed monotone operators $F : X \to X$ satisfying a contractive condition which is significantly more general that the corresponding conditions (1.1) in [4], thus extending many other related results in literature. We also provide illustrative example in support of our results.

2. Coupled fixed points for generalized weakly contractive mappings

We start with an example which shows the weakness of Theorem 1.8.

Example 2.1. Let $X = \mathbb{R}^+$ be a set endowed with order $x \preceq y \Leftrightarrow x \leq y$. Let $p(x, y) = \max\{x, y\}$, then (X, p) is a partial metric space. Define the mapping $F : X \times X \to X$ by

$$F(x,y) = \frac{5x - 2y}{8} \text{ for all } x, y \in X.$$

Then the following properties hold:
  (1) F is mixed monotone;
  (2) the condition (1.1) does not hold.
Indeed, we show that F does not satisfy condition (1.1). Assume on the contrary, that there exist ϕ and ψ, such that (1.1) holds. Therefore it implies that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that, for all $x \geq u$ and $y \leq v$, we have

$$\begin{aligned}
\phi\left(\frac{5x-2y}{8}\right) &= \phi\left(\max\left\{\frac{5x-2y}{8}, \frac{5u-2v}{8}\right\}\right) \\
&= \phi(p(F(x,y), F(u,v)))
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}\phi(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \\ &= \frac{1}{2}\phi(x+v) - \psi\left(\frac{x+v}{2}\right). \end{aligned}$$

Setting $x = 5, y = 1/2$ and $v = 1$ to the last inequality, we get, since $\psi(3) > 0$, that

$$\phi(3) \leq \left(\frac{1}{2}\right)\phi(6) - \psi(3) < \left(\frac{1}{2}\right)\phi(6) \leq \left(\frac{1}{2}\right)2\phi(3) = \phi(3)$$

which gives a contradiction. Hence F does not satisfy condition (1.1). Notice, however, that $(0,0) \in X^2$ is the coupled fixed point of F. □

We now state and prove our first result which successively guarantees the existence of a coupled fixed point and generalizes Theorem 3.1 in [4].

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$M_F^{\phi,\psi}(x,y,u,v) \leq \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$, where

$$M_F^{\phi,\psi}(x,y,u,v) = \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))).$$

If there exist two elements $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0),$$

and also suppose either
(a) *F is continuous or*
(b) *X has the following property:*
(i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$,then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$,then $y \preceq y_n$ for all n,*
then F has a coupled fixed point in X, that is there exists $(x,y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. We construct sequence $\{x_n\}$ and $\{y_n\}$ in X as

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(x_n,y_n) \text{ for all } n \geq 0. \tag{2.2}$$

Next, we show that

$$x_n \preceq x_{n+1} \text{ for all } n \geq 0 \tag{2.3}$$

and

$$y_n \succeq y_{n+1} \text{ for all } n \geq 0. \tag{2.4}$$

For this we shall use mathematical induction.

Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (2.3) and (2.4) hold for $n = 0$.

Suppose now that (2.3) and (2.4) hold for some fixed $n \geq 0$, then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{2.5}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_{n+1}, x_n) \preceq F(y_n, x_n) = y_{n+1}. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$x_{n+1} \preceq x_{n+2} \text{ and } y_{n+1} \succeq y_{n+2}.$$

Hence, by the induction method we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{2.7}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots. \tag{2.8}$$

For each $n \geq 0$, let

$$\xi_{n+1} = p(x_{n+1}, x_n) + p(y_n, y_{n+1}).$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, using (2.1) and (2.2), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n)) + \phi(p(y_n, y_{n+1})) &= \phi(p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\quad + \phi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &= M_F^{\phi,\psi}(x_n, y_n, x_{n-1}, y_{n-1}) \\ &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big). \end{aligned} \tag{2.9}$$

By property ($\phi 3$), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n) + p(y_n, y_{n+1})) &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big), \end{aligned} \tag{2.10}$$

which implies, since ψ is a non-negative function,

$$\phi(p(x_{n+1},x_n)+p(y_n,y_{n+1})) \leq \phi(p(x_n,x_{n-1})+p(y_n,y_{n-1})); \tag{2.11}$$

that is

$$\phi(\xi_{n+1}) \leq \phi(\xi_n) \quad \text{for all } n \geq 0. \tag{2.12}$$

Using the fact that ϕ is a non-decreasing, we get $\xi_{n+1} \leq \xi_n$ for all $n \geq 0$. This shows that $\{\xi_n\}$ is decreasing. Therefore is some $\xi \geq 0$ such that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1},x_n)+p(y_{n+1},y_n)] = \xi. \tag{2.13}$$

We shall prove that $\xi = 0$. Suppose, to the contrary, that $\xi > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\xi_n \to \xi$) of both sides of (2.10) and remembering $\lim_{t\to r}\psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\begin{aligned}\phi(\xi) &= \lim_{n\to\infty}\phi(\xi_n) \leq \lim_{n\to\infty}\left[\phi(\xi_{n-1}) - 2\psi\left(\frac{\xi_{n-1}}{2}\right)\right] \\ &= \phi(\xi) - 2\lim_{\xi_{n-1}\to\xi}\psi\left(\frac{\xi_{n-1}}{2}\right) < \phi(\xi),\end{aligned}$$

which gives a contradiction. Thus $\xi = 0$, that is,

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1},x_n)+p(y_{n+1},y_n)] = 0. \tag{2.14}$$

Let

$$\xi_n^p = d_p(x_{n+1},x_n)+d_p(y_n,y_{n+1})$$

for all $n \in \mathbb{N}$. From the definition of d_p, it is clear that $\xi_n^p \leq 2\xi_n$ for all $n \in \mathbb{N}$. Using (2.14), we get

$$\lim_{n\to+\infty} \xi_n^p = \lim_{n\to+\infty} [d_p(x_{n+1},x_n)+d_p(y_{n+1},y_n)] = 0.$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the partial metric space (X,p). From Lemma 1.5 (a), it is sufficient to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (X,d_p). Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$, $\{y_{m_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$d_p(x_{n_k},x_{m_k})+d_p(y_{n_k},y_{m_k}) \geq \varepsilon. \tag{2.15}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.15). Then

$$d_p(x_{n_k-1},x_{m_k})+d_p(y_{n_k-1},y_{m_k}) < \varepsilon. \tag{2.16}$$

Using (2.15), (2.16) and the triangle inequality, we have

$$\begin{aligned}\varepsilon &\le r_k^p := d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k})\\ &\le d_p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k}) + d_p(y_{n_k}, y_{n_k-1}) + d_p(y_{n_k-1}, y_{m_k})\\ &\le d_p(x_{n_k}, x_{n_k-1}) + d_p(y_{n_k}, y_{n_k-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.14), we get

$$\lim_{k\to\infty} r_k^p = \lim_{k\to\infty} [d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k})] = \varepsilon. \tag{2.17}$$

By the triangle inequality,

$$\begin{aligned} r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k})\\ &\le d_p(x_{n_k}, x_{n_k+1}) + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(x_{m_k+1}, x_{m_k})\\ &+ d_p(y_{n_k}, y_{n_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}) + d_p(y_{m_k+1}, y_{m_k})\\ &= \xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}).\end{aligned}$$

Using the properties of ϕ, we have

$$\begin{aligned}\phi(r_k^p) &\le \phi(\xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}))\\ &\le \phi(\xi_{n_k}^p + \xi_{m_k}^p) + \phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1})).\end{aligned} \tag{2.18}$$

Now, let

$$r_k = p(x_{n_k}, x_{m_k}) + p(y_{n_k}, y_{m_k}).$$

By the definition of r_k^p, we have

$$\begin{aligned} r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k})\\ &= 2p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k})\\ &+ 2p(y_{n_k}, y_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k})\\ &= 2r_k - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k}).\end{aligned} \tag{2.19}$$

In view of property (p2) and (2.14), we have

$$\begin{aligned}\lim_{k\to+\infty} p(x_{n_k}, x_{n_k}) &= \lim_{k\to+\infty} p(x_{m_k}, x_{m_k})\\ &= \lim_{k\to+\infty} p(y_{n_k}, y_{n_k})\\ &= \lim_{k\to+\infty} p(y_{m_k}, y_{m_k}) = 0.\end{aligned}$$

Therefore, letting $k \to +\infty$ in (2.19) and using (2.17), we get

$$\lim_{k\to+\infty} r_k = \frac{\varepsilon}{2}.$$

Since $x_{n_k} \succeq x_{m_k}$ and $y_{n_k} \preceq y_{m_k}$, we have

$$\phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1}))$$

$$\begin{aligned}
&\leq \phi(2p(x_{n_k+1}, x_{m_k+1})) + \phi(2p(y_{n_k+1}, y_{m_k+1})) \\
&\leq 2\phi(p(x_{n_k+1}, x_{m_k+1})) + 2\phi(p(y_{n_k+1}, y_{m_k+1})) \\
&= 2\phi(p(F(x_{n_k}, y_{n_k})), p(F(x_{m_k}, y_{m_k}))) + 2\phi(p(F(y_{n_k}, x_{n_k})), p(F(y_{m_k}, x_{m_k}))) \\
&= 2M_F^{\phi,\psi}(x_{n_k}, y_{n_k}, x_{m_k}, y_{m_k}) \\
&\leq 2\phi(p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))) \\
&\quad -4\psi\Big(\frac{p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))}{2}\Big) \\
&= 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big). \qquad (2.20)
\end{aligned}$$

Thus, from (2.18), we have

$$\phi(r_k^p) \leq \phi(\xi_{n_k}^p + \xi_{m_k}^p) + 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big).$$

Letting $k \to +\infty$, and using the properties of ϕ and ψ together with the inequalities established above, we have

$$\begin{aligned}
\phi(\varepsilon) \leq \phi(0) + 2\phi(\frac{\varepsilon}{2}) - 4\lim_{k\to+\infty}\psi(\frac{r_k}{2}) &\leq \phi(\varepsilon) - 4\lim_{\frac{r_k}{2}\to\frac{\varepsilon}{4}}\psi(\frac{r_k}{2}) \\
&\leq \phi(\varepsilon) - 4\lim_{t\to\frac{\varepsilon}{4}}\psi(t) \\
&< \phi(\varepsilon) \qquad (2.21)
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete metric space (X, d_p). Thus, there are $x, y \in X$ such that

$$\lim_{n\to+\infty} d_p(x_n, x) = \lim_{n\to+\infty} d_p(y_n, y) = 0, \qquad (2.22)$$

which implies that

$$\begin{aligned}
&\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} x_n = x \\
&\lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} y_n = y. \qquad (2.23)
\end{aligned}$$

Therefore, from Lemma 1.5 (b), using (2.14) and the property (p2), we have

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(x_n, x_n) = 0, \qquad (2.24)$$

$$p(y, y) = \lim_{n\to+\infty} p(y_n, y) = \lim_{n\to+\infty} p(y_n, y_n) = 0. \qquad (2.25)$$

On utilizing $p(x, x) = p(y, y) = 0$ in (2.1), we get

$$\begin{aligned}
&\phi(p(F(x, y), F(x, y))) + \phi(p(F(y, x), F(y, x))) \\
&\quad \leq \phi(p(x, x) + p(y, y)) - 2\psi\Big(\frac{p(x, x) + p(y, y)}{2}\Big)
\end{aligned}$$

$$= \quad \phi(0) - 2\psi(0) = -2\psi(0) \leq 0,$$

which implies $\phi(p(F(x,y), F(x,y))) + \phi(p(F(y,x), F(y,x))) = 0$. Therefore, from the property ($\phi 2$), we get

$$p(F(x,y), F(x,y)) = 0 = p(F(y,x), F(y,x)). \tag{2.26}$$

We now show that $x = F(x,y)$ and $y = F(y,x)$. Suppose that the assumption (a) holds. For any given $\varepsilon > 0$, the commutativity of F at a point (x,y) implies that there exists $\xi > 0$ such that if $(u,v) \in X \times X$ with $q((x,y),(u,v)) < q((x,y),(x,y)) + \xi = \xi$, meaning that

$$p(x,u) + p(y,v) < p(x,x) + p(y,y) + \xi = \xi,$$

because $p(x,x) = p(y,y) = 0$, then we have

$$p(F(x,y), F(u,v)) < p(F(x,y), F(x,y)) + \frac{\varepsilon}{2}. \tag{2.27}$$

Since $\lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(y_n, y) = 0$, and $\xi = \min\{\frac{\xi}{2}, \frac{\varepsilon}{2}\} > 0$, there exist $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$p(x_n, x) < \xi \text{ and } p(y_n, y) < \xi,$$

which gives that

$$p(x_n, x) + p(y_n, y) < 2\xi < \xi.$$

Using (2.27), we get that

$$p(F(x,y), F(x_n, y_n)) < p(F(x,y), F(x,y)) + \frac{\varepsilon}{2}. \tag{2.28}$$

Then, for any $n \geq n_0$, we have

$$\begin{aligned} p(F(x,y), x) &\leq p(F(x,y), x_{n+1}) + p(x_{n+1}, x) \\ &= p(F(x,y), F(x_n, y_n) + p(x_{n+1}, x) \\ &\leq p(F(x,y), F(x,y)) + \frac{\varepsilon}{2} + \xi \\ &\leq p(F(x,y), F(x,y)) + \varepsilon. \end{aligned} \tag{2.29}$$

From (2.26), we have

$$p(F(x,y), x) < \varepsilon.$$

Since ε is arbitrary, we can conclude that

$$p(F(x,y), x) = 0. \tag{2.30}$$

Similarly, we show that $p(F(y,x), y) = 0$. These together with (2.26) and (p1) imply that

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Hence, (x,y) is a coupled fixed point of F.

Finally, suppose that (b) holds. By (2.3), (2.22) and (2.24), we have $\{x_n\}$ is a non-decreasing sequence, $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence, $y_n \to y$ as $n \to \infty$. Hence, by the assumption (b), we have for all $n \geq 0$,

$$x_n \preceq x \text{ and } y \preceq y_n. \tag{2.31}$$

By property (p4), we have

$$p(F(x,y),x) \leq p(F(x,y),x_{n+1}) + p(x_{n+1},x) = p(F(x,y),F(x_n,y_n)) + p(x_{n+1},x)$$

and

$$p(F(y,x),y) \leq p(F(y,x),y_{n+1}) + p(y_{n+1},y) = p(F(y,x),F(y_n,x_n)) + p(y_{n+1},y)$$

Therefore,

$$\begin{aligned}
&\phi(p(F(x,y),x)) + \phi(p(F(y,x),y)) \\
&\quad \leq \quad \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(F(x,y),F(x_n,y_n))) + \phi(p(F(y,x),F(y_n,x_n))) \\
&\quad \leq \quad \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(x,x_n) + p(y,y_n)) - 2\psi\Big(\frac{(p(x,x_n) + p(y,y_n))}{2}\Big).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (2.24) and (2.25) and the properties of ϕ and ψ, we get $\phi(p(x,F(x,y))) = 0 = \phi(p(y,F(y,x)))$, which implies

$$p(x,F(x,y)) = 0 = p(y,F(y,x)).$$

These together with (2.26), we have

$$x = F(x,y) \text{ and } y = F(y,x)$$

Hence, (x,y) is a coupled coincidence point of F. This complete the proof. □

Remark 2.3. Theorem 2.2 is more general than [4, Theorem 3.1], since the contractive condition (2.1) is weaker than (1.1), a fact which is clearly illustrated by the following example.

Example 2.4. Let us recall Example 2.1. Define the mappings $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = 2t \text{ and } \psi(t) = \frac{t}{2} \text{ for all } t \in [0,\infty),$$

We show that F, ϕ and ψ satisfy condition (2.1). For all $x \geq u$ and $y \leq v$, we observe that

$$\phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) = \frac{5x-2y}{4} + \frac{5v-2u}{4},$$

$$\phi(p(x,u) + p(y,v)) = \phi(x+v) = 2(x+v),$$

and

$$2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) = 2\psi\left(\frac{x+v}{2}\right) = \frac{x+v}{2}.$$

Furthermore, we can have the following fact

$$\begin{aligned} x+v \geq -2u-2y &\Leftrightarrow 6x+6v \geq 5x-2y+5v-2u \\ &\Leftrightarrow \frac{3}{2}(x+v) \geq \frac{5x-2y}{4} + \frac{5v-2u}{4}. \end{aligned}$$

Therefore, we arrive that

$$\begin{aligned} \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) &= \frac{5x-2y}{4} + \frac{5v-2u}{4} \\ &\leq \frac{3}{2}(x+v) \\ &= 2(x+v) - \frac{x+v}{2} \\ &= \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right). \end{aligned}$$

Hence F, ϕ and ψ satisfy (2.1). By Theorem 2.2, we conclude that F has a coupled fixed point in X. Moreover, $(0,0) \in X^2$ is a coupled fixed point of F. $\square$

As an immediate consequence of the above theorem, by taking $\phi(t) = t$, we have:

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p(F(x,y),F(u,v)) \leq \frac{1}{2}(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) *F is continuous or*
(b) *X has the following property:*
(i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-decreasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.*
Then there exist $x, y, \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x),$$

that is F has a coupled fixed point in X.

Moreover, if we take $\psi(t) = \frac{1-k}{2}t$ where $k \in [0, 1)$ in Corollary 2.5, we get:

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p(F(x, y), F(u, v)) \leq \frac{1}{2}(p(x, u) + p(y, v))$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) *F is continuous or*
(b) *X has the following property:*
(i) *if a non-decreasing sequence $x_n \to x$,then $x_n \preceq x$ for all n,*
(ii) *if a non-decreasing sequence $y_n \to y$,then $y \preceq y_n$ for all n.*
Then there exist $x, y, \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x),$$

that is F has a coupled fixed point in X.

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References

[1] Banach, S: Surles operations dans les ensembles et leur application aux equation sitegrales, Fund. Math. 3, 133181 (1922)

[2] Matthews, SG: Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. 728, 183197 (1994)

[3] Abbas, M, Khan, SH, Nazir, T: Common fixed points of R-weakly commuting maps in generalized metric spaces. Fixed Point Theory Appl. **2011**, 41 (2011)

[4] Alsulami, S, Hussain, N, Alotaibi, A: Coupled Fixed and Coincidence Points for Monotone Operators in Partial Metric Spaces, Fixed Point Theory and Applications 2012, 2012:173

[5] Abdeljawad, TH, Karapinar, E, Tas, K: Existence and uniqueness of a common fixed point on partial metric spaces, Applied Mathematics Letters 24, 19001904 (2011)

[6] Abdeljawad, TH, Karapinar, E, Tas, K: A generalized contraction principle with control functions on partial metric spaces, Computers and Mathematics with Applications 6, 716719 (2012)

[7] Abdeljawad, TH: Fixed points for generalized weakly contractive mappings in partial metric spaces, Mathematical and Computer Modelling 54 29232927 (2011)

[8] Altun, I, Erduran, A: Fixed point theorems for monotone mappings on partial metric spaces, Fixed Point Theory and Applications 2011, Article ID 508730, 10 pages (2011)

[9] Altun, I, Sola, F, Simsek, H: Generalized contractions on partial metric spaces, Topology and Its Applications, 157, 27782785 (2010)

[10] Agarwal, RP, Alghamdi, MA, Shahzad, N: Fixed point theory for cyclic generalized contractions in partial metric spaces, Fixed Point Theory and Applications 2012, 2012:40

[11] Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1–8 (2008)

[12] Aydi, H: Some fixed point results in ordered partial metric spaces, The J. Nonlinear Sci. Appl. 4, 112 (2011)

[13] Aydi, H: Some coupled fixed point results on partial metric spaces, International Journal of Mathematical Sciences 2011, Article ID 647091, 11 pages (2011)

[14] Aydi, H: Fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces, Journal of Nonlinear Analysis and Optimization: Theory and Applications 2, 33.48 (2011)

[15] Aydi, H, Karapinar, E, Shatanawi, W: Coupled fixed point results for (ψ, ϕ)-weakly contractive condition in ordered partial metric spaces, Computer and Mathematics with Applications 62, 4449.4460 (2011)

[16] Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces, Computers and Mathematics with Applications 63, 298.309 (2012)

[17] Berinde, V: Coupled coincidencepointtheorems for mixedmonotone nonlinear operators, Computers and Mathematics with Applications (2012), doi:10.1016/j.camwa.2012.02.012

[18] Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379–1393 (2006)

[19] Cho, YJ, Rhoades, BE, Saadati, R, Samet, B, Shatanawi, W: Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, Fixed Point Theory and Applications 2012, 2012:8, doi:10.1186/1687-1812-2012-8

[20] GoluboviLc, Z, Kadelburg, Z, RadenoviLc, S: Coupled coincidence points of mappings in ordered partial metric spaces, Abstract and Applied Analysis 2012, Article ID 192581, 10 pages (2012).

[21] Heckmann, R: Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7, 71.83 (1999)

[22] Karapinar, E: Generalizations of Caristi Kirkfs theorem on partial metric spaces, Fixed Point Theory and Appl. (in press) (2011).

[23] Karapinar, E, Erhan, IM: Fixed point theorems for operators on partial metric spaces, Applied Mathematics Letters (2011), 10.1016/j.aml.2011.05.013.

[24] Kirk, WA, Srinavasan PS, Veeramani P: Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory 4 79-89 (2003)

[25] Karapinar, E, Erhan, IM: Best proximity point on different type contractions, Appl. Math. Inf. Sci. 5 342-353 (2011)

[26] Karapinar, E, Erhan,IM, Ulus, AY: Fixed point theorem for cyclic maps on partial metric spaces, Appl. Math. Inf. Sci. 6, 239.244 (2012)

[27] Karapinar, E, Erhan, IM: Cyclic contractions and fixed point theorems, Filomat 26, 777-7-82 (2012)

[28] Khan, MS, Swaleh M, Sessa, S: Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30, 1.9 (1984)

[29] Lakshmikantham, V, Ćirić, LJ., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341-4349.

[30] Lakzian, H, Samet, B: Fixed points for (ψ, ϕ) -weakly contractive mappings in generalized metric spaces, Applied Mathematics Letters 25 , 902.906 (2012)

[31] Nashine, HK, Kadelburg, Z, Radenović, S: Coupled common fixed point theorems for w^*-compatible mappings in ordered cone metric spaces. Appl. Math. Comput. **218**, 5422–5432 (2012)

[32] Nashine, HK, Kadelburg, Z, RadenoviLc, S: Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces, Mathematical and Computer Modelling (2012) Article in Press.

[33] Oltra,S, Valero, O: Banachfs fixed point theorem for partial metric spaces, Rend. Istid. Math. Univ. Trieste 36, 17-26 (2004)

[34]] Rus, IA, Cyclic representations and fixed points, Ann. T. Popoviciu Seminar Funct. Eq. Approx.

Convexity 3 (2005) 171-178

[35] Romaguera, S: A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. 2010, Article ID 493298, 6 pages (2010)

[36] Radenović, S, Kadelburg, Z: Generalized weak contractions in partially ordered metric spaces. Comput. Math. Appl. **60**, 1776–1783 (2010)

[37] Samet, B, RajoviLc, M, LazoviLc, R, StoiljkoviLc, R: Common fixed point results for nonlinear contractions in ordered partial metric spaces, Fixed Point Theory Appl., 2011:71 (2011)

[38] Shatanawi, W, Abbas, M, Nazir, T: Common coupled fixed points results in two generalized metric spaces. Fixied Point Theory Appl. **2011**, 80 (2011). doi:10.1186/1687-1812-2011-80

[39] Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2011**, 81 (2011)

[40] Shatanawi, W, Al-Rawashdeh, A: Common fixed points of almost generalized (ψ, ϕ)-contractive mappings in ordered metric spaces, Accepted in Fixed Point Theory and Applications.

[41] Shatanawi, W, Mustafa, Z, Tahat, N: Some coincidence point theorems for nonlinear contraction in ordered metric spaces, Fixed point Theory and Applications 2011, 2011:68.

[42] Shatanawi, W, Samet, B: On (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, Computers and Mathematics with Applications 62, 3204.3214 (2011)

[43] Shatanawi, W, Nashine, HK: A generalization of Banachfs contraction principle for nonlinear contraction in a partial metric space, J. Nonlinear Sci. Appl. 5, 37.43 (2012)

[44] Valero, O: On Banach fixed point theorems for partial metric spaces, Appl. General Topology 6, 229-240 (2005)

A note on structures of fuzzy approximation spaces *

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Abstract: In this paper, fuzzy rough approximation operators are further established. Topological structures of fuzzy approximation spaces are given.

Keywords: Fuzzy set; Fuzzy relation; Fuzzy approximation space; Fuzzy topology.

1 Introduction

Rough set theory, proposed by Pawlak [9], is a mathematical tool for approximate reasoning about data. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [10, 14, 15, 16].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules.

Various fuzzy generalizations of rough approximations have been proposed [1, 8, 18, 20]. The most common fuzzy rough set is obtained by replacing the crisp relations with fuzzy relations on the universe and crisp subsets with fuzzy sets.

An interesting and natural research topic in rough set theory is to study the relationship between rough sets or approximation spaces and topologies. Many authors studied topological properties of rough sets or approximation spaces [3, 6, 13, 22]. In the study of topological properties of fuzzy rough sets or fuzzy

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approximation spaces, Qin et al. [17] investigated the topological properties of fuzzy rough sets.

The purpose of this paper is to investigate topological structures of fuzzy approximation spaces.

2 Preliminaries

Throughout this paper, X denotes a nonempty set. I denotes $[0,1]$. $F(X)$ denotes the set of all fuzzy sets in X. For $a \in I$, $\bar{a}$ denotes the constant fuzzy set in X.

For each $A \in F(X)$, we denote

$$R_A = \{(x,y) \in X \times X : A(x) > A(y)\}.$$

Obviously, $R_A = \emptyset \Longleftrightarrow A = \bar{a}$ for some $a \in I$.

A fuzzy set is called a fuzzy point in X, if it takes the value 0 for each $y \in X$ except one, say, $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its support and λ is called its height (see [4, 12]). Denote

$$P(X) = \{x_\lambda : x \in X, \lambda \in (0,1]\}.$$

For a fuzzy point x_λ and $A \in F(X)$, we defined $x_\lambda \in A$ by $x_\lambda \subseteq A$.

Obviously,

$$x_\lambda \in A \iff \lambda \leq A(x).$$

Definition 2.1 ([2]). *$\sigma \subseteq F(X)$ is called a fuzzy topology on X, if*

(i) For each $a \in I$, $\bar{a} \in \sigma$.

(ii) $A, B \in \sigma \implies A \cap B \in \sigma$,

(iii) $\{A_i : i \in J\} \subseteq \sigma \Longrightarrow \bigcup\limits_{i \in J} A_i \in \sigma$.

In this case the pair (X, σ) is called a fuzzy topological space. Every member of σ is called a fuzzy open set in X. Its complement is called a fuzzy closed set in X.

We denote $\sigma^c = \{A \in F(X) : A^c \in \sigma\}$.

Interior and closure of A denoted respectively by $int_\sigma(A)$ and $cl_\sigma(A)$ for each $A \in F(X)$, are defined as follows:

$$int_\sigma(A) = \bigcup\{B \in \sigma : B \subseteq A\}, \quad cl_\sigma(A) = \bigcap\{B \in \sigma^c : B \supseteq A\}.$$

A fuzzy topology σ is called Alexandrov [5] if (ii) in Definition 2.2 is replaced by

$(ii)'$ $\{A_i : i \in J\} \subseteq \sigma \Longrightarrow \bigcap\limits_{i \in J} A_i \in \sigma$.

Definition 2.2 ([4]). *Let (X, σ) be a fuzzy topological space and let $x_\lambda \in P(X)$ and $A \in F(X)$. A is called a closed remote-neighborhood of x_λ, if $A \in \sigma^c$ and $x_\lambda \notin A$.*

Definition 2.3 ([4]). *Let (X, σ) be a fuzzy topological space.*

(1) *(X, σ) is called T_{-1} space, if for any $x_\lambda, x_\mu \in P(X)$ and $\mu < \lambda$, there exists A such that $x_\mu \in A$ and A is a closed remote-neighborhood of x_λ, or, there exists B such that $x_\lambda \in B$ and B is a closed remote-neighborhood of x_μ.*

(2) *(X, σ) is called sub-T_0 space, if for any $x, y \in X$ and $x \neq y$, there exist $\lambda \in (0, 1]$ and A such that $y_\lambda \in A$ and A is closed remote-neighborhood of x_λ, or, there exist $\lambda \in (0, 1]$ and B such that $x_\lambda \in B$ and B is a closed remote-neighborhood of y_λ.*

Definition 2.4 ([21]). *Let R be a crisp relation on X. For each $x \in X$, denote*

$$R_p(x) = \{y \in X : (y, x) \in R\} \quad and \quad R_s(x) = \{y \in X : (x, y) \in R\}.$$

$R_p(x)$ and $R_s(x)$ are called the predecessor and successor neighborhood of x, respectively.

3 Fuzzy approximation spaces and fuzzy rough sets

Recall that R is called a fuzzy relation on X if $R \in F(X \times X)$.

Definition 3.1. *Let R be a fuzzy relation on X. Then R is called*

(1) *reflexive, if $R(x, x) = 1$ for each $x \in X$.*

(2) *symmetric, if $R(x, y) = R(y, x)$ for any $x, y \in X$.*

(3) *transitive, if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in X$.*

Let R be a fuzzy relation on X. R is called preorder (resp. equivalence) if R is reflexive and transitive (resp. reflexive, symmetric and transitive).

Definition 3.2 ([11, 17]). *Let R be a fuzzy relation on X. The pair (X, R) is called a fuzzy approximation space. For each $A \in F(X)$, the fuzzy lower and the fuzzy upper approximation of A with respect to (X, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are respectively, defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y \in X} (A(y) \vee (1 - R(x, y))) \quad (x \in X)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in X} (A(y) \wedge R(x, y)) \qquad (x \in X).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the fuzzy rough set of A with respect to (X, R).

Remark 3.3.

$$\overline{R}(x_1)(y) = R(y, x) \quad and \quad \underline{R}((x_1)^c)(y) = 1 - R(y, x) \quad (x, y \in X).$$

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Proposition 3.4 ([17, 19]). *Let (X, R) be a fuzzy approximation space. Then for any $A, B \in F(X)$, $\{A_i : i \in J\} \subseteq F(X)$ and $\lambda \in I$,*

(1) $\underline{R}(\bar{1}) = \bar{1}$, $\overline{R}(\bar{0}) = \bar{0}$.

(2) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.

(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.

(4) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$.

(5) $\underline{R}(\bigcap_{i \in J} A_i) = \bigcap_{i \in J} (\underline{R}(A_i))$, $\overline{R}(\bigcup_{i \in J} A_i) = \bigcup_{i \in J} \overline{R}(A_i)$.

Theorem 3.5 ([7, 11, 17]). *Let (X, R) be a fuzzy approximation space. Then*

$$\begin{array}{lll}
(1)\ R\ is\ reflexive & \Longleftrightarrow & (ILR)\ \forall A \in F(X), \underline{R}(A) \subseteq A. \\
 & \Longleftrightarrow & (IUR)\ \forall A \in F(X), A \subseteq \overline{R}(A). \\
(2)\ R\ is\ symmetric & \Longleftrightarrow & (ILS)\ \forall (x, y) \in X \times X, \underline{R}((x_1)^c)(y) = \underline{R}((y_1)^c)(x). \\
 & \Longleftrightarrow & (IUS)\ \forall (x, y) \in X \times X, \overline{R}(x_1)(y) = \overline{R}(y_1)(x). \\
(3)\ R\ is\ transitive & \Longleftrightarrow & (ILT)\ \forall A \in F(X), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)). \\
 & \Longleftrightarrow & (IUT)\ \forall A \in F(X), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).
\end{array}$$

Remark 3.6. (1) *For each $a \in I$, $\overline{R}(\bar{a}) \subseteq \ \bar{a} \ \subseteq \underline{R}(\bar{a})$;*

(2) *If R is reflexive, then for each $a \in I$, $\underline{R}(\bar{a}) = \bar{a} = \overline{R}(\bar{a})$.*

Proposition 3.7. *Let (X, R) be a fuzzy approximation space. Then for each $A \in F(X)$ with $R_A \neq \emptyset$,*

(1) *a)* $\underline{R}(A) \supseteq A \Longleftrightarrow (FLO) \forall (x, y) \in R_A, 1 - R(x, y) \geq A(x) \vee A(y)$.

b) $\overline{R}(A) \subseteq A \Longleftrightarrow (FUO) \forall (x, y) \in R_A,\ R(y, x) \leq A(x) \wedge A(y)$.

(2) *If R is reflexive, then*

a) $\underline{R}(A) = A \Longleftrightarrow (FLR) \forall (x, y) \in R_A, 1 - R(x, y) \geq A(x) \vee A(y)$.

b) $\overline{R}(A) = A \Longleftrightarrow (FUR) \forall (x, y) \in R_A, R(y, x) \leq A(x) \wedge A(y)$.

Proof. (1) a) Necessity. Suppose that $\underline{R}(A) \supseteq A$. Note that for each $x \in X$,

$$\bigwedge_{y \in U} (A(y) \vee (1 - R(x, y))) = (\underline{R}(A))(y) \geq A(x).$$

Then $A(y) \vee (1 - R(x, y)) \geq A(x)$ for any $x, y \in X$. Since $A(x) > A(y)$ for each $(x, y) \in R_A$, we have

$$1 - R(x, y) \geq A(x) = A(x) \vee A(y) \quad ((x, y) \in R_A).$$

Sufficiency. Suppose that (FLO) holds. Let $x \in X$.

(i) If $y \in (R_A)_s(x)$, then

$$A(y) \vee (1 - R(x, y)) \geq A(y) \vee (A(x) \vee A(y)) \geq A(x).$$

(ii) If $y \notin (R_A)_s(x)$, then $A(y) \geq A(x)$ and so

$$A(y) \vee (1 - R(x, y)) \geq A(y) \geq A(x).$$

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Hence $\underline{R}(A)(x) = \bigwedge_{y\in U}(A(y) \vee (1 - R(x, y))) \geq A(x)$.

Thus $\underline{R}(A) \supseteq A$.

b) Necessity. Suppose that $\overline{R}(A) \subseteq A$. Note that for each $y \in X$,

$$\bigvee_{x\in X}(A(x) \wedge R(y, x)) = \overline{R}(A)(y) \leq A(y).$$

Then $A(x) \wedge R(y, x) \leq A(y)$ $(x, y \in X)$ for any $x, y \in X$. Since $A(x) > A(y)$ for each $(x, y) \in R_A$, we have

$$R(y, x) \leq A(y) = A(x) \wedge A(y) \quad ((x, y) \in R_A).$$

Sufficiency. Suppose that (FLO) holds. Let $y \in X$.

(i) If $x \in (R_A)_p(y)$, then $(x, y) \in R_A$ and so

$$A(x) \vee R(y, x) \leq A(x) \wedge (A(x) \wedge A(y)) \leq A(y).$$

(ii) If $x \notin (R_A)_p(y)$, then $A(x) \leq A(y)$ and so

$$A(x) \wedge R(y, x) \leq A(x) \leq A(y).$$

Hence $(\overline{R}(A))(y) = \bigvee_{x\in X}(A(x) \wedge R(y, x)) \leq A(y)$. Thus $\overline{R}(A) \subseteq A$.

(2) This holds by (1) and Theorem 3.5(1). □

4 Topological structures of fuzzy approximation spaces

Let (X, R) be a fuzzy approximation space. We denote

$$\sigma_R = \{A \in F(X) : A \subseteq \underline{R}(A)\};$$

$$s_R = \bigwedge_{x,y\in X} R(x, y), \quad t_R = \bigvee_{x,y\in X,\ x\neq y} R(x, y).$$

Theorem 4.1. *Let (X, R) be a fuzzy approximation space.*

(1) *σ_R is an Alexandrov fuzzy topology on X.*

(2) *$int_{\sigma_R}(A) \subseteq \underline{R}(A)$ and $\overline{R}(A) \subseteq cl_{\sigma_R}(A)$ $(A \in F(X))$.*

(3) *$A \in (\sigma_R)^c \iff A \supseteq \overline{R}(A)$.*

(4) *For each $a \in I$, $\bar{a} \in (\sigma_R)^c$.*

Proof. (1) (i) For each $a \in I$, by Remark 3.6(1), $\bar{a} \subseteq \underline{R}(\bar{a})$. Then $\bar{a} \in \sigma_R$.

(ii) Let $\{A_i : i \in J\} \subseteq \sigma_R$. Then $A_i \subseteq \underline{R}(A_i)$ for each $i \in J$. By Proposition 3.4(5),

$$\bigcap_{i\in J} A_i \subseteq \bigcap_{i\in J} \underline{R}(A_i) = \underline{R}(\bigcap_{i\in J} A_i).$$

Hence $\bigcap_{i\in J} A_i \in \sigma_R$.

(iii) Let $\{A_i : i \in J\} \subseteq \sigma_R$. Then $A_i \subseteq \underline{R}(A_i)$ for each $i \in J$. By Proposition 3.4(2),

$$\bigcup_{i\in J} A_i \subseteq \bigcup_{i\in J} \underline{R}(A_i) \subseteq \underline{R}(\bigcup_{i\in J} A_i).$$

Then $\bigcup_{i\in J} A_i \in \sigma_R$.

Hence σ_R is an Alexandrov fuzzy topology on X.

(2) For each $A \in F(X)$, by Proposition 3.4(2),

$$\begin{aligned} int_{\sigma_R}(A) &= \bigcup\{B \in \sigma_R : B \subseteq A\} \\ &\subseteq \bigcup\{B \in \sigma_R : \underline{R}(B) \subseteq \underline{R}(A)\} \\ &= \bigcup\{B \in F(X) : B \subseteq \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A). \end{aligned}$$

By Proposition 3.4(3),

$$cl_{\sigma_R}(A) = (int_{\sigma_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A) \quad (A \in F(X)).$$

(3) This holds by Proposition 3.4(3).

(4) This holds by (3) and Remark 3.6(1). □

Theorem 4.2. *Let (X, R) be a fuzzy approximation space.*

(1) *(X, σ_R) is not connected.*

(2) *(X, σ_R) is T_{-1}.*

(3) *a) If $t_R < 1$, then (X, σ_R) is sub-T_0.*

b) If (X, σ_R) is sub-T_0, then for any $x, y \in U$ with $x \neq y$, $R(x, y) \wedge R(y, x) < 1$.

(4) *If R is preorder, then*

$$(X, \sigma_R) \text{ is } T_1 \iff \sigma_R = F(X).$$

Proof. (1) This holds by Theorem 4.1(4).

(2) For any $x_\lambda, x_\mu \in P(X)$ and $\mu < \lambda$, put $u \in (\mu, \lambda)$, we have $x_\lambda \notin \bar{u}$. By Theorem 4.1(4), $\bar{u}$ is a closed remote-neighborhood of x_λ. Note that $x_\mu \in \bar{u}$. Then (X, σ_R) is T_{-1}.

(3) a) Let $x, y \in X$ with $x \neq y$. By $t_R < 1$, there exist $\lambda \in (t_R, 1]$. Put $A \in F(X)$ such that

$$A(t) = \begin{cases} t_R, & t = x, \\ \lambda, & t \neq x \end{cases},$$

then $R_A = \{(t, x) : t \in X - \{x\}\}$. For each $(t, x) \in R_A$,

$$R(x, t) \leq \bigvee_{x,y \in X, x\neq y} R(x, y) = t_R = \lambda \wedge t_R = A(t) \wedge A(x).$$

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By Proposition 3.8(1), $\overline{R}(A) \subseteq A$. By Theorem 4.1(3), $A \in (\sigma_R)^c$. Note that $A(x) = t_R < \lambda$. Then A is a closed remote-neighborhood of x_λ. Obviously, $y_\lambda \in A$. Hence (X, σ_R) is sub-T_0.

(4) b) For any $x, y \in X$ and $x \neq y$, by (X, σ_R) is sub-T_0, then there exist $\lambda \in (0, 1]$ and A such that $y_\lambda \in A$ and A is a closed remote-neighborhood of x_λ, or, there exist $\lambda \in (0, 1]$ and B such that $x_\lambda \in B$ and B is a closed remote-neighborhood of y_λ.

(i) If there exist $\lambda \in (0, 1]$ and A such that $y_\lambda \in A$ and A is a closed remote-neighborhood of x_λ, we can obtain that $A(y) \geq \lambda$ and $A(x) < \lambda$. By Theorem 4.1(3) and $A \in (\sigma_R)^c$, $\overline{R}(A) \subseteq A$. By $(y, x) \in R_A$ and Proposition 3.8(1),

$$R(x, y) \leq A(x) \wedge A(y) = A(x) < \lambda \leq 1.$$

Then $R(x, y) < 1$.

(ii) If there exist $\lambda \in (0, 1]$ and B such that $x_\lambda \in B$ and B is a closed remote-neighborhood of y_λ, similarly, we can prove that $R(y, x) < 1$.

So $R(x, y) \wedge R(y, x) < 1$. □

Definition 4.3. *Let R be a fuzzy relation on X. R is called pseudo-constant if there exists $a \in I$ such that for any $x, y \in X$,*

$$R(x, y) = \begin{cases} 1, & \text{if } x = y, \\ a, & \text{if } x \neq y. \end{cases}$$

We write R by a^*.

Obviously, every pseudo-constant fuzzy relation is an equivalence fuzzy relation.

Remark 4.4. (1) *For any $a, b \in I$, $a \leq b$ implies $a^* \subseteq b^*$.*

(2) *For each $a \in I$, $\theta_{a^*} = \sigma_{a^*}$.*

(3) $\sigma_{0^*} = F(X)$, $\sigma_{1^*} = \{\bar{a} : a \in I\}$.

Remark 4.5. *Let R be a fuzzy relation on X. Then*

(1) $R \subseteq t^*_R$.

(2) *R is reflexive $\Longleftrightarrow$ $s^*_R \subseteq R$.*

Lemma 4.6. *Let R_1 and R_2 be two fuzzy relations on U. If $R_1 \subseteq R_2$, then $\sigma_{R_2} \subseteq \sigma_{R_1}$.*

Proof. Let $A \in \sigma_{R_2}$. Then $A \subseteq \underline{R_2}(A)$. Note that $\underline{R_2}(A) \subseteq \underline{R_1}(A)$ by $R_1 \subseteq R_2$ and Proposition 3.9(1). Then $A \subseteq \underline{R_1}(A)$ and so $A \in \sigma_{R_1}$.

Thus $\sigma_{R_2} \subseteq \sigma_{R_1}$.

□

Theorem 4.7. *Let (X, R) be a fuzzy approximation space. Then*

(1) $\sigma_R \supseteq \sigma_{t^*_R}$.

(2) *If R is reflexive, then $\sigma_{t^*_R} \subseteq \tau_R \subseteq \sigma_{s^*_R}$.*

(3) $\sigma_R = \sigma_{1^*} \cup \{A \in F(X) : \forall (x, y) \in R_A, A(x) \vee A(y) \leq 1 - R(x, y)\}$.

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Proof. (1) This holds by Remark 4.5(1) and Lemma 4.6.

(1) This holds by Remark 4.5 and Lemma 4.6.

(3) This holds by Proposition 3.8(1), Theorem 4.1(1) and Remark 4.4(3). □

References

[1] D.Dubois, H.Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems, 17(1990), 191-208.

[2] R.Lowen, Fuzzy topological spaces and fuzzy compactness, Journal of Mathematical Analysis and Applications, 56(1976), 621-633.

[3] E.F.Lashin, A.M.Kozae, A.A.Abo Khadra, T.Medhat, Rough set theory for topological spaces, International Journal of Approximate Reasoning, 40(2005), 35-43.

[4] Y.Liu, M.Luo, Fuzzy Topology, World Scientific Publishing, Singapore, 1998.

[5] H.Lai, D.Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets and Systems, 157(2006), 1865-1885.

[6] Z.Li, T.Xie, Q.Li, Topological structure of generalized rough sets, Computers and Mathematics with Applications, 63(2012), 1066-1071.

[7] J.Mi, W.Wu, W.Zhang, Constructive and axiomatic approaches for the study of the theory of rough sets, Pattern Recognition Artificial Intelligence, 15(2002), 280-284.

[8] S.Nanda, Fuzzy rough sets, Fuzzy Sets and Systems, 45(1992), 157-160.

[9] Z.Pawlak, Rough sets, International Journal of Computer and Information Science, 11(1982), 341-356.

[10] Z.Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Dordrecht, 1991.

[11] D.Pei, A generalized model of fuzzy rough sets, International Journal of General Systems, 34(5)(2005), 603-613.

[12] B.Pu, Y.Liu, Fuzzy Topology I. Neighborhood strucure of a fuzzy point and Moore-Smiith convergence, Journal of Mathematical Analysis and Applications, 79(1980), 571-599.

[13] Z.Pei, D.Pei, L.Zheng, Topology vs generalized rough sets, International Journal of Approximate Reasoning, 52(2011), 231-239.

[14] Z.Pawlak, A.Skowron, Rudiments of rough sets, Information Sciences, 177(2007), 3-27.

[15] Z.Pawlak, A.Skowron, Rough sets: some extensions, Information Sciences, 177(2007), 28-40.

[16] Z.Pawlak, A.Skowron, Rough sets and Boolean reasoning, Information Sciences, 177(2007), 41-73.

[17] K.Qin, Z.Pei, On the topological properties of fuzzy rough sets, Fuzzy Sets and Systems, 151(2005), 601-613.

[18] A.M. Radzikowska, E.E.Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems, 126(2002), 137-155.

[19] H.Thiele, Fuzzy rough sets versus rough fuzzy setsAn interpretation and a comparative study using concepts of modal logics, in: Proc. 5th Europ. Congr. on Intelligent Technigues and Soft Computing, Aachen, Germany, September, 1997, pp.159-167.

[20] W.Wu, J.Mi, W.Zhang, Generalized fuzzy rough sets, Information Sciences, 151(2003), 263-282.

[21] Y.Y.Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Information Sciences, 111(1998), 239-259.

[22] L.Yang, L.Xu, Topological properties of generalized approximation spaces, Information Sciences, 181(2011), 3570-3580.

SOME OSTROWSKI TYPE INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS FOR $h-$CONVEX FUNCTIONS

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Abstract. In this paper, some Ostrowski type inequalities via Riemann-Liouville fractional integrals for $h-$convex functions, which are super-multiplicative or super-additive, are given. These results not only generalize those of [24, 25], but also provide new estimates on these types of Ostrowski inequalities for fractional integrals.

1. Introduction

Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \le M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall\, x \in [a,b]. \tag{1.1}$$

This is the well-known Ostrowski inequality (see [19] or [18, page 468]), which gives an upper bound for the approximation of the integral average $\frac{1}{(b-a)} \int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$. In recent years, a number of authors have written about generalizations, extensions and variants of such inequalities (see [1, 7, 8, 14, 15, 16, 21]).

Let us recall definitions of some kinds of convexity as follows.

Definition A. [11] *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$, one has*

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition B. [9] *We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P-$function or that f belongs to the class $P(I)$ if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$, one has*

$$f(tx + (1-t)y) \le f(x) + f(y).$$

Definition C. [13] *We say that $f : (0, \infty] \to [0, \infty]$ is $s-$convex in the second sense, or that f belongs to the class K_s^2, if for all $x, y \in (0, b]$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$, one has*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y).$$

Definition D. [26] *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$, one has*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y). \tag{1.2}$$

If inequality (1.2) is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$.

If $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$ for $s \in (0, 1]$, then $SX(h, I) \supseteq K_s^2$.

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Remark 1. [26] *Let h be a non-negative function such that $h(t) \geq t$ for all $t \in (0,1)$. If f is a non-negative convex function on I, then for $x, y \in I$, $t \in (0,1)$, one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.3}$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(t) \leq t$ for all $t \in (0,1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

Definition E. [26] *We say that $h : J \to \mathbb{R}$ is a super-multiplicative function, if for all $x, y \in J$, one has*

$$h(xy) \geq h(x)h(y).$$

Definition F. [2] *We say that $h : J \to \mathbb{R}$ is a super-additive function, if for all $x, y \in J$, one has*

$$h(x+y) \geq h(x) + h(y).$$

For recent results concerning $h-$convex functions see [5, 23, 25, 26] and references therein. More recently, Tunc [25] established some new Ostrowski type inequalities for the class of $h-$convex functions which are super-multiplicative or super-additive.

We then recall some definitions and mathematical preliminaries of fractional calculus theory which will be used throughout this paper.

Definition G. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. Here, $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities we refer the reader to the papers [3, 4, 6, 10, 12, 17, 20, 22] and the reference cited therein. In [24], Set established some new Ostrowski type inequalities for $s-$convex functions in the second sense via Riemann-Liouville fractional integral.

Motivated by these results, in the present paper, we establish some Ostrowski type inequalities via Riemann-Liouville fractional integrals for $h-$convex functions, which are super-multiplicative or super-additive. So, new estimates on these types of Ostrowski inequalities via fractional integrals are provided and the results of [24, 25] are generalized.

2. Ostrowski type fractional integral inequalities for h-convex functions

To prove our main theorems, we need the following identity established by Set in [24]:

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$, one has*

$$\begin{aligned}
&\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) \right] \\
=& \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx + (1-t)a)\, dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx + (1-t)b)\, dt.
\end{aligned} \tag{2.1}$$

Using this lemma, we can obtain the following fractional integral inequalities for $h-$convex functions.

Theorem 1. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0,1] \subseteq J$) be a non-negative and super-multiplicative function, $h(t) \geq t$ for $0 \leq t \leq 1$, $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $|f'|$ is $h-$convex on $[a,b]$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequalities for fractional integrals with $\alpha > 0$ hold:*

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \\
\leq&\frac{M\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]}{b-a}\int_0^1\left[t^{\alpha}h(t)+t^{\alpha}h(1-t)\right]dt \qquad (2.2)\\
\leq&\frac{M\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]}{b-a}\int_0^1\left[h\left(t^{\alpha+1}\right)+h\left(t^{\alpha}(1-t)\right)\right]dt. \qquad (2.3)
\end{aligned}$$

Proof. From (2.1) and since $|f'|$ is $h-$convex, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \\
\leq&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt \\
\leq&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^{\alpha}h(t)\left|f'(x)\right|+t^{\alpha}h(1-t)\left|f'(a)\right|\right]dt \\
&+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^{\alpha}h(t)\left|f'(x)\right|+t^{\alpha}h(1-t)\left|f'(b)\right|\right]dt \\
\leq&\frac{M(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^{\alpha}h(t)+t^{\alpha}h(1-t)\right]dt+\frac{M(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^{\alpha}h(t)+t^{\alpha}h(1-t)\right]dt,
\end{aligned}$$

which completes the proof of (2.2).

By using the additional properties of h in the assumptions, we further have

$$\begin{aligned}
\int_0^1\left[t^{\alpha}h(t)+t^{\alpha}h(1-t)\right]dt &\leq \int_0^1\left[h(t^{\alpha})h(t)+h(t^{\alpha})h(1-t)\right]dt \\
&\leq \int_0^1\left[h\left(t^{\alpha+1}\right)+h\left(t^{\alpha}(1-t)\right)\right]dt. \qquad (2.4)
\end{aligned}$$

Hence, the proof of (2.3) is complete. □

Remark 2. *We note that in the proof of (2.2) we does not use the additional super-multiplicative property of h and the condition "$h(t) \geq t$ for $0 \leq t \leq 1$". In Theorem 1, if we choose $\alpha = 1$, then (2.3) reduces the inequality [25, (2.1)], i.e.,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{M\left[(x-a)^2+(b-x)^2\right]}{b-a}\int_0^1\left[h\left(t^2\right)+h\left(t-t^2\right)\right]dt,$$

which can be better than the inequality (1.1) provide that h is chosen such that

$$\int_0^1\left[h\left(t^2\right)+h\left(t-t^2\right)\right]dt < \frac{1}{2}.$$

In Theorem 1, if we choose $h(t) = t$, then (2.2) and (2.3) reduce the inequality in [24, Corollary 1].

In the next corollary, we will also make use of the Beta function of Euler type, which is defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall\, x,y > 0.$$

Corollary 1. *If we choose $h(t) = t^s$, $s \in (0, 1]$, in Theorem 1, then we have*

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right| \\
\leq &\frac{M}{b-a}\left[1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right]\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1} \\
\leq &\frac{M}{b-a}\left[1+\frac{\Gamma(\alpha s+1)\Gamma(s+1)}{\Gamma(\alpha s+s+1)}\right]\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha s+s+1},
\end{aligned}$$

due to the fact that

$$\begin{aligned}
&\int_0^1\left[h\left(t^{\alpha+1}\right)+h\left(t^\alpha(1-t)\right)\right]dt=\int_0^1 t^{s(\alpha+1)}dt+\int_0^1 t^{\alpha s}(1-t)^s dt \\
=&\frac{1}{\alpha s+s+1}+\frac{\Gamma(\alpha s+1)\Gamma(s+1)}{\Gamma(\alpha s+s+2)}=\frac{1}{\alpha s+s+1}\left[1+\frac{\Gamma(\alpha s+1)\Gamma(s+1)}{\Gamma(\alpha s+s+1)}\right].
\end{aligned}$$

The first inequality is the same as the one established in [24, Theorem 7].

Theorem 2. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ $([0, 1] \subseteq J)$ be a non-negative and super-additive function, and $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L_1[a, b]$. If $|f'|^q$ is $h-$convex on $[a, b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $|f'(x)| \leq M$, $x \in [a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right| \\
\leq &\frac{M\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]}{(1+p\alpha)^{\frac{1}{p}}(b-a)}\left(\int_0^1[h(t)+h(1-t)]\,dt\right)^{\frac{1}{q}} \qquad (2.5)\\
\leq &\frac{M\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]}{(1+p\alpha)^{\frac{1}{p}}(b-a)}h^{\frac{1}{q}}(1). \qquad (2.6)
\end{aligned}$$

Proof. From Lemma 1 and using the well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right| \\
\leq&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)b)\right|dt \\
\leq&\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is $h-$convex and $|f'(x)| \leq M$, we get

$$\begin{aligned}
\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\leq&\int_0^1\left[h(t)\left|f'(x)\right|^q+h(1-t)\left|f'(a)\right|^q\right]dt \\
\leq&M^q\int_0^1[h(t)+h(1-t)]\,dt
\end{aligned}$$

and similarly

$$\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\leq M^q\int_0^1[h(t)+h(1-t)]\,dt.$$

By simple computation, we have

$$\int_0^1 t^{p\alpha}dt = \frac{1}{p\alpha+1}.$$

Using these results, we complete the proof of (2.5).

By using the super-additive property of h in the assumptions, we further have

$$\int_0^1 [h(t)+h(1-t)]\,dt \le \int_0^1 h(1)dt = h(1).$$

Hence, the proof of (2.6) is complete. □

Remark 3. *We note that in the proof of* (2.5) *we does not use the additional super-additive property of h. In Theorem 2, if we choose $h(t)=t$, then* (2.5) *reduces the inequality in* [24, Corollary 2]*; in Theorem 2, if we choose $\alpha=1$, then* (2.5) *becomes*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M\left[(x-a)^2+(b-x)^2\right]}{(1+p)^{\frac{1}{p}}(b-a)}\left(\int_0^1 [h(t)+h(1-t)]\,dt\right)^{\frac{1}{q}}, \tag{2.7}$$

which can be better than the inequality (1.1) *provide that p,q and h are chosen such that*

$$\left(\int_0^1 [h(t)+h(1-t)]\,dt\right)^{\frac{1}{q}} < \frac{1}{2}(1+p)^{\frac{1}{p}}.$$

Corollary 2. *If we choose $h(t)=t^s$, $s\in(0,1]$, in Theorem 2, then we have*

$$\begin{aligned}&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right|\\ \le&\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a},\end{aligned}$$

due to the fact that

$$\int_0^1 [h(t)+h(1-t)]\,dt = \frac{2}{s+1}.$$

This is the inequality established in [24, Theorem 8]*.*

Theorem 3. *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative and super-multiplicative function, $h(t)\ge t$ for $0\le t\le 1$, $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is $h-$convex on $[a,b]$, $q\ge 1$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right|\\ \le&\frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 [t^\alpha h(t)+t^\alpha h(1-t)]\,dt\right)^{\frac{1}{q}} &(2.8)\\ \le&\frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left[h\left(t^{\alpha+1}\right)+h\left(t^\alpha(1-t)\right)\right]dt\right)^{\frac{1}{q}}. &(2.9)\end{aligned}$$

Proof. From Lemma 1 and using the well-known power mean inequality, we have

$$\begin{aligned}&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right|\\ \le&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)b)\right|dt\end{aligned}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is $h-$convex on $[a,b]$ and $|f'(x)|\leq M$, we get

$$\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt \leq \int_0^1 \left[t^\alpha h(t)\left|f'(x)\right|^q + t^\alpha h(1-t)\left|f'(a)\right|^q\right] dt$$
$$\leq M^q \int_0^1 \left[t^\alpha h(t) + t^\alpha h(1-t)\right] dt$$

and similarly

$$\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt \leq M^q \int_0^1 \left[t^\alpha h(t) + t^\alpha h(1-t)\right] dt.$$

Using these inequalities, we complete the proof of (2.8).

By using the additional properties of h in the assumptions, we further have (2.4). Hence, the proof of (2.9) is complete. □

Remark 4. *We note that in the proof of* (2.8) *we does not use the additional super-multiplicative property of h and the condition "$h(t)\geq t$ for $0\leq t\leq 1$". In Theorem 3, if we choose $\alpha=1$, then* (2.9) *reduces the inequality* [25, (2.4)]*; in Theorem 3, if we choose $h(t)=t$, then* (2.8) *and* (2.9) *reduce the inequality in* [24, Corollary 3].

Corollary 3. *If we choose $h(t)=t^s$, $s\in(0,1]$, in Theorem 3, then we have*

$$\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\right]\right|$$
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\frac{1}{(\alpha+s+1)^{\frac{1}{q}}}\left[1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha s+s+1)}\right]^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}$$
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\frac{1}{(\alpha+s+1)^{\frac{1}{q}}}\left[1+\frac{\Gamma(\alpha s+1)\Gamma(s+1)}{\Gamma(\alpha s+s+1)}\right]^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}.$$

The first inequality is the same as the one established in [24, Theorem 9].

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References

[1] M. Alomari et al., Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. **23** (2010), no. 9, 1071–1076.

[2] H. Alzer, A superadditive property of Hadamard's gamma function, Abh. Math. Semin. Univ. Hambg. **79** (2009), no. 1, 11–23.

[3] G. Anastassiou et al., Montgomery identities for fractional integrals and related fractional inequalities, JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 4, Article 97, 6 pp.

[4] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 3, Article 86, 5 pp.

[5] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl. **58** (2009), no. 9, 1869–1877.

[6] Z. Dahmani, New inequalities in fractional integrals, Int. J. Nonlinear Sci. **9** (2010), no. 4, 493–497.

[7] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, Bull. Austral. Math. Soc. **60** (1999), no. 3, 495–508.

[8] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, Comput. Math. Appl. **38** (1999), no. 11-12, 33–37.

[9] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. **21** (1995), no. 3, 335–341.

[10] B. Dyda, Fractional Hardy inequality with a remainder term, Colloq. Math. **122** (2011), no. 1, 59–67.

[11] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, in *Numerical mathematics and mathematical physics (Russian)*, 138–142, 166, Moskov. Gos. Ped. Inst., Moscow.

[12] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in *Fractals and fractional calculus in continuum mechanics (Udine, 1996)*, 223–276, CISM Courses and Lectures, 378 Springer, Vienna.

[13] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math. **48** (1994), no. 1, 100–111.

[14] W. J. Liu and Q. -A. Ngô, A generalization of Ostrowski inequality on time scales for k points, Appl. Math. Comput. **203** (2008), no. 2, 754–760.

[15] Z. Liu, Some companions of an Ostrowski type inequality and applications, JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 2, Article 52, 12 pp.

[16] Z. X. Lü, On sharp inequalities of Simpson type and Ostrowski type in two independent variables, Comput. Math. Appl. **56** (2008), no. 8, 2043–2047.

[17] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.

[18] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Mathematics and its Applications (East European Series), 53, Kluwer Acad. Publ., Dordrecht, 1991.

[19] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, Comment. Math. Helv. **10** (1937), no. 1, 226–227.

[20] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.

[21] M. Z. Sarikaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenian. (N.S.) **79** (2010), no. 1, 129–134.

[22] M. Z. Sarikaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, Abstr. Appl. Anal. 2012, Art. ID 428983, 1-10.

[23] M. Z. Sarikaya, A. Saglam and H. Yildirim, On some Hadamard-type inequalities for h-convex functions, J. Math. Inequal. **2** (2008), no. 3, 335–341.

[24] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Comput. Math. Appl. **63** (2012), no. 7, 1147–1154.

[25] M. Tunc, Ostrowski-type inequalities via h-convex functions with applications to special means, J. Inequal. Appl. **2013** (2013), no. 326, 1-10.

[26] S. Varošanec, On h-convexity, J. Math. Anal. Appl. **326** (2007), no. 1, 303–311.

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SOME SIMPSON TYPE INEQUALITIES FOR $h-$CONVEX AND $(\alpha, m)-$CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some Simpson type inequalities for functions whose third derivatives in the absolute value are $h-$convex and $(\alpha, m)-$convex, respectively.

1. INTRODUCTION

The following inequality is well known in the literature as Simpson's inequality:

$$\left| \int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5, \tag{1.1}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is supposed to be four time differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. This inequality gives an error bound for the classical Simpson quadrature formula, which, actually, is one of the most used quadrature formulae in practical applications. In recent years, such inequalities were studied extensively by many researchers and numerious generalizations, extensions and variants of them appeared in a number of papers (see [1, 5, 6, 10, 11, 12, 13, 19, 21]).

Let us recall definitions of some kinds of convexity as follows.

Definition A. *[8] We say that $f : I \to \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{1.2}$$

Definition B. *[7] We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P-$function or that f belongs to the class $P(I)$ if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{1.3}$$

Definition C. *[9] Let $s \in (0, 1]$. A function $f : (0, \infty] \to [0, \infty]$ is said to be $s-$convex in the second sense if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \tag{1.4}$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$. This class of $s-$convex functions is usually denoted by K_s^2.

Definition D. *[22] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y). \tag{1.5}$$

If inequality (1.5) is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$.

Obviously, if $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1]$, then $SX(h, I) \supseteq K_s^2$. For recent results concerning $h-$convex functions see [3, 4, 14, 18, 22] and references therein.

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Definition E. [20] *The function $f:[0,b]\to\mathbb{R}$ is said to be $m-$convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have*

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0,b]$ for which $f(0)\le 0$.

Definition F. [15] *The function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be $(\alpha,m)-$convex, where $(\alpha,m)\in[0,1]^2$, if for all $x,y\in[0,b]$ and $t\in[0,1]$ we have*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y).$$

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha,m)-$convex functions on $[0,b]$ for which $f(0)\le 0$.

If we choose $(\alpha,m)=(1,m)$, it can be easily seen that $(\alpha,m)-$convexity reduces to $m-$convexity and for $(\alpha,m)=(1,1)$, we have ordinary convex functions on $[0,b]$.

Recently, Özdemir et al. [16] established some Simpson type inequalities for functions whose third derivatives in the absolute value are $m-$convex. In [17], Özdemir et al. established the following inequalities for functions whose third derivatives in the absolute value are $s-$convex in the second sense.

Theorem A. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'''\in L_1[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f'''|$ is $s-$convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ &\le\frac{(b-a)^4}{6}\left[\frac{2^{-4-s}\left((1+s)(2+s)+34+2^{4+s}(-2+s)+11s+s^2\right)}{(1+s)(2+s)(3+s)(4+s)}\right]\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right].\end{aligned}\tag{1.6}$$

Theorem B. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'''\in L_1[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f'''|^q$ is $s-$convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ \le&\frac{(b-a)^4}{48}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{\frac{1}{p}}\left\{\left[\frac{1}{2^{s+1}(s+1)}\left|f'''(a)\right|^q+\frac{2^{s+1}-1}{2^{s+1}(s+1)}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{2^{s+1}-1}{2^{s+1}(s+1)}\left|f'''(a)\right|^q+\frac{1}{2^{s+1}(s+1)}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{1.7}$$

Theorem C. *Suppose that all the assumptions of Theorem B are satisfied. Then*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ \le&\frac{(b-a)^4}{6}\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\\ &\times\left\{\left(\frac{2^{-4-s}}{(3+s)(4+s)}\left|f'''(a)\right|^q+\frac{2^{-4-s}\left(34+2^{4+s}(-2+s)+11s+s^2\right)}{(1+s)(2+s)(3+s)(4+s)}\left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{2^{-4-s}\left(34+2^{4+s}(-2+s)+11s+s^2\right)}{(1+s)(2+s)(3+s)(4+s)}\left|f'''(a)\right|^q+\frac{2^{-4-s}}{(3+s)(4+s)}\left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{1.8}$$

The main purpose of this paper is to establish some new Simpson type inequalities for functions whose third derivatives in the absolute value are $h-$convex and $(\alpha,m)-$convex, respectively.

2. Simpson type inequalities for h-convex functions

To prove our main theorems, we need the following identity established in [2]:

Lemma 1. *Let $f : I \to \mathbb{R}$ be a function such that f''' be absolutely continuous on I°, the interior of I. Assume that $a, b \in I^\circ$, with $a < b$ and $f''' \in L_1[a,b]$. Then, the following equality holds:*

$$\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = (b-a)^4 \int_0^1 p(t) f'''(ta + (1-t)b)dt,$$

where

$$p(t) = \begin{cases} \frac{1}{6}t^2\left(t - \frac{1}{2}\right), & t \in [0, \frac{1}{2}], \\ \frac{1}{6}(t-1)^2\left(t - \frac{1}{2}\right), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Using this lemma, we can obtain the following inequalities for $h-$convex functions.

Theorem 1. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ $([0,1] \subseteq J)$ be a non-negative function, and $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f''' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'''|$ is $h-$convex on $[a,b]$, then*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$
$$\leq \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(t)dt + \int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(1-t)dt\right]\left[\left|f'''(a)\right| + \left|f'''(b)\right|\right]. \tag{2.1}$$

Proof. From Lemma 1 and $h-$convexity of $|f'''|$, we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\
\leq &(b-a)^4\left\{\int_0^{\frac{1}{2}}\left|\frac{1}{6}t^2\left(t - \frac{1}{2}\right)\right|\left|f'''(ta + (1-t)b)\right|dt \right.\\
&\left. + \int_{\frac{1}{2}}^1\left|\frac{1}{6}(t-1)^2\left(t - \frac{1}{2}\right)\right|\left|f'''(ta + (1-t)b)\right|dt\right\} \\
\leq &\frac{(b-a)^4}{6}\left\{\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)\left(h(t)\left|f'''(a)\right| + h(1-t)\left|f'''(b)\right|\right)dt \right.\\
&\left. + \int_{\frac{1}{2}}^1 (t-1)^2\left(t - \frac{1}{2}\right)\left(h(t)\left|f'''(a)\right| + h(1-t)\left|f'''(b)\right|\right)dt\right\} \\
= &\frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(t)dt + \int_{\frac{1}{2}}^1 (t-1)^2\left(t - \frac{1}{2}\right)h(t)dt\right]\left[\left|f'''(a)\right| + \left|f'''(b)\right|\right],
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(t)dt + \int_{\frac{1}{2}}^1 (t-1)^2\left(t - \frac{1}{2}\right)h(t)dt \\
= &\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(1-t)dt + \int_{\frac{1}{2}}^1 (t-1)^2\left(t - \frac{1}{2}\right)h(1-t)dt \\
= &\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(t)dt + \int_0^{\frac{1}{2}} t^2\left(\frac{1}{2} - t\right)h(1-t)dt.
\end{aligned}$$

Hence, the proof of (2.1) is complete. □

Remark 1. *In Theorem 1, if we choose $h(t) = t^s$, $s \in (0,1]$, then* (2.1) *reduces to* (1.6).

Theorem 2. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0,1] \subseteq J$) be a non-negative function, and $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f''' \in L_1[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'''|^q$ is $h-$convex on $[a,b]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq &\frac{(b-a)^4}{48}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\left(\int_0^{\frac{1}{2}} h(t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} h(1-t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\int_0^{\frac{1}{2}} h(1-t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} h(t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right\}. \qquad (2.2)
\end{aligned}$$

Proof. From Lemma 1, and using the $h-$convexity of $|f'''|^q$ and the well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq &\frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left((t-1)^2\left(t-\frac{1}{2}\right)\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^4}{6}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left[h(t)|f'''(a)|^q+h(1-t)|f'''(b)|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left[h(t)|f'''(a)|^q+h(1-t)|f'''(b)|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^4}{48}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\left(\int_0^{\frac{1}{2}} h(t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} h(1-t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\int_{\frac{1}{2}}^1 h(t)dt\right)|f'''(a)|^q+\left(\int_{\frac{1}{2}}^1 h(1-t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where we have used the fact that

$$\int_0^{\frac{1}{2}}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt = \int_{\frac{1}{2}}^1\left((t-1)^2\left(t-\frac{1}{2}\right)\right)^p dt = \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)} \qquad (2.3)$$

and Γ is the Gamma function. Hence, the proof of (2.2) is complete. □

Remark 2. *In Theorem 2, if we choose $h(t) = t^s$, $s \in (0,1]$, then* (2.2) *reduces to* (1.7).

A different approach leads to the following result.

Theorem 3. *Suppose that all the assumptions of Theorem 2 are satisfied. Then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq&\frac{(b-a)^4}{6}\left(\frac{1}{192}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(1-t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(1-t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(t)dt\right)|f'''(b)|^q\right]^{\frac{1}{q}}\right\}. \qquad (2.4)
\end{aligned}$$

Proof. From Lemma 1 and using the well-known power-mean inequality we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq&\frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)|f'''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)|f'''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'''|^q$ is $h-$convex, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)|f'''(ta+(1-t)b)|^q\,dt \\
\leq&\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left(h(t)|f'''(a)|^q+h(1-t)|f'''(b)|^q\right)dt \\
=&\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(1-t)dt\right)|f'''(b)|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)|f'''(ta+(1-t)b)|^q\,dt \\
\leq&\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)\left(h(t)|f'''(a)|^q+h(1-t)|f'''(b)|^q\right)dt \\
=&\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(1-t)dt\right)|f'''(a)|^q+\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)h(t)dt\right)|f'''(b)|^q.
\end{aligned}$$

Hence, the proof of (2.4) is complete. □

Remark 3. *In Theorem 3, if we choose* $h(t)=t^s$, $s\in(0,1]$, *then* (2.4) *reduces to* (1.8).

3. Simpson type inequalities for $(\alpha,m)-$convex functions

We use the following modified identity:

Lemma 2. [16, Lemma 2] *Let* $f: I\to\mathbb{R}$ *be a function such that* f''' *be absolutely continuous on* I°*, the interior of* I*. Assume that* $a,b\in I^\circ$*, with* $a<b$*,* $m\in(0,1]$ *and* $f'''\in L_1[a,b]$*. Then, the*

following equality holds:

$$\begin{aligned}&\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]\\ =&(mb-a)^4\int_0^1 p(t)f'''(ta+m(1-t)b)dt,\end{aligned}$$

where

$$p(t)=\begin{cases}\frac{1}{6}t^2\left(t-\frac{1}{2}\right), & t\in[0,\frac{1}{2}],\\ \frac{1}{6}(t-1)^2\left(t-\frac{1}{2}\right), & t\in(\frac{1}{2},1].\end{cases}$$

Using this lemma, we can obtain the following inequalities for $(\alpha,m)-$convex functions.

Theorem 4. *Let $f: I\subset[0,b^*]\to\mathbb{R}$, be a differentiable function on I° such that $f'''\in L_1[a,b]$ where $a,b\in I$ with $a<b$, $b^*>0$. If $|f'''|^q$ is $(\alpha,m)-$convex on $[a,b]$ for $(\alpha,m)\in[0,1]^2$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\left|\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]\right|\\ \leq&\frac{(mb-a)^4}{96}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{\frac{1}{p}}\left\{\left[\frac{|f'''(a)|^q+m[2^\alpha(1+\alpha)-1]\,|f'''(b)|^q}{2^\alpha(1+\alpha)}\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{(2^{1+\alpha}-1)\,|f'''(a)|^q+m[2^\alpha(1+\alpha)-(2^{1+\alpha}-1)]\,|f'''(b)|^q}{2^\alpha(1+\alpha)}\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.1}$$

Proof. From Lemma 2 and using Hölder's inequality we have

$$\begin{aligned}&\left|\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]\right|\\ \leq&\frac{(mb-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}}\left(t^2\left(\frac{1}{2}-t\right)\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{1}{2}}^1\left((t-1)^2\left(t-\frac{1}{2}\right)\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Due to the $(\alpha,m)-$convexity of $|f'''|^q$, we have

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|f'''(ta+m(1-t)b)\right|^q dt\leq&\int_0^{\frac{1}{2}}\left[t^\alpha\left|f'''(a)\right|^q+m(1-t^\alpha)\left|f'''(b)\right|^q\right]dt\\ =&\frac{|f'''(a)|^q+m[2^\alpha(1+\alpha)-1]\,|f'''(b)|^q}{2^{1+\alpha}(1+\alpha)}\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^1\left|f'''(ta+m(1-t)b)\right|^q dt\leq&\int_{\frac{1}{2}}^1\left[t^\alpha\left|f'''(a)\right|^q+m(1-t^\alpha)\left|f'''(b)\right|^q\right]dt\\ =&\frac{(2^{1+\alpha}-1)\,|f'''(a)|^q+m[2^\alpha(1+\alpha)-(2^{1+\alpha}-1)]\,|f'''(b)|^q}{2^\alpha(1+\alpha)}.\end{aligned}$$

The proof of (3.1) is complete by combining the above inequalities and (2.3). □

Remark 4. *In Theorem 4, if we choose $\alpha=1$, we get the inequality in* [16, Theorem 4].

Theorem 5. *Let the assumptions of Theorem 4 hold with $q \geq 1$. Then*

$$\begin{aligned}
&\left|\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb)\right]\right| \\
\leq &\frac{(mb-a)^4}{1152}\left\{\left(\frac{12\left|f'''(a)\right|^q + m[2^\alpha(3+\alpha)(4+\alpha)-12]\left|f'''(b)\right|^q}{2^\alpha(3+\alpha)(4+\alpha)}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{12[\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)]}{2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}\left|f'''(a)\right|^q\right. \\
&\left.\left.+m\left[1-\frac{12[\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)]}{2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}\right]\left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right\}. \qquad (3.2)
\end{aligned}$$

Proof. From Lemma 2, using the well known power-mean inequality and $(\alpha, m)-$convexity of $|f'''|^q$, we have

$$\begin{aligned}
&\left|\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb)\right]\right| \\
\leq &\frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left|f'''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)\left|f'''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left[t^\alpha\left|f'''(a)\right|^q + m(1-t^\alpha)\left|f'''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)\left[t^\alpha\left|f'''(a)\right|^q + m(1-t^\alpha)\left|f'''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using the fact that

$$\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)t^\alpha dt = \frac{1}{16\times 2^\alpha(3+\alpha)(4+\alpha)},$$

$$\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)(1-t^\alpha)dt = \frac{2^\alpha(3+\alpha)(4+\alpha)-12}{192\times 2^\alpha(3+\alpha)(4+\alpha)},$$

$$\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)t^\alpha dt = \frac{\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)}{16\times 2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}$$

and

$$\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)(1-t^\alpha)dt = \frac{2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)-12[\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)]}{192\times 2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)},$$

we obtain

$$\begin{aligned}
&\left|\int_a^{mb} f(x)dx - \frac{mb-a}{6}\left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb)\right]\right| \\
\leq &\frac{(mb-a)^4}{6}\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left\{\left(\frac{12\left|f'''(a)\right|^q + m[2^\alpha(3+\alpha)(4+\alpha)-12]\left|f'''(b)\right|^q}{192\times 2^\alpha(3+\alpha)(4+\alpha)}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)}{16\times 2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}\left|f'''(a)\right|^q\right.
\end{aligned}$$

$$+m\left[\frac{1}{192}-\frac{\alpha^2+11\alpha+34-2^{4+\alpha}(2-\alpha)}{16\times 2^\alpha(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}\right]\left|f'''(b)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\},$$

which implies the desired result. □

Remark 5. *In Theorem 5, if we choose $\alpha = 1$, we have the inequality in* [16, Theorem 5].

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References

[1] M. Alomari and M. Darus, On some inequalities of Simpson-type via quasi-convex functions and applications, Transylv. J. Math. Mech. **2** (2010), no. 1, 15–24.

[2] M. Alomari and S. Hussain, Two inequalities of Simpson type for quasi-convex functions and applications, Appl. Math. E-Notes **11** (2011), 110–117.

[3] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl. **58** (2009), no. 9, 1869–1877.

[4] P. Burai and A. Házy, On approximately h-convex functions, J. Convex Anal. **18** (2011), no. 2, 447–454.

[5] S. S. Dragomir, On Simpson's quadrature formula for mappings of bounded variation and applications, Tamkang J. Math. **30** (1999), no. 1, 53–58.

[6] S. S. Dragomir, R. P. Agarwal and P. Cerone, On Simpson's inequality and applications, J. Inequal. Appl. **5** (2000), no. 6, 533–579.

[7] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. **21** (1995), no. 3, 335–341.

[8] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, in *Numerical mathematics and mathematical physics (Russian)*, 138–142, 166, Moskov. Gos. Ped. Inst., Moscow.

[9] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math. **48** (1994), no. 1, 100–111.

[10] V. N. Huy and Q. -A. Ngô, New inequalities of Simpson-like type involving n knots and the mth derivative, Math. Comput. Modelling **52** (2010), no. 3-4, 522–528.

[11] W. J. Liu, Y. N. Sun and Q. L. Zhang, Some new error inequalities for a generalized quadrature rule of open type, Comput. Math. Appl. **62** (2011), no. 5, 2218–2224.

[12] Z. Liu, An inequality of Simpson type, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2059, 2155–2158.

[13] Z. Liu, Some sharp modified Simpson type inequalities and applications, Vietnam J. Math. **39** (2011), no. 2, 135–144.

[14] Z. Liu, A note on Ostrowski type inequalities related to some s-convex functions in the second sense, Bull. Korean Math. Soc. **49** (2012), no. 4, 775–785.

[15] V.G. Miheşan, A generalization of the convexity, Seminar of Functional Equations, Approx. and Convex, Cluj-Napoca (Romania) (1993).

[16] M. E. Özdemir, M. Avci and H. Kavurmaci, Simpson type inequalities for $m-$convex functions, arXiv: 1112.3559v1 [math.FA].

[17] M. E. Özdemir, M. Avci and H. Kavurmaci, Simpson type inequalities for functions whose third derivatives in the absolute value are $s-$convex and $s-$concave functions, arXiv:1206.1193v1 [math.CA].

[18] M. Z. Sarikaya, A. Saglam and H. Yildirim, On some Hadamard-type inequalities for h-convex functions, J. Math. Inequal. **2** (2008), no. 3, 335–341.

[19] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On new inequalities of Simpson's type for s-convex functions, Comput. Math. Appl. **60** (2010), no. 8, 2191–2199.

[20] G. H. Toader, Some generalisations of the convexity, Proc. Colloq. Approx. Optim (1984), 329–338.

[21] K.-L. Tseng, G.-S. Yang and S. S. Dragomir, On weighted Simpson type inequalities and applications, J. Math. Inequal. **1** (2007), no. 1, 13–22.

[22] S. Varošanec, On h-convexity, J. Math. Anal. Appl. **326** (2007), no. 1, 303–311.

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